Bifurcation scenario of a network of two coupled rings of cells

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Abstract. We study the bifurcation scenario appearing in systems of two coupled rings of cells with \( \mathbb{Z}_3 \times \mathbb{Z}_5 \) exact symmetry, and \( \mathbb{Z}_3 \) interior symmetry. This study was motivated by previous work by Antoneli, Dias and Pinto, on two rings of cells coupled through a ‘buffer’ cell, with \( \mathbb{Z}_3 \times \mathbb{Z}_5 \) and \( D_3 \times D_5 \) exact and interior symmetry groups. There, quasi-periodic behavior was found through a sequence of Hopf bifurcations. We questioned if an analogous mechanism could explain the appearance of quasi-periodic motion in the examples considered here. Surprisingly, we observe periodic and quasi-periodic states appearing also through Hopf bifurcations. We compute the relevant states numerically.

Keywords: Hopf bifurcation, exact symmetry, interior symmetry, coupled cells systems.

1 Introduction

Stewart, Golubitsky and Pivato [24] and Golubitsky, Stewart and Török [17] have developed a new theory for networks of coupled cells systems. They focused in patterns of synchrony and associated bifurcations.

Networks of coupled cells may be represented schematically by a directed graph, where the nodes correspond to the individual cells and the edges to the couplings between them. The term ‘cells’ means nonlinear dynamical systems of ordinary differential equations.

There has been considerable development on the study of synchrony, phase-relations, quasi-periodic motion, synchronized chaos, amongst others [5,6,12,20,18], in networks of coupled cells. Graphs architecture appear to be an important part in the explanation of these phenomena.

Networks of coupled cells may arise as models of animal and robot locomotion, speciation, visual perception, electric power grids, internet communication [8,9,21,11,22,23,10,7], and many others.

There are special networks of coupled cells that possess some degree of symmetry. We divide these networks in two groups: (i) networks with exact symmetry group; and (ii) networks with interior symmetry group. A symmetry of a network is a permutation on the nodes that preserves the network architecture (including cell-types and arrow-types). An interior symmetry
generalizes the notion of symmetry. It has been introduced by Golubitsky et al [13]. It is a permutation in a subset of the cells that partially preserves the network architecture. In this case, ‘forgetting’ about some arrows leads to a subnetwork whose symmetry group is the interior symmetry group of the entire network.

In this paper we study interesting dynamical features occurring in two coupled systems of two unidirectional rings, with \( \mathbb{Z}_3 \times \mathbb{Z}_5 \) exact symmetry and \( \mathbb{Z}_3 \) interior symmetry, see Fig. 1. We were motivated by previous work in the study of quasi-periodic motion in four examples of networks of two rings coupled through a ‘buffer’ cell, with \( \mathbb{Z}_3 \times \mathbb{Z}_5 \) and \( \mathbb{D}_3 \times \mathbb{D}_5 \) exact and interior symmetry [2–4]. We questioned if the bifurcation scenario observed in those cases was seen in the networks considered here. Surprisingly, here too, we find quasi-periodic states appearing through a sequence of Hopf bifurcations, analogously to what was found in [2–4]. We also obtain a curious feature appearing further away of the third Hopf bifurcation point, similarly to what was found in [2–4] and [12].

![Networks of two coupled unidirectional rings](image)

**Fig. 1.** Networks of two coupled unidirectional rings, one with three cells and the other with five. The network on the left (a) has exact \( \mathbb{Z}_3 \times \mathbb{Z}_5 \)-symmetry, the network on the right (b) has interior \( \mathbb{Z}_3 \)-symmetry.

### 1.1 Outline of the paper

In section 2, we give a brief summary of the coupled cells networks formalism. In section 2.2, we simulate the coupled cells systems associated to the networks of two coupled rings of cells in Fig. 1. We consider the cases of exact and interior symmetry. In section 3, we state the main conclusions and unravel future research directions.

### 2 Coupled cells network

A coupled cells network consists of a (i) finite set of nodes (or cells) \( \mathcal{C} \); (ii) an equivalence relation on cells in \( \mathcal{C} \), where the equivalence class of \( c \) is the type
of cell $c$; (iii) an input set of cells $\mathcal{I}(c)$, that consists of cells whose edges have cell $c$ as head; (iv) an equivalence relation on the edges (or arrows), where the equivalence class of $c$ is the type of edge $e$; (v) and satisfies the condition that ‘equivalent edges have equivalent tails and edges’.

