Symmetry-Breaking of Interfacial Polygonal Patterns and Synchronization of Travelling Waves within a Hollow-Core Vortex

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Abstract: A hollow vortex core in shallow liquid, produced inside a cylindrical reservoir using a rotating disk near the bottom of the container, exhibits interfacial polygonal patterns. These pattern formations are to some extent similar to those observed in various geophysical, astrophysical and industrial flows. In this study, the dynamics of rotating waves and polygonal patterns of symmetry-breaking generated in a laboratory model by rotating a flat disc near the bottom of a cylindrical tank is investigated experimentally. The goal of this paper is to describe in detail and to confirm previous conjecture on the generality of the transition process between polygonal patterns of the hollow vortex core under shallow water conditions. Based on the image processing and an analytical approach using power spectral analysis, we generalize in this work – using systematically different initial conditions of the working fluids – that the transition from any $N$-gon to $(N+1)$-gon pattern observed within a hollow core vortex of shallow rotating flows occurs in an universal two-step route: a quasi-periodic phase followed by frequency locking (synchronization). The present results also demonstrate, for the first time, that all possible experimentally observed transitions from $N$-gon into $(N+1)$-gon occur when the frequencies corresponding to $N$ and $N+1$ waves lock at a ratio of $(N-1)/N$.

Keywords: Swirling flow, patterns, transition, quasi-periodic, synchronization.

1. Introduction

Swirling flows produced in closed or open stationary cylindrical containers are of fundamental interest; they are considered as laboratory model for swirling flows encountered in nature and industries. These laboratory flows exhibit patterns which resemble to a large extent the ones observed in geophysical, astrophysical and industrial flows. In general, the dynamics and the stability of such class of fluid motion involve a solid body rotation and a shear layer flow. Because of the cylindrical confining wall, the shear layer flow forms the outer region while the inner region is a solid body rotation flow. The interface between the flow regimes can undergo Kelvin-Helmholtz instability because of the jump in velocity at the interface between the inner and outer regions, which manifests as azimuthal waves. These waves roll up into satellite vortices which impart the interface polygonal shape (e.g., see Hide & Titman 1967; Niño &
Misawa 1984; Rabaud & Couder 1983; Chomaz et al. 1988; Poncet and Chauve 2007). The inner solid body rotation region can also be subjected to inertial instabilities which manifest as Kelvin’s waves and it is this type of waves that will be investigated in this paper. In our experiment a hollow core vortex, produced by a rotating disk near the bottom of a vertical stationary cylinder, is within the inner solid body rotation flow region and acts as a wave guide to azimuthal rotating Kelvin’s waves. The shape of the hollow core vortex was circular before it breaks into azimuthal rotating waves (polygonal patterns) when some critical condition was reached.

A fundamental issue that many research studies were devoted to the study of rotating waves phenomena is the identification and characterization of the transition from symmetrical to non-symmetrical swirling flows within cylindrical containers. Whether confined or free surface flow, the general conclusion from all studies confirmed that, the Reynolds number and aspect ratio (water initial height $H$/cylinder container radius $R$) are generally the two dominant parameters influencing the symmetry breaking phenomenon’s behaviour. Vogel (1968) and Escudier (1984) studied the transitional process in confined flows and found that symmetry breaking occurs when a critical Reynolds number was reached for each different aspect ratio. Vogel used water as the working fluids in his study where he observed and defined a stability range, in terms of aspect ratio and Reynolds number, for the vortex breakdown phenomenon which occurred in the form of a moving bubble along the container’s axis of symmetry. Escudier (1984) later extended the study by using an aqueous glycerol mixture (3 to 6 times the viscosity of water) and found that varying the working fluid viscosity caused changes in the critical Reynolds number values. He also observed that for a certain range of aspect ratio and viscosity, the phenomenon of vorticity breakdown has changed in behaviour, revealing more vortices breakdown stability regions than the conventional experiments using water as the working fluid. Where in open free surface containers under shallow liquid conditions using water as the working fluid, Vatistas (1990) studied the transitional flow visually and found that the range of the disc’s RPM where the transitional process occurs shrinks as the mode shapes number increased. Jansson et al. (2006) concluded that the endwall shear layers as well as the minute wobbling of the rotating disc are the main two parameters influencing the symmetry breaking phenomenon and the appearance of the polygonal patterns. Vatistas et al. (2008) studied the transition between polygonal patterns from $N$ to $N+1$, using image processing techniques, with water as the working fluid and found that the transition process from $N$ to a higher mode shape of $N+1$ occurs when their frequencies ratio locks at $(N-1)/N$; therefore following a devil staircase scenario which also explains the fact that the transition process occurs within a shorter frequency range as the mode shapes increase. Speculating the transition process as being a bi-periodic state, the only way for such system to lose its stability is through frequency locking (Bergé et al. 1984). From nonlinear dynamics consideration, Ait Abderrahmane et al. (2009) proposed the transition between equilibrium states under similar configurations using classical nonlinear dynamic theory approach and found that
the transition occurs in two steps being, a quasi-periodic and frequency locking stages, i.e., the transition occurs through synchronization of the quasi-periodic regime formed by the co-existence of two rotating waves with wave numbers $N$ and $N+1$. Their studies however was built mainly on the observation of one transition, from 3-gon to 4-gon.

In the present paper, we provide further details on the symmetry-breaking pattern transitions and confirm the generalized mechanism on the transition from $N$-gon into $(N+1)$-gon using power spectra analysis. This study systematically investigates different mode transitions, the effect of working fluid with varying viscosity, liquid initial height on the polygonal pattern instability observed within the hollow core.

2. Experimental Setup and Measurement Technique

The experiments were conducted in a 284 mm diameter stationary cylindrical container with free surface (see Fig. 1). A disk, located at 20 mm from the bottom of the container, with radius $R_d = 126$ mm was used and experiments with three initial water heights above the disk, $h_o = 20, 30$ and $h_o = 40$ mm, were conducted. Similar experiment was conducted by Jansson et al. (2006) within a container of different size where the distance of the disk from the bottom of the container is also much higher than in the case of our experiment. In both experiments similar phenomenon – formation of a polygonal pattern at the surface of the disk – was observed. It appears therefore that the dimension of the container and the distance between the disk and the container bottom do not affect the mechanism leading to the formation of the polygon patterns. In our experiment, the disk was covered with a thin smooth layer of white plastic sheet. It is worth noting that the roughness of the disk affects the contact angle between the disk and the fluid; this can delay the formation of the pattern. However, from our earlier observation in many experiments, roughness of the disk does not seem to influence prominently the transition mechanism.

![Fig. 1. Experimental setup.](image)
The disk speed, liquid initial height and viscosity were the control parameters in this study. The motor speed, therefore the disc’s speed, was controlled using a PID controller loop implemented on LABVIEW environment. Experiments with tape water and aqueous glycerol mixtures, as the working fluids, were conducted at three different initial liquid heights of 20, 30 and 40 mm above the rotating disc. The viscosity values of the used mixtures were obtained through technical data provided by a registered chemical company (Dow Chemical Company 1995-2010). Eight different aqueous glycerol mixtures were used in the experiments with viscosity varying from 1 to 22 (0 ~ 75% glycerol) times the water’s at room temperature (21°C). The detailed points of study were: 1, 2, 4, 6, 8, 11, 15 and 22 times the water’s viscosity ($\mu_{water}$) at room temperature. Although the viscosity of the mixture varied exponentially with the glycerol concentration (see Fig. 2), closer points of study were conducted at low concentration ratios since significant effects have been recognized by just doubling the viscosity of water as it will be discussed later. The temperature variation of the working fluid was measured using a mercury glass thermometer and recorded before and just after typical experimental runs and was found to be stable and constant (i.e. room temperature). Therefore, the viscosity of the mixture was ensured to be constant and stable during the experiment. Phase diagrams had been conducted and showed great approximation in defining the different regions for existing patterns in terms of disc’s speed and initial height within the studied viscosity range.

A digital CMOS high-speed camera (pco.1200hs) with a resolution of 1280 x 1024 pixels was placed vertically above the cylinder using a tripod. Two types of images were captured: colored and 8-bit gray scale images, at 30 frames per second, for the top view of the formed polygonal patterns (see Fig. 3 for example). The colored images were used as illustration of the observed stratification of the hollow vortex core where each colored layer indicates a water depth within the vortex core. It is worth noticing that the water depth increases continuously as we move away from the center of the disk (due to the
applied centrifugal force). The continuous increase in the water depth, depicted in the Fig. 3 by the colored layers, indicates momentum stratification in the radial direction (i.e., starting with the central white region which corresponds to a fully dry spot of the core and going gradually through different water depth phases until reaching the black color region right outside the polygonal pattern boundary layer). For subsequent quantitative analysis, the data was conducted with grey images as those are simpler for post-processing.

The transition mechanism is investigated using image processing techniques. First the images were segmented; the original 8-bit gray-scale image is converted into a binary image, using a suitable threshold, to extract the polygonal contours (Gonzalez et al. 2004). This threshold value is applied to all subsequent images in a given run. In the image segmentation process, all the pixels with gray-scale values higher than the threshold were assigned 1’s (i.e. bright portions) and the pixels with gray-scale values lower than or equal to the threshold were assigned 0’s (i.e. dark portions). The binary image obtained after segmentation is filtered using a low-pass Gaussian filter to get rid of associated noises. In the next step, the boundaries of the pattern were extracted using the standard edge detection procedure. The pattern contours obtained from the edge detection procedure were then filtered using a zero-phase filter to ensure that the contours have no phase distortion. The transformations of the vortex core are analyzed using Fast Fourier Transform (FFT) of the time series of the radial displacement for a given point on the extracted contour, defined by its radius and its angle in polar coordinates with origin at the centroid of the pattern; see Ait Abderrahmane et al. (2008, 2009) for further details.

Fig. 3. Polygonal vortex core patterns. The inner white region is the dry part of the disk and the dark spot in the middle of the image is the bolt that fixes the disk to the shaft. The layers with different colors indicate the variation of water depth from the inner to the outer flow region.
3. Results and Discussion

We first discuss results obtained at an initial height $h_i = 40$ mm where transitions from $N = 2 \rightarrow N = 3$ and $N = 3 \rightarrow N = 4$ were recorded and analyzed using power spectral analysis. Starting with stationary undisturbed flow, the disc speed was set to its starting point of 50 RPM and was then increased with increments of 1 RPM. Sufficient buffer time was allowed after each increment for the flow to equilibrate. At a disc speed of 2.43 Hz the first mode shape (oval) appeared on top of the disc surface. At the beginning of the $N = 2$ equilibrium state, the vortex core is fully flooded. While increasing the disc speed gradually, several sets of 1500 8-bit gray-scale images were captured and recorded. Recorded sets ranged 3 RPM in between. Systematic tracking of the patterns speed and shape evolution were recorded and the recorded images were processed. The evolution of the oval equilibrium state shape and rotating frequency is shown in Figs. 4a to 4d. Starting with a flooded core at $f_p = 0.762$ Hz in figure 4a where the vertex of the inverted bell-like shape free surface barely touched the disc surface, Fig. 4b then shows the oval pattern after gaining more centrifugal force by increasing the disc speed by 9 RPM. The core became almost dry and the whole pattern gained more size both longitudinally and
transversely with a rotating frequency of $f_p = 0.791$ Hz. It is clearly shown that at this instance, one of the two lobes of the pattern became slightly fatter than the other. Fig. 4c shows shape development and rotational speed downstream the $N=2$ range of existence. It is important to mention that once the oval pattern is formed, further increase in the disc speed, therefore the centrifugal force applied on the fluid, curved up the oval pattern and one of the lobes became even much fatter giving it a quasi-triangular shape. Fig. 4d features the end of the oval equilibrium pattern in the form of a quasi-triangular pattern and therefore the beginning of the first transition process ($N = 2$ to $N = 3$). The transition process is recorded, processed and the corresponding power spectrum was generated (see Fig. 4d). The power spectral analysis revealed two dominant frequencies from the extracted time series function of the captured images; frequency $f_m$ corresponds to the original oval pattern and frequency $f_s$ corresponds to the growing subsequent wave $N = 3$, which is a travelling soliton-like wave superimposed on the original oval pattern therefore forming the quasi-triangular pattern (Ait Abderrahmane et al. 2009). Further increase of the disc speed resulted in the forming and stabilizing of the triangular mode shape ($N = 3$) with a flooded core; both the troughs and apexes of the polygonal pattern receded and the core area shrank significantly.

