Stability of solutions to some evolution problems

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Abstract

Large time behavior of solutions to abstract differential equations is studied. The corresponding evolution problem is:

$$\dot{u} = A(t)u + F(t, u) + b(t), \quad t \geq 0; \quad u(0) = u_0. \qquad (*)$$

Here $\dot{u} := \frac{du}{dt}$, $u = u(t) \in H$, $t \in \mathbb{R}_+ := [0, \infty)$, $A(t)$ is a linear dissipative operator: $\mathrm{Re}(A(t)u, u) \leq -\gamma(t)(u, u)$, $\gamma(t) \geq 0$, $F(t, u)$ is a nonlinear operator, $\|F(t, u)\| \leq c_0\|u\|^p$, $p > 1$, c_0, p are constants, $\|b(t)\| \leq \beta(t)$, $\beta(t) \geq 0$ is a continuous function.

Sufficient conditions are given for the solution $u(t)$ to problem (*) to exist for all $t \geq 0$, to be bounded uniformly on $\mathbb{R}_+$, and a bound on $\|u(t)\|$ is given. This bound implies the relation $\lim_{t\to\infty} \|u(t)\| = 0$ under suitable conditions on $\gamma(t)$ and $\beta(t)$.

The basic technical tool in this work is the following nonlinear inequality:

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t),\ t \geq 0; \quad g(0) = g_0,$$

which holds on any interval $[0, T)$ on which $g(t) \geq 0$ exists and has bounded derivative from the right, $\dot{g}(t) := \lim_{s\to+0} \frac{g(t+s)-g(t)}{s}$. It is assumed that $\gamma(t)$, and $\beta(t)$ are nonnegative continuous functions of t defined on $\mathbb{R}_+ := [0, \infty)$, the function $\alpha(t, g)$ is defined for all $t \in \mathbb{R}_+$, locally Lipschitz with respect to g uniformly with respect to t on any compact subsets $[0, T]$, $T < \infty$, and non-decreasing with respect to g. If there exists a function $\mu(t) > 0$, $\mu(t) \in C^1(\mathbb{R}_+)$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \forall t \geq 0; \quad \mu(0)g(0) \leq 1,$$

then $g(t)$ exists on all of $\mathbb{R}_+$, that is $T = \infty$, and the following estimate holds:

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0.$$

If $\mu(0)g(0) < 1$, then $0 \leq g(t) < \frac{1}{\mu(t)}, \quad \forall t \geq 0$.

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