We define, for each cell $c$ an internal phase space $P_c$, the total phase space of the network being $P = \prod_{i=1}^{n} P_c$. Coordinates on $P_c$ are denoted by $x_c$, and thus coordinates on $P$ are $(x_1, x_2, \ldots, x_n)$. At time $t$, the system is at state $(x_1(t), x_2(t), \ldots, x_n(t))$.

A vector field $f$ on $P$ that is compatible with the network architecture is said to be admissible for that network, and satisfies two conditions: (1) the domain and (2) the pull-back condition. Moreover, condition (1) states that each component $f_i$ corresponding to cell $c_i$ is a function of the cells in $\mathcal{I}(c_i)$. Condition (2) says that if cells $c_i$ and $c_j$ have isomorphic input cells then their corresponding components $f_i$ and $f_j$ are identical up to a suitable permutation of the relevant variables [14].

2.1 Symmetry groups

A symmetry of a coupled cells system is the group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on $P$ is by permutation of cell coordinates. Formally, we have a coupled system given by

$$\dot{x} = f(x)$$

where $f(x)$ is an admissible vector field for the a given network. If $f$ is $\Gamma$ symmetric, then $f(\gamma x) = \gamma f(x), \gamma \in \Gamma$ (equivariance condition). It follows from the “pull-back condition” that this equivariance condition is satisfied for all $\gamma \in \Gamma$, with respect to the action of the symmetry group $\Gamma$ on the phase space $P$, by commuting cells coordinates. A symmetry is thus a transformation of the phase space that sends solutions to solutions.

The network in Figure 1(a) is an example of a network with exact $\mathbb{Z}_3 \times \mathbb{Z}_5$ symmetry.

An interior symmetry generalizes the concept of exact symmetry and it was introduced by Golubitsky et al [13]. It is a group of permutations that acts in a subset of cells (but not on the entire set of cells) and partially preserves the network structure (cell-types and edges-types).

The network in Figure 1(b) is an example of a coupled cells system with $\mathbb{Z}_3$ ‘interior symmetry’. Moreover, if we ignore the couplings from cells $x_1$, $x_2$, $x_3$ to cells $y_1$, $y_2$, $y_3$, $y_4$, $y_5$, then the resulting network is $\mathbb{Z}_3$-exactly symmetric. Moreover, the network has interior $\mathbb{Z}_3$-symmetry on the set of cells $\{x_1, x_2, x_3\}$.

2.2 Numerical results

In this section we simulate the coupled cells systems associated with the two networks depicted in Fig. 1. We use the following function for the internal
dynamics of each of the eight cells \([2,12]\): 

\[ f(x) = \mu x - \frac{1}{10}x^2 - x^3 \]

where \(\mu\) is a real parameter.

The coupled cells system of equations associated to the network (a) in Fig. 1 is given by:

\[
x_j' = f(x_j) + c_1 (x_j - x_{j+1}) \quad j = 1, \ldots, 3 \\
y_j' = f(y_j) + c_2 (y_j - y_{j+1}) + d (y_j - x_1) + d (y_j - x_2) + d (y_j - x_3) \quad j = 1, \ldots, 5
\]

where \(c_1 = 0.75\), \(c_2 = 0.60\), \(d = 0.2\), and the indexing assumes \(x_4 \equiv x_1\) and \(y_6 \equiv y_1\).

The coupled cells system of equations associated to the network (b) in Fig. 1 is given by:

\[
x_j' = f(x_j) + c_1 (x_j - x_{j+1}) \quad j = 1, \ldots, 3 \\
y_j' = f(y_j) + c_2 (y_j - y_{j+1}) + d_1 (y_j - x_1) + d_2 (y_j - x_2) + d_3 (y_j - x_3) \quad j = 1, \ldots, 5
\]

where \(d_1 = 0.1\), \(d_2 = 0.2\), \(d_3 = 0.3\) and all other parameters and indexes are as above.

Note that if \(d_1 = d_2 = d_3\) then the structure of the coupled cell system (3) is consistent with the network of Figure 1(a) and thus has \(\mathbb{Z}_3 \times \mathbb{Z}_5\) exact symmetry.

We vary parameter \(\mu \in [-1.0, 2.0]\), going from positive values to negative values. We obtain a branching pattern similar to the schematic bifurcation diagram presented in Fig. 2.

**Fig. 2.** Schematic (partial) bifurcation diagram for the coupled cell systems in Fig. 1, near the equilibrium point. Solid lines represent stable solutions, dashed lines correspond to unstable solutions [2].