Following the same procedure, the development of the triangular pattern and its transition to square ($N = 4$) shape were recorded, image processed and analyzed. Figs. 5a to 5e show the power spectra plots and their corresponding sample image from the set recorded and used in generating each of the power spectra. The behaviour of the oval pattern’s shape development and transition was also respected for the triangular pattern evolution.

Ait Abderrahmane et al. (2009) described the transition process in the form of a rotating solid body $N$ shape associated with a traveling “soliton”-like wave along the vortex core boundary layer. The evidence of such soliton-like wave is revealed here. Fig. 6 shows a sample set of colored RGB images during the transition process described above; these images feature the quasi-periodic state during $N = 3$ to $N = 4$ transition described earlier. Giving a closer look at the sequence of images, one could easily figure out the following: the three lobes or apexes of the polygonal pattern are divided into one flatten apex and two almost identical sharper apexes. Keeping in mind that the disc, therefore the polygonal pattern, is rotating in the counter clockwise direction and that the sequence of images is from left to right, by tracking the flatten lobe, one could easily recognize that an interchange between the flatten lobe and the subsequent sharp lobe (ahead) takes place (see third row of images). In other words, now the flattened apex receded to become a sharp stratified apex and the sharp lobe gained a more flattened shape. Such phenomenon visually confirms the fact that transition takes place through a soliton-like wave travelling along the vortex core boundary but with a faster speed than the parent pattern. This first stage of the transition process was referred to as the quasi-periodic stage by Ait Abderrahmane et al. (2009). The quasi-periodic stage takes place in all transitions until the faster travelling soliton-like wave synchronizes with the patterns rotational frequency forming and developing the new higher state of
equilibrium pattern. Vatistas et al. (2008) found that the synchronization process takes place when the frequencies ratio of both pattern \( N \) and the subsequent pattern developed by the superimposed soliton wave \( (N+1) \) lock at a ratio of \( (N-1)/N \). Therefore, for transition from \( N = 2 \) to \( N = 3 \), the synchronization takes place when the frequencies ratio is rationalized at 1/2. And the transition \( N = 3 \) to \( N = 4 \), takes place when the ratio between both frequencies are equal to 2/3. In the above illustrated two transition processes, the frequency ratio for first transition was equal to \( f_N / f_{N+1} = f_m / f_s = 1.69/3.04 = 0.556 \approx 1/2 \). On the other hand, the second transition took place when \( f_N / f_{N+1} = f_m / f_s = 3.28/4.92 = 0.666 \approx 2/3 \).

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**Fig. 5.** (a), (b), (c) Triangular pattern progression and corresponding power spectra; (d) Transitional process from triangular to square pattern; and (e) square pattern and corresponding power spectra.
Following the same trend, the second experiment was conducted using water at an initial height of 20 mm. At this low aspect ratio, transition between higher mode shapes was tracked and recorded. Using similar setup and experimental procedure, the transition from square mode ($N = 4$) to pentagonal pattern ($N = 5$) and from pentagonal to hexagonal pattern ($N = 6$) were recorded and image-processed for the first time in such analysis. Following the same behavior, the transition occurred at the expected frequency mode-locking ratio. Fig. 7a shows the third polygonal transition, from $N = 4$ to $N = 5$. The frequency ratio of the parent pattern to the soliton-like wave is $f_{m}/f_{s} = 4.102/5.449 = 0.753 \approx 3/4$.

Similarly, Fig. 7b shows the transition power spectrum for the last transition process observed between polygonal patterns, which is from $N = 5$ to $N = 6$ polygonal patterns. The frequency ratio $f_{m}/f_{s} = 5.625/6.973 = 0.807$ which is almost equal to the expected rational value $4/5$. With these two experimental runs, the explanation of the transition process between polygonal patterns observed within hollow vortex core of swirling flows within cylinder containers under shallow water conditions is confirmed for all transitional processes.
Fig. 7. (a) Square to pentagonal transition; and (b) pentagonal to hexagonal transition.

<table>
<thead>
<tr>
<th>Transition (N) - (N+1)</th>
<th>( h_i = 20 \text{ mm} )</th>
<th>( h_i = 30 \text{ mm} )</th>
<th>( h_i = 40 \text{ mm} )</th>
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<tr>
<td>3 - 4</td>
<td>0.697, 0.787, 0.829</td>
<td>0.545, 0.68, 0.74</td>
<td>0.558, 0.69</td>
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<tr>
<td>4 - 5</td>
<td>4.6%, 4.9%, 3.6%</td>
<td>9.0%, 2.0%, 7.5%</td>
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</tr>
<tr>
<td>5 - 6</td>
<td>0.667, 0.747, --</td>
<td>0.558, 0.671, --</td>
<td>0.557, 0.678</td>
</tr>
<tr>
<td>Transition (N) - (N+1)</td>
<td>( h_i = 30 \text{ mm} )</td>
<td>( h_i = 40 \text{ mm} )</td>
<td></td>
</tr>
<tr>
<td>2 - 3</td>
<td>0.1%, 0.4%, --</td>
<td>11.6%, 0.7%, --</td>
<td>11.4%, 1.7%</td>
</tr>
<tr>
<td>3 - 4</td>
<td>0.64, --, --</td>
<td>0.671, --</td>
<td>0.557, 0.686</td>
</tr>
<tr>
<td>4 - 5</td>
<td>4.0%, --, --</td>
<td>0.7%, --</td>
<td>11.4%, --</td>
</tr>
<tr>
<td>5 - 6</td>
<td>--, --, --</td>
<td>0.6667, --</td>
<td>0.55, --</td>
</tr>
<tr>
<td>Visco. x 10^3 ( \mu )</td>
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<td>10.0%</td>
<td>--</td>
</tr>
<tr>
<td>8</td>
<td>--</td>
<td>0.536</td>
<td>7.2%</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>0.58</td>
<td>16.0%</td>
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<tr>
<td>15</td>
<td></td>
<td>0.552</td>
<td>10.4%</td>
</tr>
<tr>
<td>22</td>
<td></td>
<td>0.559</td>
<td>11.8%</td>
</tr>
</tbody>
</table>

Table 1. Transition mode-locking frequencies for different liquid viscosities.
The influence of the liquid viscosity on the transitional process from any \( N \) mode shape to a higher \( N+1 \) mode shape is also investigated. As described earlier, eight different liquid viscosities were used in this study ranging from 1 up to 22 times the viscosity of water. All transitional processes between subsequent mode shapes were recorded, and acquired images were processed. Using the same procedure as in the last section, the frequency ratio of the parent pattern \( N \) and the subsequent growing wave \( N+1 \) has been computed and tabulated in Table 1. As shown in Table 1, the maximum deviation from the expected mode-locking frequency ratio (\( f_m/f_s \)) always appeared in the first transition (\( N = 2 \) to \( N = 3 \)). A reasonable explanation for such induced error is the fact that, the higher the number of apexes per full pattern rotation, the more accurate is the computed speed of the pattern using the image processing technique explained before. Therefore, throughout the conducted analysis, the most accurate pattern’s speed is the hexagon and the least accurate is the oval pattern. Apart from that significant deviation, one can confidently confirm that even at relatively higher viscous swirling flows, the transition between polygonal patterns instabilities takes place when the parent pattern \( (N) \) frequency and the developing pattern \( (N+1) \) frequency lock at a ratio of \( (N-1)/N \) (Vatistas et al. 2008).

As explained earlier, transition has been found to occur in two main stages being the quasi-periodic and the frequency-locking stages (Ait Abderrahmane et al. 2009). It is also confirmed that frequency mode-locking does exist in polygonal patterns transition irrespective of the mode shapes, liquid heights and the liquid viscosity (within the studied region). In this section, the quasi-periodic phase will be further elucidated and confirmed. Earlier in this paper the quasi-periodic state in the transition of \( N = 3 \) to \( N = 4 \), using water as the working fluid, was observably described in Fig. 6. To further analyze the quasi-periodic stage, a technique has been developed which animates the actual polygonal patterns instabilities but without the existence of the speculated travelling soliton-like wave along the patterns boundary layer. Using MAPLE plotting program, all mode shapes replica have been plotted and printed. Table 2 shows the plots and their corresponding plotting functions. Printed images were glued to the rotating disc under dry conditions one at a time. The disc was rotated with corresponding pattern’s expected speeds under normal working conditions. Such
technique gave full control of the rotating pattern. Therefore, both speed and geometry of the patterns were known at all times. Sets of 1500 8-bit images were captured and processed using similar computing procedure.

<table>
<thead>
<tr>
<th>N</th>
<th>Pattern plot</th>
<th>Plot function</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td><img src="image1" alt="Pattern" /></td>
<td>$r = 1 + 0.2 \sin(2 \theta)$</td>
</tr>
<tr>
<td>2 - 3</td>
<td><img src="image2" alt="Pattern" /></td>
<td>$r = 1 + 0.2 \sin(2 \theta) + 0.1 \sin(3 \theta +1)$</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Pattern" /></td>
<td>$r = 1 + 0.1 \sin(3 \theta)$</td>
</tr>
<tr>
<td>3 - 4</td>
<td><img src="image4" alt="Pattern" /></td>
<td>$r = 1 + 0.1 \sin(3 \theta) + 0.15 \sin(4 \theta +1)$</td>
</tr>
<tr>
<td>4</td>
<td><img src="image5" alt="Pattern" /></td>
<td>$r = 1 + 0.15 \sin(4 \theta)$</td>
</tr>
</tbody>
</table>

Table 2. Patterns replica with corresponding functions.
Power spectra of the processed sets of images revealed similar frequency plots. Starting with the oval-like shape, the disc was rotated at a constant speed of 1 Hz and the power spectrum was generated from the extracted images and plotted as shown in Fig. 8. Since the oval pattern speed is controlled in this case (by disc speed), the frequency extracted could have been presumed to be double the disc frequency (2 Hz). The actual frequency extracted is shown in Fig. 8, $f_m = 1.934$ Hz (3.3% error). Following the same procedure, other polygonal patterns replica were printed to the disc, rotated, captured and processed subsequently. Figs 9a and 9b show the power spectra generated from rotating the quasi-triangular and the quasi-square patterns, respectively.

Fig. 9a shows a power spectrum generated from the set of pictures featuring a quasi-triangular pattern captured at 30 fps. The power spectrum revealed two dominant frequencies being $f_m = 3.809$ Hz and $f_s = 5.742$ Hz corresponding to the oval and triangular patterns, respectively. Since the quasi-triangular pattern is stationary and under full control, it could have been presumed that the frequency ratio would have a value of 2/3 since the replica pattern is generated by superimposing the oval and triangular functions. The actual extracted frequency was $f_m/f_s = 3.81/5.74 = 0.663 \approx 2/3$. Comparing this frequency ratio with the real polygonal patterns mode-locking ratio of 1/2 described earlier, it is clear that the ratio is totally different which proves that both patterns are not behaving equivalently although having generally similar instantaneous geometry. Therefore, the actual rotating pattern does not rotate rigidly as the pattern replica does, but rather deforms in such a way that the ratio of the two frequencies is smaller which confirms the idea of the existence of the fast rotating soliton-like wave ($f_s$). Moving to the second transition process, triangular to square, as shown in Fig. 9b, the frequency ratio was found to be 3/4 as expected since the function used to plot the quasi-square pattern is the superposition of both functions used in plotting the pure triangular and square patterns given in Table 2. Comparing this ratio with the actual mode-locking ratio of 2/3 observed with real polygonal patterns, it is obvious that the ratio is still smaller which respects the existence of a faster rotating wave along the
triangular pattern boundary that eventually develops the subsequent square pattern as visualized earlier using the colored images. From these two experiments, along with the visual inspection discussed earlier, the existence of the fast rotating soliton-like wave \((N+1)\) along the parent pattern boundary layer \((N)\) is verified, therefore, the quasi-periodic stage.

