Stability of solutions to some evolution problems

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Abstract

Large time behavior of solutions to abstract differential equations is studied. The corresponding evolution problem is:

\[ \dot{u} = A(t)u + F(t, u) + b(t), \quad t \geq 0; \quad u(0) = u_0. \quad (\ast) \]

Here \( \dot{u} := \frac{du}{dt} \), \( u = u(t) \in H, \quad t \in \mathbb{R}_+ := [0, \infty) \), \( A(t) \) is a linear dissipative operator: \( \text{Re}(A(t)u, u) \leq -\gamma(t)(u, u) \), \( \gamma(t) \geq 0 \), \( F(t, u) \) is a nonlinear operator, \( \|F(t, u)\| \leq c_0\|u\|^p, \quad p > 1, \quad c_0, p \) are constants, \( \|b(t)\| \leq \beta(t), \quad \beta(t) \geq 0 \) is a continuous function.

Sufficient conditions are given for the solution \( u(t) \) to problem \( (\ast) \) to exist for all \( t \geq 0 \), to be bounded uniformly on \( \mathbb{R}_+ \), and a bound on \( \|u(t)\| \) is given. This bound implies the relation \( \lim_{t \to \infty} \|u(t)\| = 0 \) under suitable conditions on \( \gamma(t) \) and \( \beta(t) \).

The basic technical tool in this work is the following nonlinear inequality:

\[ \dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \quad t \geq 0; \quad g(0) = g_0, \]

which holds on any interval \([0, T]\) on which \( g(t) \geq 0 \) exists and has bounded derivative from the right, \( \dot{g}(t) := \lim_{s \to +0} g(t+s) - g(t) \). It is assumed that \( \gamma(t) \), and \( \beta(t) \) are nonnegative continuous functions of \( t \) defined on \( \mathbb{R}_+ := [0, \infty) \), the function \( \alpha(t, g) \) is defined for all \( t \in \mathbb{R}_+ \), locally Lipschitz with respect to \( g \) uniformly with respect to \( t \) on any compact subsets \([0, T]\), \( T < \infty \), and non-decreasing with respect to \( g \).

If there exists a function \( \mu(t) > 0, \mu(t) \in C^1(\mathbb{R}_+), \) such that

\[ \alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{1}{\mu(t)} \left( \gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \quad \forall t \geq 0; \quad \mu(0)g(0) \leq 1, \]

then \( g(t) \) exists on all of \( \mathbb{R}_+ \), that is \( T = \infty \), and the following estimate holds:

\[ 0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0. \]

If \( \mu(0)g(0) < 1 \), then \( 0 \leq g(t) < \frac{1}{\mu(t)}, \quad \forall t \geq 0. \)
References.