In Table 1, we give a summary of the values of the Hopf bifurcation points and the corresponding solutions in the two rings for the networks in Fig. 1. The first branch of Hopf bifurcation, 1\(^{st}\) (HB1), comes from a trivial branch of equilibria. The solutions corresponding to the primary branch

Fig. 3 shows the time series after (HB1) in the coupled cells systems (2)-(3). On the panel on the left we plot the time series for the network with $Z_3 \times Z_5$ exact symmetry and on the right panel we plot the time series for the network with $Z_3$ interior symmetry. In both cases, we observe a rotating wave on the 5-ring (periodic solution in which the cells in the 5-ring have the same wave form but they are 1/5 out of phase) and the cells in the 3-ring stay in equilibrium.

By varying further the parameter $\mu$, there is a secondary Hopf bifurcation point (HB2) where the time series of the cells in the 3-ring appear to show a rotating wave (periodic solution in which the cells in the 3-ring have the same wave form but they are 1/3 out of phase). Figures 4-5 (left) show the time series after the secondary Hopf bifurcation (HB2) in the coupled cell systems (2)-(3). The Hopf bifurcation “occurs” in the 3-ring, leading to a rotating wave on the 3-ring. Cells in both rings appear to be at a rotating wave state. The full solution is quasi-periodic (solution fills in the visible region), see Figures 4-5 (right).

Figures 6-7 show the time series after the tertiary Hopf bifurcation (HB3) in the coupled cells systems (2)-(3). Cells in the 3- and 5-rings appear to be at a rotating wave state. The full solution is quasi-periodic.

Table 1. Summary of the dynamical behavior of coupled cell systems associated to the networks in Fig. 1. In the first column we indicate some branches of solutions with the respective bifurcation points. The second, third and fourth columns show the type of asymptotic stable solutions in the rings and the full systems in the corresponding branch. See text for more details.
Fig. 3. Simulation of the coupled systems (2) and (3). Time series from the eight cells after the first Hopf bifurcation point (HB1). (Left) Exact symmetry $\mathbb{Z}_3 \times \mathbb{Z}_5$. Cells in the 3-ring are at equilibrium and cells in the 5-ring display a rotating wave. (Right) Interior symmetry $\mathbb{Z}_5$. Cells in the 3-ring are at equilibrium and cells in the 5-ring display a rotating wave.

Fig. 4. Simulation of the coupled system (2) with $\mathbb{Z}_3 \times \mathbb{Z}_5$ exact symmetry, after the second Hopf bifurcation point (HB2). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

Fig. 5. Simulation of the coupled system (3) with $\mathbb{Z}_3$ interior symmetry, after the second Hopf bifurcation point (HB2). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

Figures 8-9 show the time series further away from the tertiary Hopf bifurcation (HB3) in the coupled cell systems (2)-(3). In Figures 8-9, we plot,
Fig. 6. Simulation of the coupled system (2) with $\mathbb{Z}_3 \times \mathbb{Z}_5$ exact symmetry, after the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

Fig. 7. Simulation of the coupled system (3) with $\mathbb{Z}_3$ interior symmetry, after the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$.

on the left panel, the time series for the eight cells and on the right panel cell $x_1$ vs cell $y_5$, for the cases with exact symmetry and interior symmetry, respectively.

Fig. 8. Simulation of the coupled system (2) with $\mathbb{Z}_3 \times \mathbb{Z}_5$ exact symmetry, further away of the third Hopf bifurcation point (HB3). (Left) Time series from the eight cells. (Right) Cell $x_1$ vs cell $y_5$. 
The full solution is quasi-periodic that is, the time series on the 3-ring looks like a (approximate) rotating wave and the time series on the 5-ring a (approximate) rotating wave.

3 Conclusion

In this paper we study the dynamical behavior of two networks consisting of two coupled rings of cells that admit $Z_3 \times Z_5$ exact and $Z_3$ interior symmetry groups.

We find equilibria, rotating waves, quasi-periodic motion, and relaxation oscillations. The bifurcation diagram that explains the occurrence of these phenomena is similar to the one found in Antoneli et al [2–4]. There, authors study two rings coupled through a ‘buffer’ cell with $Z_3 \times Z_5$ and $D_3 \times D_5$ exact and interior symmetry groups. Analogously of what was found in [2–4], here too, the exotic behavior found further away of the third Hopf bifurcation point, reveals itself when a relaxation oscillation occurs. Relaxation oscillations are solutions that appear through canard explosions [19,25].