4. Conclusion

Through the analysis of the present experimental results from different initial conditions, we confirmed with further evidences and generalized the mechanism leading to transition between two subsequent polygonal instabilities waves, observed within the hollow vortex core of shallow rotating flows. The transition follows the universal route of quasi-periodic regime followed by synchronization of the two waves’ frequencies. We shows, for the first time, all observed transitions from \(N\)-gon to a subsequent \((N+1)\)-gon occur when the frequencies corresponding to \(N\) and \(N+1\) waves lock at a ratio of \((N-1)/N\). The effect of varying the working fluid viscosity on the transitional processes between subsequent polygonal patterns was also addressed in this paper.

Both stages of the transitional process were further explored in this work. The quasi-periodic stage was first tackled using two different techniques, a visual method and an animated method. The deformation of the colored stratified boundary layers of polygonal patterns were inspected during transition process of polygonal patterns and the existence of a fast rotating wave-like deformation was recognized which confirms the idea of the co-existence of a soliton-like wave that initiates the quasi-periodic stage at the beginning of the transition. In order to further materialize this observation, experiments were re-conducted using fixed patterns replica featuring the quasi-periodic geometry of polygonal patterns under dry conditions. Such technique allowed full control of the patterns geometry and speed at all time, therefore working as a reference to the real experiment performed under wet conditions. The experiments revealed an interesting basic idea that was useful when addressing the significant difference in behavior associated with the real patterns transitions. The second part of the transition process included the frequency mode-locking ratio of subsequent patterns. Dealing with the first part of the transition process as being a bi-periodic state or phase, in order for such state to lose its stability, a synchronization event has to occur (Bergé et al. 1984). This synchronization has been confirmed to occur when the frequency ratio of the parent pattern \(N\) to the subsequent pattern \(N+1\) rationalized at \((N-1)/N\) value (Vatistas et al. 2008). The frequency mode-locking phenomenon was found to be respected even at relatively higher viscosity fluids when mixing glycerol with water.

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References


The Influence of Charge Traps in Semiconductor Diode on Complex Dynamics in Non-autonomous RL-Diode Circuit

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Abstract: In this work the results of numerical simulation of the complex charge dynamics in the well known model system - p-n junction semiconductor diode connected with non-autonomous RL-diode circuit are presented. Nonlinear charge dynamics was shown as changes of the oscillation regimes maps topology in the presence of charge traps in diode and without ones. The effects under consideration were explained on the base of detailed description of p-n junction functioning in terms of accumulation and relaxation of non-equilibrium charge carriers at diode base. As well, the study of the influence of the charge accumulation and recombination processes on the traps on excitation of complex current oscillations in the circuit was carried out. We discuss the possibility the application of the comparative analysis of oscillation regimes maps topology as a method for express traps diagnostics in semiconductor devices.

Keywords: p-n junction, semiconductor diode, complex oscillations, numerical simulation, charge traps, non-equilibrium charge carriers.

1. Introduction

Electrical active defects – so-called traps and states - play an important role in processes of thermodynamic non-equilibrium charge transition in semiconductor structures. Such defects define dynamic characteristics and fast recovery of Schottky diodes, MIS-structures and transistors [1]. At present the methodological basis for detection and investigation of characteristics of defects is Deep-Level Transient Spectroscopy (DLTS method [2]). DLTS method is standard for laboratory researches of traps in multilayer structures and surfaces barriers and interfaces states. However that method possesses a number of such essential shortcomings as complexity of practical realization, sometimes insufficient resolution, impossibility of application in the presence of thick dielectric layers and in some others cases. Consequently, creation of an effective alternative method for detection and studying of states and traps in the basic types of barrier semiconductor structures is actual.

In this article a sensitive method of diagnostics of accumulation and relaxation processes of a non-equilibrium charge of the majority and minority carriers on states and capture levels on the "semiconductor-dielectric" interface is offered and considered. Suggested method is based on application of methods of the nonlinear theory of oscillations to thermodynamic non-equilibrium object that is on dynamical system [3]. Such dynamical system is formed from the
in series connected semiconductor device, inductance (an inertial element), resistance (an element of dissipative losses) and source of the external periodic influence.

2. The Subject of Study

Let's consider semiconductor structure of the diode. The semiconductor diode [1] consists of two areas with a various alloying of the semiconductor: hole area (p-type) with dominating concentration of holes and electronic area (n-type) with dominating concentration of electrons. The anode is connected to p-type area, and the cathode - to n-type area. The impurity added in a semiconductor material at manufacturing; define type of impurity conduction of each of areas. Take into consideration a Shokli-Rid-Hall carriers recombination, which is the basic process of a recombination through traps, being in the forbidden zone of the semiconductor. Let's assume that there are traps with only one level. Change of filling of traps by carriers will take place in characteristic time (relaxation of a non-equilibrium charge) \( \tau = R_sC_S \), where \( C_S \) – capacity of traps. Thus the equivalent scheme of p-n-junction of the diode will look like it is shown on fig.1. Diagnostics of traps parameters is carried out by observation of a current or voltage relaxation in a chain containing the investigated diode [2]. In the presence of a trap with small capacity the dynamics of a current in a diode chain practically coincides with a case when traps are absent as the current of a relaxation of a charge of a trap is rather small. This rather complicates the process of measurements because it limits sensitivity of a method. Further in work the new method of research of non-equilibrium processes of a recharge of traps and conditions, free from the specified shortcomings will be offered.

Fig. 1. Equivalent circuit of p-n-junction of the diode. The capacity of traps \( C_S \), barrier and diffusion capacities of junction \( C_{b,d}(U) \), resistance of a recharge of trap \( R_S \), the voltage controlled current generator \( i(U) \) are shown.
3. Method of Diagnostics of Charge Relaxation
To increase a current of a recharge of traps or to provide effective accumulation of a charge in traps it is necessary to eliminate defects of a known method of research of traps.

The increase in a current of a recharge is related to increase voltage applied to p-n-junction of the diode. Hence it relates to change of conditions of capture and a charge recombination on a trap. The second approach can be realized at the expense of inclusion in an external chain of some inertial element. Such element enters certain shift of phases between a current injected to p-n-junction of the diode in an external circuit and the voltage. As the specified element the inductance providing phase delay of a current from voltage on the diode can be used. Thus, sufficient concentration of non-equilibrium carriers in the field of a space charge of junction of the diode during the time sufficient for effective capture of carriers on traps is provided.

4. The Model and Numerical Experiment
Let's construct the mathematical model of the modified nonlinear RLD-circuit in which p-n-junction is used as diode D. This model will be presented in the form of the system of the ordinary non-autonomous nonlinear differential equations:

\[
\begin{align*}
\frac{dI}{d\tau} &= \frac{L}{\Omega} (E(\Omega) + V_d) + R RI \\
\frac{dV_d}{d\tau} &= \frac{C}{C} \left( I - e^{\frac{f}{f_0}} + 1 \right) \\
\frac{dV}{d\tau} &= \Theta(V_d - V_s) \\
\frac{d\Omega}{d\tau} &= f
\end{align*}
\]

where I – normalized current of circuit, \(V_d\) - normalized voltage of p-n junction and \(V_s\) - normalized voltage of trap, \(L_0/L\) – normalized inductance, \(C_0/C\) – normalized capacity and \(R_0/R\) – normalized resistance, \(\tau\) and \(\Omega\) – time and phase of external force, \(f/f_0\) - relative frequency of external force, \(\Theta\) – relative time overcharge of trap. Rationing of all models is executed according to scale factors: \(T=1/f_0\) - the period of external influence, \(I_0\) – current of saturation of p-n-junction, \(\varphi\) – thermal potential. The constructed model has been investigated numerically for a following set of parameters: \(L=2\) mH, \(C_0=200\) pF, \(R=50\) Ohm and a range of change of parameter \(f/f_0=0.1\ldots1.0\). For the comparative analysis of solutions the projections of phase space of system and bifurcation diagrams with and without traps was used.

For modeling of a case of absence of traps the modified system was used. The third equation for recharge dynamics of a trap has been removed from this system.
5. Discussion and Conclusions

Simulated results are presented in a fig. 2. Bifurcation diagrams for a situation of absence of a trap are resulted on fig. 2a-c, they coincide with known results for a nonlinear RLD-circuit [3]. As we can see, typical bifurcation scenarios are realized in the system – period doubling cascade and period adding (including the return), which lead to occurrence chaotic/stochastic attractor. In experiments typical frequencies ratio $f/f_0 = 0.1; 0.25; 0.75$ were used.

Insertion of a trap with parameters $\tau = R_S C_S = 10^3 s$ leads to transformation of structure of space of parameters (fig. 2d-f). So, the extended area of existence of a mode $2T$ disappears at relative small frequency of influence (fig. 2d). Within the given area the scenario $<2T-3T-2T>$ on the basis of sequence soft bifurcation is realized. The similar situation is observed at excitation of a nonlinear circuit on subharmonic frequency in the field of realization of the cascade of period doubling. Transitions $<2T-3T>$ occur on the basis of a cycle of the doubled period and transitions to period adding on the basis of a 4T-cycle. Bifurcation diagram constructed at frequency of excitation $f/f_0 = 0.75$ in case of a trap presence differs from bifurcation diagrams of a "classical" nonlinear oscillator insignificantly. Thus "the ladder" structure of the cascade of resonances - of additions of the period $<nT - (n+1)T>$ is realized on which base under the scenario of doubling of the period chaotic attractor occurs (fig. 2f). Presence of a "slow" trap ($\tau = R_S C_S = 10^4 s$) insignificantly transforms dynamics of a nonlinear circuit (fig. 2a-c). Thus all features in a known circuit «semiconductor diode – inductance - resistance» remain. The resulted features of transformation of dynamics of a current in a circuit are connected with current distribution on a trap capacity, and, as consequence, with capture of a part of a charge by a trap synchronously with oscillatory process of a recharge of barrier and diffusive capacities of the diode. At the realization in system bifurcation transition $<T-2T>$ the important role is played by a steady-state equilibrium distribution of charges of minority carriers in junction. Trap presence leads to disturbance ("tightening") of this process, generating longer cycles of a relaxation of a charge that is shown in the form of transitions $<2T-3T-2T>$. Significant sensitivity of the given process to parameters $\tau, R_S, C_S$ allows studying bifurcation diagrams of nonlinear circuit with the included investigated semiconductor element as a sensitive method for detection and an estimation of characteristics of traps and conditions.
Fig. 2. Bifurcation diagrams for nonlinear circuit, which involve the investigated semiconductor diode (numerical experiment)

References


Particle based method for shallow landslides: modeling sliding surface lubrication by rainfall

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Abstract. Landslides are a recurrent phenomenon in many regions of Italy: in particular, the rain-induced shallow landslides represent a large percentage of this type of phenomenon, responsible of human life loss, destruction of assets and infrastructure and other major economical losses. In this paper a theoretical computational mesoscopic model based on interacting particles has been developed to describe the features of a granular material along a slope. We use a Lagrangian method similar to molecular dynamic (MD) for the computation of the movement of particles after and during a rainfall. In order to model frictional forces, the MD method is complemented by additional conditions: the forces acting on a particle can cause its displacement if they exceed the static friction between them and the slope surface, based on the failure criterion of Mohr-Coulomb, and if the resulting speed is larger that a given threshold. Preliminary results are very satisfactory; in our simulations emerging phenomena such as fractures and detachments can be observed. In particular, the model reproduces well the energy and time distribution of avalanches, analogous to the observed Gutenberg-Richter and Omori distributions for earthquakes. These power laws are in general considered the signature of self-organizing phenomena. As in other models, this self organization is related to a large separation of time scales between rain events and landslide movements. The main advantage of these particle methods is given by the capability of following the trajectory of a single particle, possibly identifying its dynamical properties.