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References


Bifurcation Analysis of Multibody Parafoil-Payload System

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Abstract: The bifurcation theory is suitable to predict nonlinear dynamics, and stability of parafoil-payload system caused due to multibody dynamics, and presence of vertical offset between the parafoil aerodynamic center and the suspended payload center of gravity. Parafoil is high performance gliding parachute with sufficient control authority to steer it along desired path to achieve guided payload delivery for variety of application. But nonlinear dynamics and stability prediction is necessary before designing guidance law for a guided parafoil-payload delivery system. This paper rightly highlights the importance and application of Bifurcation theory for nonlinear dynamics and stability prediction using multibody Nine Degree-Of-Freedom (9-DOF) parafoil-payload concept. The advantage of 9-DOF model is that it allows to model the parafoil-payload system as multibody system with provision to incorporate aerodynamic forces and moments of parafoil and payload at their respective e locations.

Keywords: Bifurcation Methods, Multibody System, 9-DOF system, Parafoil-payload System, Stability, Trim.

1. Introduction

The dynamics of the parafoil-payload system is inherently nonlinear. Since the payload is usually heavier, the center of gravity of the system is closer to the payload than the parafoil. Large vertical offset between parafoil canopy AC(Aerodynamic center) and parafoil-payload system CG (Center of Gravity), together with the nonlinear parafoil canopy aerodynamics play a key role in making the system trim and stability characteristics significantly nonlinear. Also, parafoil-payload system has drag-dominant control characteristics, i.e., deflection of trailing edge control surface (brake) essentially acts to increase canopy drag, thereby causing a significant nonlinear effect on trim and stability.

Inertia and kinematic coupling terms in the parafoil and payload equations of motion, and coupling between parafoil and payload equations of motion arising out of joint constraints while considering total system as multibody system are other sources of nonlinearity. Nonlinearities present in parafoil-payload coupled system equations of motions manifest themselves as nonlinear dynamic behavior of the system in flight. Nonlinear behavior in longitudinal motion is
usually connected with canopy stall, e.g., stall onset at high angle of attack, and parafoil-payload coupling, e.g., unintentional trim during dynamic maneuver. The rigging angle which decides horizontal location of payload with respect to parafoil canopy in longitudinal axis, is a critical design parameter. Improper choice of rigging angle can lead to adverse dynamics during deployment, poor inflation, stall onset at high angle of attack, leading edge collapse at low angle of attack, and unintentional trim during dynamic maneuvers [1].

2. Bifurcation Methodology

Bifurcation methods are a convenient tool for numerical analysis of trim and stability of dynamical systems, including nonlinear effects. Bifurcation methods were introduced to flight dynamics by Carroll and Mehra [2], Zagainov and Goman [3], and Jahnke and Culick [4]. Bifurcation methods have been used to study constrained flight maneuvers, such as level trims[5]. Nonlinear behavior may be best understood in terms of bifurcations of the dynamical system, which is a record of all the critical points where equilibriums are either created or destroyed, or undergo a change in their stability.

The bifurcation method requires a model of the system formulated as set of first-order differential equations. Therefore, the set of nonlinear differential equations of 9-DOF parafoil-payload flight dynamic model derived in chapter 3 is suitable for use in a bifurcation analysis. The 9-DOF model uses apparent mass terms from standard literature and longitudinal aerodynamic data in tabular look-up form, including all nonlinearities, with angle of attack range of -10 to +80 deg. The lateral aerodynamics is modeled in terms of standard stability and control derivatives with no angle of attack dependency. The longitudinal aerodynamic data is linearly interpolated as a function of angle of attack in 1-deg intervals.
Trim and stability analysis using bifurcation methods essentially requires a dynamical system to be represented by a set of first-order ordinary differential equations as

\[ x = f(x, U) \quad 2.1 \]

where \( x \) is a vector of \( n \) state variables, \( U \) is a vector of \( m \) control parameters, and \( f \) is a vector of nonlinear mathematical functions. It is normal practice in bifurcation analysis to vary one parameter at a time while keeping all other parameters fixed. Let \( P \) be the set of \( m-1 \) fixed control parameters, and \( u \) be the scalar control parameter which is required to be varied. Hence, the bifurcation method requires tracking of all possible trim states \( x^* \), such that \( f(x^*, u, P) = 0 \) is satisfied, as a free control parameter \( u \) is varied within prescribed limits while keeping control parameter \( P \) fixed. This is achieved by use of a continuation algorithm.