Keywords: Landslide, molecular dynamics, lagrangian modelling, particle based method, power law.

1 Introduction

Predicting natural hazards such as landslides, floods or earthquake is one of the challenging problems in earth science. With the rapid development of computers and advanced numerical methods, detailed mathematical models are increasingly being applied to the study of complex dynamical processes such as flow-like landslides and debris flows.

The term landslide has been defined in the literature as a movement of a mass of rock, debris or earth down a slope under the force of gravity (Varnes [1958], Cruden [1991]). Landslides occur in nature in very different ways. It is possible to classify them on the bases material involved and type of movement (Varnes [1978]).
Landslides can be triggered by different factors but in most cases the trigger is an intense or long rain. Rainfall-induced landslides deserved a large interest in the international literature in the last decades with contributions from different fields, such as engineering geology, soil mechanics, hydrology and geomorphology (Crosta and Frattini [2007]). In the literature, two approaches have been proposed to evaluate the dependence of landslides on rainfall measurements. The first approach relies on dynamical models while the second is based on the definition of empirical rainfall thresholds over which the triggering of one or more landslides can be possible (Segoni et al. [2009]). At present, several methods has been developed to simulate the propagation of a landslide; most of the numerical methods are based on a continuum approach using an Eulerian point of view (Crosta et al. [2003], Patraa et al. [2005]).

An alternative to these continuous approaches is given by discrete methods for which the material is represented as an ensemble of interacting but independent elements (also called units, particles or grains). The model explicitly reproduces the discrete nature of the discontinuities, which correspond to the boundaries of each element. The commonly adopted term for the numerical methods for discrete systems made of non deformable elements, is the discrete element method (DEM) and it is particularly suitable to model granular materials, debris flows and flow-like landslide (Iordanoff et al. [2010]). The DEM is very closely related to molecular dynamics (MD), the former method is generally distinguished by its inclusion of rotational degrees-of-freedom as well as stateful contact and often complicated geometries. As usual, the more complex the individual element, the heavier is the computational load and the “smaller” is the resulting simulation, for a given computational power. On the other hand, the inclusion of a more detailed description of the units allows for more realistic simulations. However, the accuracy of the simulation has to be compared with the experimental data available. While for laboratory experiments it is possible to collect very accurate data, this is not possible for real-field landslides. And, finally, the proposed model is just an approximation of a much more complex dynamics. These arguments motivated us in exploring the consequences of reducing the complexity of the model as much as possible.

In this paper we present a simplified model, based on the MD approach, applied to the study of the starting and progression of shallow landslides, whose displacement is induced by rainfall. The main hypothesis of the model is that the static friction decreases as a result of the rain, which acts as a lubricant and increases the mass of the units. Although the model is still schematic, missing known constitutive relations, its emerging behavior is quite promising.
2 The model and simulation methodology

We limit the study to two-dimensional simulations (seen from above) along a slope, modeling shallow landslides. We consider $N$ particles, initially arranged in a regular grid (Fig. 5), all of radius $r$ and mass $m$.

The idea is to simulate the dynamics of these particles during and after a rainfall. In the model the rain has two effects: the first causes an increase in the mass of particles, while the second involves a reduction in static friction between the particle and the surface below.

The equation of Mohr-Coulomb,

$$\tau_f = c' + \sigma' \tan(\phi'),$$  \hspace{1cm} (1)

says that the shear stress $\tau_f$ on the sliding surface is given by an adhesive part $c'$ plus a frictional part $\tan(\phi')$. In the our model we want to find a trigger condition of the particle that is based on the law of Mohr-Coulomb (Eq. (1)). The coefficient of cohesion, $c'$ in the Eq. (1), has been modeled by a random coefficient that depends on the position of the surface. On the other hand, the term $\sigma' \tan(\phi')$ in the Eq.(1), has been modeled by a theoretical force of static friction $F_i^{(s)}$ which is described later.

The static-dynamic transition is based on the following trigger conditions:

$$|F_i^{(a)}| < F_i^{(s)} + c',$$
$$|v_i| < v_i^{(threshold)} \rightarrow 0,$$

(2)

then the motion of the single block will not be triggered until the active forces $F_i^{(a)}$ (gravity forces + contact forces) do not exceed the static friction $F_i^{(s)}$ plus the cohesion term $c'$ and until the velocity $|v_i|$ not overcomes the threshold velocity $v_i^{(threshold)}$ (Eq. (2)). The irregularities of the surface are modeled by means of the friction coefficients, which depends stochastically on the position (quenched disorder).

In Eq. (2), the force $F_i^{(a)}$ is given by the sum of two components: the gravity $F_i^{(g)}$ and the interaction between the particles $F_i^{(i)}$.

$$F_i^{(a)} = F_i^{(g)} + F_i^{(i)}.$$  \hspace{1cm} (3)

The gravity $F_i^{(g)}$ is given by

$$F_i^{(g)} = g \sin(\alpha)(m_i + w_i(t)),$$  \hspace{1cm} (4)

where $g$ is the acceleration of gravity, $\alpha$ the slope (supposed constant) of the surface, $m_i$ the dry mass of block $i$ and $w_i$ the absorbed water cumulated in time. The quantity $w_i(t)$ is a stochastic variable (corresponding to rainfall events $\sigma^{(w)}(t)$),

$$w_i(t) = \int \sigma_i^{(w)}(t) \, dt.$$  \hspace{1cm} (5)
Fig. 1. (a) Particles in the computational domain: the maximum radius of iteration defined in the algorithm is equal to the side $L$ of the cell. Considering the black particle in the center of the circumference, it can interact only with the neighboring blue particles.

Fig. 2. (b) Cells considered when calculating the forces: if a particle is in cell $(x, y)$, the interaction forces will be calculated considering only the particles located in cells $(x + 1, y)$, $(x + 1, y + 1)$, $(x + 1, y)$ and $(x - 1, y)$. This method halves the number of interactions because it calculates 4 cells instead of 8.

The interaction force between two particles is defined through a potential that, in the absence of experimental data, we modeled after the Lennard-Jones one. The corresponding interaction force $F_{ij}^{(i)}$ that acts on block $i$ due to block $j$ is given by

\[
F_{ij}^{(i)} = -F_{ji}^{(i)} = -\nabla V(R_{ij}) = -\nabla \left( 4\varepsilon \cdot \left[ \left( \frac{r}{R_{ij}} \right)^{-12} - \left( \frac{r}{R_{ij}} \right)^{-6} \right] \right), \quad (6)
\]

where $R_{ij}$ is the distance between the particles,

\[
R_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}, \quad (7)
\]

$r$ is the radius of the particles and $\varepsilon$ is a constant.

The computational strategy for calculating the interaction forces between the particles is similar to the Verlet neighbor list algorithm (Verlet [1967]). In the code the computational domain is divided in square cells of side $L$ (see Fig. 1), corresponding to the length at which the interaction force is truncated. The truncation has a very little effect on the dynamics, so we did not correct the potential by setting $V(L) = 0$, as usual in MD.

Thanks to the Newton’s third law it is possible reduce the number interaction and consider the only particle that has not been considered in the previous step (see Fig. 2).
The condition of motion for a given particle is governed by Eq. 2. The static friction $F_i^{(s)}$ is given by

$$F_i^{(s)} = \mu_s(m_i + w_i(t)) \cos(\alpha).$$

(8)

The Equation 8 is expressed by the friction’s coefficient $\mu_s$. We assumed that the rain has a lubricating effect between the particles and underlying surface; the friction coefficient has therefore been defined as,

$$\mu_s = \mu_s^{(\infty)} + (\mu_s^{(0)} - \mu_s^{(\infty)}) \exp(-w_0t),$$

(9)

where $\mu_s^{(0)}$ and $\mu_s^{(\infty)}$ are, respectively, the initial (dry) friction coefficient at $t = 0$ (starting of rainfall) and the final (wet) for $t \to \infty$. The effect of rainfall is to lubricate the sliding surface of the landslide, at a constant speed $w_0$ in this example.

When the active forces exceed the static friction plus the quenched stochastic coefficient of cohesion $\sigma'$, the particle start to move. In this case the force acting on the particle $i$ is given by

$$F_i = F_i^{(a)} - F_i^{(d)},$$

(10)

where $F_i^{(a)}$ are the active forces, and $F_i^{(d)}$ is the force of dynamic friction,

$$F_i^{(d)} = \mu_d(m_i + w_i(t)) \cos(\alpha).$$

(11)

Eq. (11) is of the same type as Eq. (8); the coefficient of dynamic friction $\mu_d$ is defined similarly to the static one (Eq. (9)). The friction coefficients (static and dynamic) varies from point to point of the computational domain this choice serves to model the sliding surface like a rough surface.

When a particle exceed the threshold condition (Eq. 2), it moves on the slope with an acceleration $a$ equal to

$$a = \frac{F_i}{(m_i + w_i(t))},$$

(12)
Fig. 5. Initial configuration of simulations. The 2500 particles are arranged on a regular grid of 50x50 cells of size 1 \times 1.

In MD the most widely used algorithm for time integration is the Verlet algorithm. This algorithm allows a good numerical approximation and is very stable. It also does not require a large computational power because the forces are calculated once for each time step. The model was implemented using the second-order Verlet algorithm. We first compute the displacement of particles, and half of the velocity updates,

\[ r_i' = r_i + v_i \Delta t + \frac{F_i}{2m_i} \Delta \tau^2, \]
\[ v_i' = v_i + \frac{F_i}{2m_i} \Delta t, \]

then compute the forces \( F_i' \) as function of the new positions \( r_i' \), and finally compute the second half of velocities,

\[ v_i'' = v_i' + \frac{F_i'}{2m_i} \Delta t. \]

We have to define a landslide-triggering time, for instance the time of the first moving block. In this case it is very simple to obtain the trigger time theoretically for an uniform rain of intensity \( w_0 \). We can write, in equilibrium conditions, for a given mass

\[ |F_i| = F_i^{(s)} + c' \]
\[ F_i = F_i^{(g)} + F_i^{(i)} \]

We assume that the first movement of the particle is only due to the effect of gravity, so that we can set the interaction forces equal to zero, and therefore the equilibrium condition is given by

\[ |F_i| = F_i^{(g)} + c'. \]
Table 1. Parameter values used in simulations

<table>
<thead>
<tr>
<th>Sim</th>
<th>m</th>
<th>r</th>
<th>cell</th>
<th>µ_s^(0)</th>
<th>µ_s^(∞)</th>
<th>µ_d^(0)</th>
<th>µ_d^(∞)</th>
<th>c'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0001</td>
<td>0.5</td>
<td>1x1</td>
<td>1.15</td>
<td>0.7</td>
<td>0.65</td>
<td>0.34</td>
<td>0.01+ε</td>
</tr>
<tr>
<td>1b</td>
<td>0.0001</td>
<td>0.5</td>
<td>1x1</td>
<td>1.15</td>
<td>0.7</td>
<td>0.65</td>
<td>0.34</td>
<td>0.01+ε</td>
</tr>
<tr>
<td>2</td>
<td>0.0001</td>
<td>0.5</td>
<td>1x1</td>
<td>1.15</td>
<td>0.7</td>
<td>0.65</td>
<td>0.34</td>
<td>1+ε</td>
</tr>
<tr>
<td>3</td>
<td>0.0001</td>
<td>0.5</td>
<td>1x1</td>
<td>0.85</td>
<td>0.4</td>
<td>0.35</td>
<td>0.14</td>
<td>0.01+ε</td>
</tr>
</tbody>
</table>

i.e.,

\[ \hat{m} g \sin(\alpha) = \hat{m} \cdot g \cos(\alpha) \{ \mu_s^{(0)} \exp(w_0 \cdot t) + \mu_s^{(\infty)}[1 - \exp(-w_0 t)] \} + c' \]  \hspace{1cm} (17)

where \( \hat{m} = m + w(t) = m + w_0 t \).

Using Eq. 17, we can define the trigger time \( T \) as

\[ T = -\frac{1}{w_0} \log \left( \frac{\tan(\alpha) - \frac{c'}{mg \cos(\alpha)} - \mu_s^{(\infty)}}{\mu_s^{(0)} - \mu_s^{(\infty)}} \right). \]  \hspace{1cm} (18)

3 Results

In order to simulate a landslide along an inclined plane, we use the theoretical model as described above with different parameters.

In the Table 1 we illustrate the parameters used in different simulations, where Sim is the number of simulation, \( m \) and \( r \) are respectively the mass and the radius of the particles, \( \mu_s^{(0)} \), \( \mu_s^{(\infty)} \), \( \mu_d^{(0)} \), \( \mu_d^{(\infty)} \) are the coefficients of static and dynamic friction and \( c' \) is the coefficient of cohesion. In the our simulations the time \( dt \) of simulation is set to 0.01: then the effective time \( t \) is different from the simulation time \( T \).

3.1 Simulation 1

The position of the particles at \( t = 3000 \) is reported in Fig. 6. The rain starts with the particles at rest. We suppose that the speed of the landslide is much bigger than the rain flux, so that the computation of sliding is performed without the contribution of rain (i.e., instantaneously). The rain increases the mass of the particle with a factor between 0 and 0.0001. The graph of the kinetic energy (Fig. 7) shows a "stick-slip" dynamic. The distribution \( f(x) \) the kinetic energy (Fig. 8) is well approximated by an exponential

\[ f(x) = a \cdot e^{bx}, \]  \hspace{1cm} (19)

with \( a \approx 3.2 \cdot 10^4 \) and \( b \approx -0.1042 \).

In Fig. 9 the statistical distribution of the intervals between trigger times is reported. This distribution is well fitted by a power law

\[ f(x) = a \cdot x^b, \]  \hspace{1cm} (20)
Fig. 6. (a) Position of particles in Simulation 1 at $t = 3000$.

Fig. 7. (b) Kinetic energy vs. time.

Fig. 8. (a) Frequency distribution of the kinetic energy in Simulation 1. The plot in semi-log axes shows an exponential distribution.

Fig. 9. (b) Frequency distribution of trigger intervals in Simulation 1. The plot in log-log axes shows a power-law distribution.

with $a \approx 691.1$ and $b \approx -0.4295$.

Several authors (Turcotte and Malamud [2004], Turcotte [1997], Malamud et al. [2004]) have observed that some natural hazards such as landslides, earthquakes and forest fires exhibit a power law distribution.

### 3.2 Simulation 1b

In this simulation we use the same parameters as in simulation 1, but we stop the rain event at time $t = 20$. This is a special case: we want to study the effect of a steady rain until a fixed time. Fig. 10 shows the arrangement of the particles and Fig. 11 the kinetic energy at $t = 300$.

One can note that the maximum kinetic energy is much greater in this simulation. In the case 1 the maximum value of kinetic energy is $5.74 \cdot 10^{-4}$ while here it is $2.6 \cdot 10^{-3}$. Many small events are observed in the first case while in the present one we observe a single large event.
Fig. 10. Position of particles in Simulation 1b at $t = 300$.

Fig. 11. Kinetic energy versus time. We observe that the "stick-slip" events disappear and the fixed duration of precipitation changes the dynamics of the system: in particular, there is peak at $t = 20$ at the end of the rain event.

3.3 Simulation 2

In order to explore the dependence of the system behavior on the coefficient of cohesion $c'$, we vary it from 0.01 to 1. The other parameters are the same of Simulation 1. We observe that the final disposition of the particles (Fig. 12) is not too different from Simulation 1 (Fig. 6), however, it occurs at time $t = 7500$ versus $t = 3000$ of Simulation 1.

As reported in Fig. 13, the increase of the cohesion coefficient $c'$ causes a time dilatation, i.e., a translation of the time at which similar events occur.

Fig. 12. Position of particles in Simulation 2 at $t = 7000$. We observe that to have a spatial arrangement of particles similar to those of the previous simulation (Fig. 6) a larger time is needed.

Fig. 13. Kinetic energy of the systems versus time. The black line is the kinetic energy of Simulation 2. Comparing it with Fig. 7 of Simulation 1, we observe that an increase in the cohesion coefficient induces a translation of the events.
3.4 Simulation 3

We explore here the behavior of the system as a function of coefficients of static and dynamic friction $\mu_s$ and $\mu_d$. Their values are shown in Table 1. The other parameters are the same of Simulation 1. The consequence of the reduction of friction causes an immediate movement of particles. Moreover the number of particles involved during the event are larger then in the previous simulations (Fig. 15).

![Fig. 14. (a) Position of particles in Simulation 3 at $t = 3000$. The gray area represents the particle position of Simulation 1 (Fig. 6).](image)

![Fig. 15. (b) Number of particles involved. The decrease of the friction coefficients leads to an increase in the number of particles in motion.](image)

![Fig. 16. (a) Kinetic energy of the systems vs. time. The black line is the kinetic energy of Simulation 3. In the last simulation the value of the kinetic energy is greater than that in Simulation 1. This is due by the number of particles involved in the event (Fig. 15).](image)

![Fig. 17. (b) Mean velocity of the system versus time after $t = 1000$ for Simulations 1 and 3. We can observe that the two values are not too different between the two simulations. The difference of the kinetic energy is due to the number of particle in movement.](image)
Fig. 18. (a) Statistical distribution of kinetic energy in Simulation 3. It follows an exponential distribution like in Simulation 1.

Fig. 19. (b) The blue line refers to Simulation 3 with parameters $a_3 \simeq 2.88 \cdot 10^5$ and $b_3 \simeq -2.365$. The black line refers to Simulation 1 with parameters $a_1 \simeq 2.83 \cdot 10^5$ and $b_1 \simeq -3.078$. The dots represent the normalized value of the respective simulations.

Fig. 18 shows that also in this case the statistical distribution of the kinetic energy follows an exponential distribution. The data fit of Eq. (19) gives $a \simeq 2.592 \cdot 10^4$ and $b \simeq -0.091$.

4 Conclusions

In this article we presented a theoretical model that may be useful for studying the effect of precipitation on granular materials. The main hypothesis is that the rain acts as a lubricant between the terrain and the granular: this effect has been modeled by a preliminary report that includes the reduction of static (or dynamic) friction when we simulate the rainfall (Eq. (8) and Eq. (11)). The reduction in friction allows to follow the evolution and change in the position of the particles during and after a rainfall. The results obtained are very encouraging as regards both the displacement and evolution of the particles and in the statistical properties of the system. The next step will be to develop an experimental setup where granular material (sand or gravel) will be placed on a sloping surface: through liquid lubricant (soap and water) we will study the dynamics of these particles. The comparison of experimental and computational model will be very useful for the analysis of the effect of lubrication of the soil caused by rainfall.
Bibliography


Algebrizing friction: a brief look 
at the Metriplectic Formalism

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Abstract: The formulation of Action Principles in Physics, and the introduction of the 
Hamiltonian framework, reduced dynamics to bracket algebræ of observables. Such a 
framework has great potentialities, to understand the role of symmetries, or to give rise to 
the quantization rule of modern microscopic Physics. 
Conservative systems are easily algebrized via the Hamiltonian dynamics: a conserved 
observable $H$ generates the variation of any quantity $f$ via the Poisson bracket \{
\}. 
Recently, dissipative dynamical systems have been algebrized in the scheme presented 
here, referred to as metriplectic framework: the dynamics of an isolated system with 
dissipation is regarded as the sum of a Hamiltonian component, generated by $H$ via a 
Poisson bracket algebra; plus dissipation terms, produced by a certain quantity $S$ via a 
new symmetric bracket. This $S$ is in involution with any other observable and is 
interpreted as the entropy of those degrees of freedom statistically encoded in friction. 
In the present paper, the metriplectic framework is shown for two original “textbook” 
examples. Then, dissipative Magneto-Hydrodynamics (MHD), a theory of major use in 
many space physics and nuclear fusion applications, is reformulated in metriplectic 
terms. 
Keywords: Dissipative systems, Hamiltonian systems, Magneto-Hydrodynamics.

1. Introduction

Hamiltonian systems play a key role in Physics, since the dynamics of 
elementary particles appear to be Hamiltonian. Hamiltonian systems are 
edowed with a bracket algebra (that of quantum commutators, or classically of 
Poisson brackets): such a scheme is of exceptional clarity in terms of 
symmetries [1], offering the opportunity of retrieving most of the information 
about the system without even trying to solve the equations of motion. 
Despite their central role, Hamiltonian systems are far from covering the main 
part of real systems: indeed, Hamiltonian systems are intrinsically conservative 
and reversible, while, as soon as one zooms out from the level of elementary 
particles, the real world appears to be made of dissipative, irreversible processes 
[2]. In most real systems there are couplings bringing energy from processes at a 
certain time- or space-scale, treated deterministically, to processes evolving at 
much “smaller” and “faster” scales, to be treated statistically, as “noise”. This is 
exactly what friction does, and this transfer appears to be irreversible.
A promising attempt of algebrizing the classical Physics of dissipation appears to be the Metriplectic Formalism (MF) exposed here [3, 4]. The MF applies to closed systems with dissipation, for which the energy conservation and entropy growth hold: the MF satisfies these two conditions [5]. The first important ingredient of the MF is the metriplectic bracket (MB):

\[ \{\{f, g\}\} = \{f, g\} + (f, g), \]

where the first term \{f, g\} is a Poisson bracket, while the term \( (f, g) \) is a symmetric bracket, bilinear and semi-definite. The total energy is represented by a Hamiltonian \( H \) which has zero symmetric bracket with any quantity (i.e. \( (f, H) = 0 \) for all \( f \)). The total entropy is mimicked by an observable \( S \) that has zero Poisson bracket with any quantity (i.e. \( \{f, S\} = 0 \) for all \( f \)). Then, a free energy \( F \) is defined as

\[ F = H + \alpha S, \]

\( \alpha \) being a coefficient that will disappear from the equations of motion, due to the suitable definition of \( (f, g) \); it coincides with minus the equilibrium temperature of the system (see below in the examples). The dynamics of any \( f \) reads:

\[ \dot{f} = \{\{f, F\}\} = \{f, H\} + \alpha \{f, S\}. \]

This dynamics conserves \( H \) and gives a monotonically varying (increasing) \( S \). Metriplectic systems admit asymptotic equilibria (due to dissipation) in correspondence to extrema of \( F \).

In this paper the MF is applied to some examples of isolated dissipative systems: two “textbook” examples and, more significantly, to visco-resistive magneto-hydrodynamics (MHD).