Continuation algorithms are based on the implicit function theorem, which says that if the Jacobian matrix with respect to the state variables of the system is non-singular at \( (x_0, u_0, P) \), then there exists a neighborhood around \( (x_0, u_0, P) \), such that for each \( u \) in the neighborhood, i.e., at \( u_0 + \Delta u \), \( f(x, u, P) = 0 \) has a unique solution. A continuation algorithm AUTO2000 [6] is available in the public domain to carry out a standard bifurcation analysis (SBA). In the SBA methodology, a continuation algorithm is used to compute all possible trim solutions of the system as a free control parameter is varied, while other control parameters are kept fixed. As part of the procedure, the continuation algorithm also computes local stability information at each of the trim points.

Bifurcation diagram is a plot of the computed trim solutions as a function of the varying control parameter, projected on a two-dimensional plane and represents the global behavior of the system. The points where the stability of a branch of trim solutions changes are known as 'bifurcation points.' Bifurcation points correspond to migration of eigenvalues of the system from left half to right half complex plane, and hence lead to unstable dynamical behavior.

There are basically two types of bifurcations, namely static bifurcation and dynamic bifurcation. A static bifurcation is between multiple trim points which occurs when a real eigenvalue of the system crosses the imaginary axis. For example, a) 'saddle-node bifurcation' (Fig. 1b) associated with change in number of trim states, i.e., two trim points, one stable and one unstable are born or disappear as the control parameter is varied, b) 'pitchfork bifurcation' (Fig. 1c) associated with change in number of trim points, i.e., the stable (unstable) trim loses (gains) stability and two new stable (unstable) trims are born which are symmetric about original trim point, and c) 'transcritical bifurcation' (Fig. 1d) associated with exchange of stability between two trim branches. The dynamic bifurcation is between a trim point of converging (diverging) oscillation and a stable (unstable) limit cycle, e.g., Hopf bifurcation (HB) point (Fig. 1f) associated with single stable trim point changed into an unstable trim point with stable periodic oscillation. HB signifies onset of periodic oscillations, or limit cycles.
The advantages of the bifurcation analysis approach are:

1) Any high-order dynamical system $x = f(x, u)$ with a wide range of nonlinear functions $f$, can be studied with equal ease. For instance, highly augmented aircraft models were studied by Planeaux et al. [7] and Avanzini and de Matteis [8] for bifurcation analysis of aircraft with control augmentation system (CAS).

2) Using bifurcation analysis, it is possible to compute multiple trims states $x^*$ which co-exist at a fixed combination of control parameters $u$ and $P$. 

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![Fig. 1. Types of Bifurcations](image)
3) As the continuation algorithm can also compute local stability information at each trim and periodic solution, therefore, bifurcation analysis becomes very useful.

4) Using bifurcation analysis one can predict the behavior of the system as different bifurcations correspond to onset of different dynamics of the states, e.g., Hopf bifurcations predict wing rock or unstable phugoid, and pitchfork bifurcation predicts departure in flight dynamics.

3. Bifurcation Analysis of Multibody 9-DOF model

The rigging angle $\mu$ indicating payload to be swung forward or back using unequal suspension lines, has a direct effect on parafoil angle of attack $\alpha_p$ which decides the $L/D$ ratio and hence the glide slope. Thus, bifurcation analysis is carried out to find best glide rigging angle within safe operating $\alpha_p$ range. For the implementation of bifurcation analysis, 9-DOF flight dynamic model is represented as set of 18 first-order differential equations:

$$x = fx, u, P$$  \hspace{1cm} (3.1)

The geometric and aerodynamic data of a particular parafoil-payload system as given in Ref. [9] are used in the dynamic model. For the present study, we choose rigging angle $\mu$ as the continuation parameter, and fix the other control parameters as $\delta a = 0$ and consider three cases of zero, half, and full symmetric brake. Thus;

$$u = [\mu]$$  \hspace{1cm} (3.2)

$$P = [\delta a = 0, \delta s = \text{Zero; Half; Full}]$$  \hspace{1cm} (3.3)

Following the usual practice, inertial position coordinates of the connection point are not included in Eq. (3.1). The canopy and payload yaw angles, $\psi_p$ and $\psi_b$, which relate inertial velocity components have been constrained to zero, which limit the model to straight longitudinal glides. Therefore, in Eq. 3.1, state vector is reduced to 13 elements:

$$x = [p_b, q_b, r_b, p_p, q_p, r_p, u_c, v_c, w_c, \phi_b, \theta_b, \phi_p, \theta_p]$$

with $\phi_b = 0$ and $\phi_p = 0$.