2. Two “textbook” examples

In order to illustrate how the MF works, two simple systems are considered. The first one is a particle of mass \( m \) dragged by the conservative force of a potential \( V \) throughout a viscous medium. A viscous friction force, proportional to the minus velocity of the particle via a coefficient \( \lambda \), converts its kinetic energy into internal energy \( U \) of the medium, with entropy \( S \). The equations of motion of the system read:

\[ \dot{x} = \frac{p}{m}, \quad \dot{p} = -\nabla V - \lambda \frac{p}{m}, \quad \dot{S} = \frac{p^2}{m^2 T}. \]
$T$ is the temperature of the medium, simply defined as the derivative of $U$ with
respect to $S$. If the MB is defined as follows:

$$\dot{f} = \langle f, F \rangle, \quad \{f, g\} = \{f, g\} + \langle f, g \rangle /$$

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x},$$

$$\langle f, g \rangle = \Gamma^{ij} \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial p^j} / \psi = (x, p, S),$$

$$\Gamma = \alpha^{-1} \left( \begin{array}{ccc}
(\nabla V)^2 & \nabla V \otimes \nabla V & 0 \\
0 & 0 & \lambda T \mathbf{1}_{3,3} - m^{-1} \lambda \mathbf{p} \\
0 & -m^{-1} \lambda \mathbf{p}^T & \frac{\lambda p^2}{m T} \end{array} \right),$$

it is easy to show that these ODEs are given by the MB of $x, p$ and $S$, with a free
energy $F$ constructed as:

$$F(x, p, S) = H(x, p, S) + \alpha S,$$

$$H(x, p, S) = \frac{p^2}{2m} + V(x) + U(S).$$

The matrix $\Gamma$ is semi-definite with the same sign as $\alpha$. The foregoing framework
conserves $H$ and increases $S$, driving the system to the asymptotic equilibrium:

$$p_{eq} = 0, \quad \nabla V(x_{eq}) = 0, \quad T_{eq} = -\alpha.$$

At the equilibrium the point particle stops at a stationary point of $V$ once its
kinetic energy has been fully dissipated into heat by friction.

The second rather simple example of metriplectic system is a piston of mass $m$
and area $A$, running along a horizontal guide pushed by a spring of elastic
constant $k$. It works against a viscous gas of pressure $P$ and mass $M$. The system
is depicted in the following Figure.
Piston moved by the spring of elastic constant $k$ and mass $m$, working against a viscous gas of density $\rho$.

If $l$ is the rest-length of the spring, then the equations of motion of the system read:

$$\begin{align*}
\dot{x} &= \frac{L}{m}, & \dot{p} &= -PA - k(x - \ell) - \lambda \frac{\rho}{m}, & \dot{S} &= \frac{\lambda p^2}{m^2 T}.
\end{align*}$$

These equations of motion may be obtained out of a metriplectic scheme assigned as

$$\begin{align*}
\dot{f} &= \left\{ \langle f, F \rangle \right\}, & \left\{ \langle f, g \rangle \right\} &= \{ f, g \} + \{ f, \dot{g} \}, & \{ f, \dot{g} \} &= \Gamma ij \frac{\partial f}{\partial \psi^i} \frac{\partial \dot{g}}{\partial \psi^j}, & \psi &= (x, p, S),
\end{align*}$$

provided the following free energy is defined

$$\begin{align*}
F(x, p, S) &= H(x, p, S) + \alpha S, \\
H(x, p, S) &= \frac{p^2}{2m} + \frac{\lambda}{2} (x - \ell)^2 + U(\rho(x), S).
\end{align*}$$

Again, this $\Gamma$ is semi-definite with the same sign as $\alpha$. The asymptotic equilibrium of the foregoing $F$ read

$$\begin{align*}
x_{eq} &= \ell - \frac{PA}{k}, & p_{eq} &= 0, & T_{eq} &= -\alpha
\end{align*}$$

(the temperature $T$ is still defined as the derivative of $U$ with respect to $S$): the piston stops where the spring equilibrates the gas pressure, its kinetic energy all dissipated by friction.

### 3. Dissipative MHD

Dissipative MHD is expected to describe many plasma processes, wherever its fundamental hypotheses apply to a highly conductive plasma interacting with its own magnetic field [6, 7]. Ideal MHD has already been cast into Hamiltonian formalism [8], here the metriplectic extension of the Poisson algebra, and the free energy extension of the Hamiltonian, is proposed to include dissipative effects [9].
The 3D visco-resistive MHD equations read:

\[
\begin{align*}
\partial_t \mathbf{v} &= -\left( \mathbf{v} \cdot \nabla \right) \mathbf{v} - \frac{\nabla p}{\rho} - \frac{\mu}{\rho} \nabla^2 \mathbf{v} + \frac{\mathbf{B} \cdot \nabla V_{\text{grav}}}{\rho} + \nabla \left( \frac{\mathbf{v} \cdot \nabla \mathbf{v}}{\rho} \right), \\
\partial_t \mathbf{B} &= -\left( \mathbf{B} \cdot \nabla \right) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{B} + \mu \nabla^2 \mathbf{B}, \\
\partial_t \rho &= -\nabla \cdot (\rho \mathbf{v}), \\
\partial_t s &= -\left( \mathbf{v} \cdot \nabla \right) s + \frac{\mu}{\rho T} \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\kappa}{\rho T} \mathbf{v} \cdot \nabla \mathbf{v}.
\end{align*}
\]

The MHD, defined on a 3D domain \( D \), with suitable boundary conditions on \( \partial D \), is a complete system described by: plasma bulk velocity \( \mathbf{v} \), magnetic induction \( \mathbf{B} \), plasma density \( \rho \) and plasma mass-specific entropy \( s \). In the foregoing field equations, \( p \) is plasma pressure, \( V_{\text{grav}} \) is an external gravitational potential; \( \sigma \) is plasma stress tensor, containing (linearly) the fluid viscosity coefficients \( \eta \) and \( \zeta \) (see below), while \( \mu \) is resistivity. \( \kappa \) is thermal conductivity, and \( T \) is temperature of the plasma. The system conserves the total energy:

\[
H = \int_D d^3x \left( \frac{\rho v^2}{2} + \frac{B^2}{2} + \rho V_{\text{grav}} + \rho U(\rho, s) \right),
\]

which is the Hamiltonian, being \( U \) the mass-specific internal energy of the plasma. In the non-dissipative limit \( \sigma = 0, \mu = 0 \) and \( \kappa = 0 \), the whole physics is given by \( H \) and the following noncanonical Poisson brackets:

\[
\{f, g\} = -\int_D d^3x \left[ \frac{\delta f}{\delta \phi} \nabla \cdot \left( \frac{\delta g}{\delta \phi} \right) + \frac{\delta f}{\delta \phi} \nabla \cdot \left( \frac{\delta g}{\delta \phi} \right) \right] +
\frac{1}{\rho} \left( \nabla \times \mathbf{v} \right) \cdot \left( \frac{\delta f}{\delta \phi} \nabla \cdot \left( \frac{\delta g}{\delta \phi} \right) \right) +
\frac{1}{\rho} \left[ \left( \frac{\delta f}{\delta \phi} \nabla \times \mathbf{B} \right) \cdot \left( \frac{\delta g}{\delta \phi} \nabla \cdot \left( \frac{\delta f}{\delta \phi} \right) \right) \right] +
\frac{\delta f}{\delta \phi} \left[ \nabla \times \left( \frac{\delta g}{\delta \phi} \nabla \cdot \left( \frac{\delta f}{\delta \phi} \right) \right) \right] +
\frac{\delta f}{\delta \phi} \left[ \nabla \times \left( \frac{\delta g}{\delta \phi} \nabla \times \left( \frac{\delta f}{\delta \phi} \right) \right) \right]
\]

(here \( \delta f/\delta \phi \) is the Fréchet derivative of the functional \( f \) with respect to the field \( \phi \)). When dissipation is considered, the Hamiltonian must be extended to free energy adding a suitable entropic term:

\[
S[\rho, s] = \int_D d^3x \left\{ f, S \right\} = 0 \quad \forall \quad f = f(\mathbf{v}, \mathbf{B}, \rho, s),
\]

\[
F[\mathbf{v}, \mathbf{B}, \rho, s] = H[\mathbf{v}, \mathbf{B}, \rho, s] + \alpha S[\rho, s];
\]

the symmetric bracket to be used to form a complete MB, together with the Poisson bracket defined before, reads:
\[
(f, g) = \alpha^{-1} \int_D T d^3x \left[ kT V \left( \frac{1}{\rho T} \frac{\partial}{\partial x} \right) \nabla \left( \frac{1}{\rho T} \frac{\partial}{\partial x} \right) + \right. \\
+ \Lambda \left[ \left( \nabla \otimes \left( \frac{\partial}{\partial \phi} \right) \nabla \otimes \eta \right) \nabla \otimes \left( \frac{\partial}{\partial \phi} \right) \right] + \\
+ \Theta \left[ \left( \nabla \otimes \left( \frac{\partial}{\partial \phi} \right) \nabla \otimes \mu \right) \nabla \otimes \left( \frac{\partial}{\partial \phi} \right) \right]
\]

Note the strict analogy between the dissipative \(v\)-terms and \(B\)-terms, which are so alike because in the equations of motion dissipation terms appear as quadratic in the gradients of \(v\) and \(B\), respectively through the rank-4 tensors \(\Lambda\) and \(\Theta\) (quadratic dissipation, see [9]):

\[
\Lambda_{knm} = \eta (\delta_{kn} \delta_{mi} + \delta_{km} \delta_{ni} - \frac{2}{3} \delta_{kn} \delta_{mn}) + \zeta \delta_{ik} \delta_{mn}, \quad \sigma = \Lambda \cdot (\nabla \otimes v), \quad \Theta_{knm} = \mu \epsilon_{ij} \epsilon^{ij}_{knm}.
\]

Due to the symmetry properties of \(\Lambda\) and \(\Theta\), the symmetric bracket \((f, g)\) just defined is semi-definite with the same sign of \(\alpha\); the functional gradient of \(H\) is a null mode of it. Finally, the quantities related to the space-time symmetries, generating the Galileo transformations

\[
P = \int_D \rho v d^3x, \quad L = \int_D \rho (x \times v) d^3x, \quad G = \int_D \rho (x - tv) d^3x
\]

via the Poisson bracket algebra given in [8] and reported above, are conserved by the metriplectic dynamics:

\[
\dot{f} = \{f, H[v, B, \rho, S]\} + \alpha (f, S[\rho, s]),
\]

provided suitable boundary conditions are assigned to all the fields.

In the above Eulerian description of MHD, the bracket is noncanonical, depends on \(s\), and the entropy \(S\) appears as a Casimir of the bracket which, by definition, belongs to the kernel of the co-symplectic form associated to the bracket [10], while in the “textbook” cases the Poisson bracket was canonical and was independent on the entropy-related variable \(S\).

The free energy \(F[v, B, \rho, S]\) constructed before is able to predict the asymptotic equilibrium state:

\[
v_{eq} = 0, \quad B_{eq} = 0, \quad T_{eq} = -\alpha, \quad p_{eq} + \rho_{eq} V_{grav} = \rho_{eq} (T_{S} - U)_{eq}.
\]

Such an equilibrium configuration has zero bulk velocity and magnetization, while pressure and gravity equilibrates the thermodynamic free energy of the gas.
4. Conclusions

In metriplectic formalism friction forces, acting within isolated systems, are algebrized. The dissipative terms in the equations of motion are given by a suitable symmetric, semi-definite bracket of the variables with the entropy of the degrees of freedom to which friction drains energy.

Two simple “textbook” examples are reported: the point particle moving through a viscous medium; a piston, moved by a spring against a viscous gas in a rigid cylinder. In both the examples the evolution is generated via the metriplectic bracket with the free energy $F = H + \alpha S$, where $H$ is the conserved Hamiltonian and $S$ is the monotonically growing entropy. $\alpha$ appears to coincide with the equilibrium temperature.

The same formalism is then applied to an isolated magnetized plasma, represented by the dissipative (i.e. viscous and resistive) MHD with suitable boundary conditions. A Hamiltonian scheme already exists for the non-dissipative limit; furthermore, the full MF had been introduced for the neutral fluid version. In this paper, we report the extension of the latter formalisms to include the magnetic forces and the dissipation due to Joule Effect [9]. The “macroscopic” level of plasma physics is described by the fluid variable $v$, but a “microscopic” level exists too, encoded effectively in the thermodynamical field $s$. The energy attributed to the macroscopic degrees of freedom $v$ is passed to the microscopic ones by friction, while the electric dissipation of Joule Effect consumes the energy pertaining to the magnetic degrees of freedom $B$. Notice that the metriplectic formulation for dissipative MHD that we found, does not require div.$B = 0$.

Dissipative MHD is mathematically much more complicated than the two “textbook” examples, nevertheless its essence is rigorously the same: the MF algebraically generates asymptotically stable motions for closed systems. At the equilibrium, mechanical and electromagnetic energies are turned into internal energy of the microscopic degrees of freedom: the asymptotic equilibria found here for the three examples are essentially entropic deaths.

Let’s conclude with few more observations.

MF is a deterministic description, but it must be possible to obtain it as an effective representation of a scenario where the superposition of the Hamiltonian and the entropic motion mirrors the Physics of a deterministic Hamiltonian system under the action of noise [8].

The appearance of MF offers potentially great chances because it drives the algebraic Physics out of the realm of Hamiltonian systems: many interesting processes in nature (as the apparent self-organization of space physics systems [12], not to mention biological or learning processes) are not expected to be even conceptually Hamiltonian. It is very stimulating to imagine dealing with algebraic formalisms describing them. MF, however, is not able to compound such processes, because it pertains to complete, i.e. closed, systems, while the processes just mentioned take place in open ones. Adapting MF to open systems will then be a necessary step to face this challenge.
Before concluding, let’s underline again the dynamical role of entropy in MF: entropy may be interpreted as an information theory quantity [13, 14], and here we find information directly included in the algebraic dynamics. Furthermore: irreversible biophysical processes appear to have something in common with learning processes [15], i.e. processes in which the information is constructed or degraded, and having a formalism where “information” is an essential function appears to offer hopes in this branch.

References
Fractal Market Time

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Abstract. The no arbitrage condition requires that market returns are martingale
and the existence of long range dependence in the squared and absolute value of
market returns (Granger et al. [9]) is consistent with Fractal Activity Time (Heyde
[12]). We model the market clock as the integrated intensity of a Cox point process
of the transaction count of stocks traded on the New York Stock Exchange (NYSE).
A comparative empirical analysis of a self-normalized version of the integrated in-
tensity is consistent with a fractal market clock with a Hurst exponent of 0.75.

Keywords: Time Deformation, Long Range Dependent, Stochastic Clock, Frac-
tal Activity Time, New York Stock Exchange, Doubly Stochastic Binomial Point
Process.

1 Introduction

Clark [7] observed that returns appear to follow a conditional Gaussian Dis-
tribution where the conditioning is taken on a latent stochastic information
flow process. As a consequence, the unconditional returns \( r(t) \) will be gener-
ated by a mixture where the returns are a Wiener process \( W(\cdot) \) subject to a
time deformation or subordination process \( \Lambda_1(t) \).

\[
    r(t) = W[\Lambda_1(t)]
\]  

Ané and Geman [1] show that the market unconditional return distri-
bution is generated from conditioning an ordinary Brownian diffusion by a
stochastic clock based on cumulative trade count \( N(t) \). We model cumulative
trade count as a Cox [8] (doubly stochastic) point process and assume that
the associated integrated intensity \( A(t) \) can be modelled as a time accelerated
baseline integrated intensity \( A(t) = \Lambda_1(Kt) \) which is an empirical proxy for
the stochastic market clock.

The empirical analysis uses intra-day cumulative trade counts from the
New York Stock Exchange (NYSE) to explore the characteristics of the inte-
grated intensity as the time deformation process by self-normalizing cumu-
lative trade count \( R(t) \) and modelling the self-normalized trade count as a
doubly stochastic binomial point process [22],

\[
    R(t) = \frac{N(t)}{N(1)}, \quad t \in [0, 1]
\]
We then show that the scaling between final trade count $K$ and the variance of the self-normalized integrated intensity $A_1(Kt)/A_1(K)$ is different for different mathematical models of stochastic market time $A_1(Kt)$.

1. If $A_1(t)$ is modelled as a finite variance Lévy subordinator then the variance of the self-normalized integrated intensity will vary approximately as the inverse of trade count $1/K$.

$$\text{Var} \left[ \frac{A_1(Kt)}{A_1(K)} \right] \propto \frac{1}{K} \quad (3)$$

2. If $A_1(t)$ is modelled as Fractal Activity Time (FAT) proposed by Heyde [12] and Heyde and Liu [14] then the variance of the self-normalized integrated intensity will vary approximately with trade count $K$ as a power of the Hurst exponent $H$ of the FAT.

$$\text{Var} \left[ \frac{A_1(Kt)}{A_1(K)} \right] \propto K^{2H-2} \quad (4)$$

3. If $A_1(t)$ is modelled as an $\alpha$-stable Lévy subordinator then the variance of the self-normalized integrated intensity will not vary with trade count $K$.

$$\text{Var} \left[ \frac{A_1(Kt)}{A_1(K)} \right] \propto 1 \quad (5)$$

The variance of the normalized integrated intensity is found to scale proportionally to the inverse square root of final trade count $1/\sqrt{K}$. This implies the Hurst exponent of the integrated intensity $A_1(t)$ is $H = 0.75$ and thus market time is fractal. This is consistent with the FAT model and excludes the Lévy subordinator models examined above.

### 1.1 Self-Normalized Integrated Intensity

The problem with using the stochastic integrated intensity $A(t)$ of different stocks to determine the aggregate statistical properties of the market stochastic clock is that stocks trade at different rates. The solution is to re-scale the intra-day trade count to between 0 and 1 by the simple expedient of dividing the intra-day count ($N(t) = k$) by the final trade count ($N(1) = K$). This defines the self-normalized trade count process $R(t)$ which is formally named the random relative counting measure.

$$R(t) = \frac{N(t)}{N(1)} = \frac{k}{K} = a, \quad a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\} \quad (6)$$
It is unsurprising that the random relative counting measure \( R(t) \) is described by a binomial point process directed by the self-normalized integrated intensity. This point process is related to a binomial point process in a way directly analogous to the relationship between a Cox point process and the Poisson point process. Formally, the probability distribution of the random relative counting measure, \( R(t) \) conditioned on the final value of the integrated intensity \( \Lambda(1) \) is a binomial point process directed by the stochastic self-normalized integrated intensity of the related Cox process (McCulloch [22]).

\[
Pr\{ R(t) = a | \Lambda(1) \} = \Pr\{ N(t) = aK | N(1) = K, \Lambda(1) \} \\
= \left( \begin{array}{c} K \\ aK \end{array} \right) \left[ \frac{\Lambda(t)}{\Lambda(1)} \right]^{aK} \left[ 1 - \frac{\Lambda(t)}{\Lambda(1)} \right]^{(1-a)K} \\
a \in \{0, \frac{1}{K}, \ldots, \frac{K-1}{K}, 1\}, \quad t \in [0,1]
\]

(7)

We can now calculate the moments of the self-normalized intensity by examining stock trade count trajectories in a 2-d histogram [22].

2 Fractal Activity Time

A stochastic process \( T \) is called wide-sense self-similar (Sato [25]) if, for each \( c > 0 \), there are a positive number \( a \) and a function \( b(t) \) such that \( T(ct) \overset{d}{=} aT(t) + b(t) \) have common finite-dimensional distributions. A wide sense self-similar stationary increment model of market activity time was introduced by Heyde [12] and Heyde and Liu [14] as consistent with empirically observed market behaviour, which they termed ‘Fractal Activity Time’ (FAT). Heyde and Leonenko [13] developed a FAT with an inverse gamma marginal distribution implying Student-t distributed returns and Finlay and Seneta [11] have defined a FAT with gamma marginal distribution implying variance-gamma distributed returns.

\[
T(t) - t \overset{d}{=} t^H (T(1) - 1), \quad \frac{1}{2} \leq H < 1
\]

(8)

\[
E[T(t)] = t + t^H (E[T(1)] - 1) = t, \quad t \in [0,1]
\]

(9)
2.1 Self-Normalized Fractal Activity Time Moments

The Taylor series approximation of the expectation of the FAT model of the time accelerated self-normalized integrated intensity has terms that scale with trade count as $K^{2H-2}$.

$$
E\left[ \frac{T(Kt)}{T(K)} \right] \approx t + \left( t - \frac{t^{2H} + 1 - (1 - t)^{2H}}{2} \right) K^{2H-2} \text{Var}[T(1)] \quad (10)
$$

We arbitrarily model the exogenous ‘S’ shaped non-linear variation in daily market time seasonality (‘U’ shaped daily trading activity) as a deterministic function with the same functional form as the expectation of the FAT model of the self-normalized integrated intensity (eqn 10). Thus market time as integrated intensity is formulated as $\Lambda_1(t) = T(\Delta(t))$ where $\Delta(t)$ is the deterministic function defined below with constant $D$ that determines the magnitude of the ‘S’ shaped non-linear variation with $\Delta(0) = 0$, $\Delta(0.5) = 0.5$ and $\Delta(1) = 1$.

$$
\Delta(t) = t + \left( t - \frac{t^{2H} + 1 - (1 - t)^{2H}}{2} \right) D, \quad t \in [0,1] \quad (11)
$$

If the baseline intensity/stochastic clock is defined as $\Lambda_1(t) = T(\Delta(t))$ then it is obvious that a stationary increment version of the baseline intensity/stochastic is $\Lambda_1(\Delta^{-1}(t)) = T(\Delta^{-1}(\Delta(t))) = T(t)$ where $\Delta^{-1}(t)$ is the inverse function of $\Delta(t)$. For a stock with $K$ observed final trades the integrated intensity is modelled using the FAT as:

$$
\Lambda(t) = \Lambda_1(Kt) = T(K\Delta(t)) \quad (12)
$$

Self-Normalized Fractal Activity Time Variance

The Taylor series approximation of the variance of the self-normalized integrated intensity has terms that scale with trade count as both $K^{2H-2}$ and $K^{4H-4}$. However, with a nominal variance of $\text{Var}[T(1)] = 0.875$ and Hurst exponent of $H = 0.75$ the $K^{4H-4}$ term is small relative to the $K^{2H-2}$ term.

$$
\text{Var}\left[ \frac{T(Kt)}{T(K)} \right] \approx \left( t^2 - t(t^{2H} + 1 - (1 - t)^{2H}) + t^{2H} \right) K^{2H-2} \text{Var}[T(1)]
$$

$$
- \left( t - \frac{t^{2H} + 1 - (1 - t)^{2H}}{2} \right)^2 K^{4H-4} (\text{Var}[T(1)])^2
$$

(13)
Fig. 1. The expectation of the self-normalized FAT with intra-day seasonality $\mathbb{E}[T(K\Delta(t))/T(K)] - t$ (linear trend removed). We model the exogenous ‘S’ shaped non-linear intra-day seasonality in market time (‘U’ shaped daily trading rate) as a deterministic function $\Delta(t)$ (eqn 11) where $D = 3$. For comparison the empirical expected intra-day variation is also displayed as thin plot lines. The empirical expected intra-day variation exhibits an asymmetry between the morning and afternoon variations that are not captured by the formal FAT model. The slight difference in intra-day variation amplitude between trade counts in the formal FAT model is due to the deterministic function $\Delta(t)$ plus the functional form of eqn 10.

3 Lévy Subordinators

Lévy subordinators are non-decreasing Lévy processes (Sato [26]). There has been considerable research proposing the use of subordinated Wiener processes, and more generally subordinated Lévy processes such as stable Paretian processes as models of stochastic market time. A number of different mixtures have been put forward to account for the observed characteristics of the unconditional return process and prominent examples of subordinated Wiener processes include the Variance Gamma model of [16], [17] and the Normal Inverse Gaussian model, [2], [6], [23], [5], [4], [3]. An example of a subordinated Lévy process is the $\alpha$-stable Gamma model of [21], [20].

3.1 Finite Variance Subordinators

Lemma 1. The following properties of finite variance Lévy subordinators are proved by examining the time dependent structure of the first two moments of a Lévy process.