Bifurcation diagrams were obtained by computing all possible trims of the system using a continuation algorithm, as rigging angle as a free control parameters $u = \mu$ was varied, with other controls kept fixed. To begin the continuation, a starting trim point was obtained from numerical simulation, and then the AUTO2000 continuation algorithm was used to compute other trims as $\mu$ was varied. The exercise was carried out for trims with zero, half and full brake deflections. The trims so computed are shown in terms of bifurcation diagrams of parafoil angle of attack $\alpha_p$, glide angle $\mu$, and payload pitch angle $\theta_b$, in Figs 2 to 4, and are discussed in the next section. These bifurcation diagrams are useful in identifying critical rigging angles at which trimmability and/or stability of the parafoil-payload system is lost. On all the bifurcation diagrams, trims represented by solid lines are stable trims, and those represented...
by dashed lines are unstable trims. Points of onset of instabilities are bifurcation points.

3. Bifurcation Results

The bifurcation results obtained from multi-body 9-DOF flight dynamics model of parafoil-payload as shown in Figs. 2 to 4 rightly highlight nonlinear flight stability and glide characteristics of parafoil-payload system with varying values of parafoil rigging angle and magnitude of symmetric brake control. The overall bifurcation diagrams for zero, half, and full symmetric brakes show a small region of oscillatory instability between two ‘Hopf bifurcations’ (solid squares), and a region of multiple trims between two ‘Saddle-node bifurcations’ (turning points). Large and small rigging angles show a single trim whereas intermediate rigging angles show multiple trims. Lower rigging angles show possibility of parafoil gliding with very large angles of attack $\alpha_p$ and smaller negative trim pitch angle of the canopy. The payload pitch angle trims at very small values showing pendulum effect on payload.

![Fig. 2. Bifurcation diagram of parafoil Angle of Attack](image)
The bifurcation diagrams clearly show highly nonlinear glide behavior of parafoil-payload system with control inputs, i.e., brake deflections, and rigging angles. The bifurcation results also highlight the nonlinearity in stability aspects of gliding characteristics of parafoil-payload system. Although system
always finds stable glide characteristics with regions of multiple glide for some values of rigging angles.

The effect of brake deflection on gliding characteristics of parafoil-payload system is obtained as highly nonlinearly dependent on the selection of rigging angle. Although, the effect of brake deflection is negligibly small on gliding characteristics of parafoil-payload system when the change in brake deflection is up to half brake.

At lower values of rigging angle effect of parafoil brake deflection shows possibility of considerable jump in trim angle of attack to a large value thereby, resulting in sudden jump in gliding characteristics to a large value glide path angle. Whereas, at higher rigging angle the effect of brake deflection as control input shows hardly any effect on change in gliding characteristics. Hence it is clearly visible the there is negligible effect on system behavior up to half brake deflection.

The operating range of rigging angles for a parafoil-payload system is surprisingly extremely narrow in order to achieve high L/D performance with small glide angle during flight with zero brake, as well as provide high descent rate with steep glide angle when necessary during flare and touch down in full brake configuration. Thus, from our bifurcation analysis only a small range of rigging angles around 9 deg is obtained where the parafoil-payload system meets both these requirements. The parafoil-payload system has poor stability and glide performance for rigging angles below 8 deg in zero and half brake configuration. In fact, still lower rigging angles may lead to stall oscillations, post-stall trims, and re-trims. On the other hand, rigging angles above 9 deg appear to show poor descent performance in full brake configuration with further deterioration with increasing rigging angle. This is a peculiar feature of parafoil systems that there is a narrow corridor of rigging angle where the system performance and stability, and inflation and safety requirements can be met. Typically, the choice of rigging angle also results in a trim angle of attack fairly close to stall as is evident in the present instance.

4. Conclusions
A bifurcation analysis of the 9-DOF model using a continuation algorithm has been used to study longitudinal trim and stability characteristics for zero brake, half brake, and full brake
deflections, for different choices of rigging angle. The bifurcation confirms the possibility of multiple flight trims as seen in standard stability analysis. If the system is properly rigged then only, the second (post-stall) trim in full symmetric brake gives large steep glide leading to faster descent. The gliding flight show the possibility of parafoil canopy being stalled which is characterized by steeper glide slope, and hence leading to faster descent. This paper rightly highlights the importance and application of Bifurcation theory for nonlinear dynamics and stability prediction using multibody Nine Degree-Of-Freedom (9-DOF) parafoil-payload concept.

References