1. Lévy subordinators with finite moments are not self-similar.
Fig. 2. The variance of the FAT model of self-normalized integrated intensity $\text{Var}[T(K\Delta(t))/T(K)]$ (eqn 13) for different trade count bands $K$. The Hurst exponent is $H = 0.75$ and nominal variance is $\text{Var}[T(1)] = 0.875$. For convenient comparison, the empirical variance $\text{Var}[\Delta(t)/\Lambda(1)]$ is also plotted as thin lines and the difference between the two is shaded. The difference between the empirical variance of the self-normalized integrated intensity and the FAT model is largely due to the symmetry of the functional form of the deterministic intra-day variation $\Delta(t)$ (eqn 11) compared to the asymmetry of the empirical intra-day variation, see figure 1 and related commentary.

2. Any self-normalized Lévy subordinator $\Gamma (Kt)$ with a finite variance scales approximately as a function of $1/K$ for values of $K \gg 1$.

$$\text{Var} \left( \frac{\Gamma(Kt)}{\Gamma(K)} \right) \propto \frac{1}{K}, \quad K \gg 1, \quad t \in [0,1]$$ (14)

We examine the closely related case where the random activity time is assumed to be an independent increment additive process (a time changed Lévy subordinator, Sato [26]). Using the results in James et al. [15] the variance of self-normalized increasing additive processes can be calculated directly. As an example, the variance of the Self-Normalized Gamma process and Self-Normalized Inverse Gaussian process are formulated explicitly.

Assuming subordinator $\Gamma(t)$ is a Gamma process, $c$ is constant for all trade counts and $\Delta(t)$ is the deterministic intra-day seasonality (eqn 11), then the variance of the self-normalized Gamma process for a stock with $K$ trades is:
\[ \text{Var} \left[ \frac{\Gamma(K \Delta(t))}{\Gamma(K)} \right] = \Delta(t) \left( 1 - \Delta(t) \right) \frac{1}{Kc + 1} \quad (15) \]

Clearly for the self-normalized Gamma process the term \( 1/(Kc + 1) \) approximates \( 1/K \) scaling for \( Kc \gg 1 \). Next we assume the additive subordinator \( \Gamma(K \Delta(t)) \) is an inverse Gaussian process and \( c \) is constant for all trade counts, then the variance\(^1\) of the self-normalized inverse Gaussian process is:

\[ \text{Var} \left[ \frac{\Gamma(K \Delta(t))}{\Gamma(K)} \right] = \Delta(t) \left( 1 - \Delta(t) \right) (Kc)^2 e^{Kc} \int_{Kc}^{\infty} \frac{e^{-u}}{u^{3/2}} \, du \quad (16) \]

The trade count term for Inverse Gaussian is less transparent than the Gamma case above but can be readily shown (figure 3) to approximate \( 1/K \) scaling for \( K \gg 1 \).

The variances of Gamma and Inverse Gaussian self-normalized Lévy subordinators are scaled as a function of trade count \( K \) and compared to the scaling of the empirical self-normalized stochastic clock and self-normalized Fractal Activity Time (FAT) process. The results are graphed in figure 3 and it is immediately clear from this graph that the Lévy subordinators scale close to \( 1/K \), whereas the FAT process with Hurst exponent \( H = 0.75 \) scales as \( 1/\sqrt{K} \) as required.

### 3.2 α-Stable Subordinators

Another class of Lévy subordinators are \( \alpha \)-stable processes \( \Gamma^\alpha \) with \( 0 < \alpha < 1 \). These processes have no defined moments (all moments are infinite) and are self-similar with \( \Gamma^{\alpha}(t) \overset{d}{=} t^{1/\alpha} \Gamma^{\alpha}(1) \) corresponding to a Hurst exponent \( H = 1/\alpha \).

Mandelbrot [18], Fama [10] and Mandelbrot and Taylor [19] introduced stable Paretian processes as models of financial market returns. These are infinite variance symmetric distributions with \( 1 \leq \alpha < 2 \) (\( \alpha = 2 \) is the Gaussian distribution). It is well known (Samorodnitsky and Taqqu [24]) that a standard Wiener process \( W(t) \) subordinated to an \( \alpha \)-stable Lévy subordinator with \( 0.5 \leq \alpha < 1 \) is distributed as a symmetric stable Paretian process with index \( 2\alpha \).

\[ \Gamma^{2\alpha}(t) \overset{d}{=} W(\Gamma^\alpha(t)) , \quad 0.5 \leq \alpha < 1 \quad (17) \]

\(^1\) The integral term is the upper incomplete gamma function \( U\Gamma(-2, Kc) \).
Fig. 3. The variance scaling of the empirical self-normalized stochastic clock $\Lambda(0.5)/\Lambda(1)$ at different trade count bands $K$ compared to the variance scaling of self-normalized versions the Fractal Activity Time (FAT) process and Lévy subordinators. It is clear from this graph that the empirical stochastic clock and FAT ($H = 0.75$) scale close to $1/\sqrt{K}$. Conversely the Gamma and Inverse Gaussian subordinators scale close to $1/K$ and are misspecified.

Although $\alpha$-stable processes with $0 < \alpha < 1$ have no defined moments the variance of the corresponding self-normalized process exists and James et al. [15] show that the variance of the self-normalized time transformed $\alpha$-stable subordinator is:

$$\text{Var}\left[\frac{\Gamma^\alpha(K \Delta(t))}{\Gamma^\alpha(K)}\right] = \Delta(t) (1 - \Delta(t)) (1 - \alpha), \quad 0 < \alpha < 1 \quad (18)$$

Therefore a self-normalized $\alpha$-stable Lévy subordinator does not scale with trade count. However, the empirical variance of the self-normalized market clock displays $1/\sqrt{K}$ scaling (figure 3) and the $\alpha$-stable Lévy subordinator model is not consistent with this evidence.

References


Abstract- The modern electric facilities are equipped by a great number of different mechanisms and devices actioned by Asynchronous electric Motor (ASM), the power of these motors is equal to the power of the generating devices, where their most complicated working regime is the starting when their power is equal to the power of the generating devices. In this regime we can have an overcharge of the generating devices by the active and reactive power. For this reason, this article is dedicated to the study of the starting methods of asynchronous motors that action the mechanisms and that are powered by Asynchronous Generating Diesel (AGD) with a limited capacity of DRY value and a given couple of resistance.

Keywords: Reliability, Autonomous asynchronous generator, starting of the asynchronous motors, Tention converter.

1. Introduction:

There are several factors that considerably influence the characteristics of the asynchronous motors starting process from a AGD, amongst these factor:
- The initial conditions of the process,
- The oscillation of frequency and amplitude of the AGD tension,
- The non linear character of the electric machines parameters used as an actioner motor for the generator,
- The mechanisms resistant couple.

If we take into account these factors, the study of the transient regime in the AGD-ASM system using analytical methods will be complex and will induce high calculations uncertainties. In this article, we will study the analysis of the regimes dynamics of the starting from AGD of mechanisms with asynchronous electric actionner using a numerical method to achieve a given precision of the calculations.

2-Mathematical model of the starting regime of mechanisms with asynchronous actionner powered by AGD

A- General characteristic of the model

Studies performed previously have shown that for a complete analysis of the common operating regime of AGD and ASM, it is necessary that the mathematical model takes into account the review of the transient regimes for the direct starting of the motor, the starting through an auxiliary resistance in the statoric circuit as well as the starting through the Tension Converters with Thyristors (TCT). In addition to considering the electromagnetic systems properties, the supplying of the consumers by a three-phased tension under neutral line, the necessity in a large interval of the regulation of the key elements starting angle value (because the actioning mechanisms has the same power that the generator) In the mathematical model, the functioning of the TCT of 3TT type that is composed of two thyristors connected head to foot in every phase of the supplying line is described.

B- Mathematical model of the asynchronous motor with short-circuited rotor.

In order to study the operating regime of ASM with the auxiliary elements connected to its statoric coil, we have to write the composed differential equations in relation to the statoric current and to the rotoric hooking flux in a simple shape [1], [2], [3] in the \((\alpha, \beta, \sigma)\) coordinate system. The ASM equations are the following:

\[
\frac{d\psi_{\alpha}}{dt} = \frac{U_{\alpha} - R_{SS\alpha}I_{\alpha} - L_mI_{\alpha} - T\psi_{\alpha} - w_{\alpha}\psi_{\beta}}{L_{\Sigma}}
\]

\[
\frac{d\psi_{\beta}}{dt} = \frac{U_{\beta} - R_{SS\beta}I_{\beta} - L_mI_{\beta} - T\psi_{\beta} - w_{\beta}\psi_{\alpha}}{L_{\Sigma}}
\]

\[
\frac{di_{\alpha}}{dt} = r_{\alpha}I_{\alpha} - T\psi_{\alpha} - w_{\alpha}I_{\beta}
\]

\[
\frac{di_{\beta}}{dt} = r_{\beta}I_{\beta} - T\psi_{\beta} - w_{\beta}I_{\alpha}
\]

where :

\[R_{SS} = R_s + R_g\]

\[L_{\Sigma} = L_{sh} + L_n\]

\[L_1 = \frac{L}{L_{mM} + L_{rM}}\]

\[T_1 = \frac{r_{rM}}{L_{mM} + L_{rM}}\]

\[L_{rM} \text{ and } L_{mM} \text{ : ASM’s rotoric and statoric flux scattering inductances.}\]

\[L_n \text{ : Inductance of the auxiliary resistance.}\]

\[L_{mM} \text{ : Mutual inductance of the ASM’s statoric and rotoric coils.}\]
$w_{rm}$ : The pulsation of the ASM’s rotor rotation speed.

$i'_M$ : ASM statoric current.

$\psi_M$ : Hooking flux of the ASM’s rotor

The calculation of the transient regime in the ASM is performed in a relative units system, the nominal current amplitudes and the generator tension are taken as basis value, the basis time is the same for the processes calculation. The instantaneous values of the motor current in the generator relative units system are transformed using of the transfer coefficients

\[ K_c = \frac{i_{at}}{i_{aG}} \]

$i_{at}$ and $i_{aG}$ : Basis values of the currents in the motor and the generator relative units system.

The electromagnetic couple developed by ASM is determined by the formula (2):

\[ C_M = \psi_{ab} \ i_{bM} - \psi_{bM} \ i_{ab} \]

The equation of the movement of the mechanism axis that action the ASM is:

\[ \frac{dw_{cm}}{d\tau} = \frac{1}{J_Z} \left( C_M - C_{at} \right) \]

$J_Z$ : Sum of the inertias couples of the ASM mobile mass and of the mechanism translated to the motor rotor

$C_{at}$ : Couple of the mechanism resistance (N.m).

$C_{ab}$ : Basis value of the ASM couple (N.m).

To take into account the influences of the rotor current and the saturation of the machine iron on the variation of the rotation frequency at the starting time we include in the mathematical model the relations linking the rotor scattering inductance and the motor rotor active resistance with the motor sliding [4].

\[ r_M = (r_{ar} - r_{am}) g_M \]

\[ L_{am} = (L_{ar} - L_{rm}) g_M \]

Where:

$r_{ar}$ : Active resistance of the rotor.

$L_{ar}$ : Rotor scattering inductance at the starting.

$g_M$ : Motor sliding ($g_M = 1$)

C - Mathematical model of the Tension Converter with Thyristors TCT For the development of the TCT mathematical model we take into account the following [3]:

The arm of each branch is composed of two thyristors connected head to foot when the command signal arrives to the thyristor trigger and become in the closed state regardless of the system tension where it is present at that given moment.

The thyristor remains closed so far the value of the current that crosses it is higher than the upholding current.

In its closed state, the thyristor is replaced by an active resistance; the drop of tension in this latter one corresponds to the drop of tension value in the thyristor in the closed state.

In its open state, the thyristor is replaced by an active resistance in which the current becomes equal to the inverse current of the chosen thyristor.

Taking into account these simplifications, the control of the thyristors state is achieved by the analysis of every step of the command tension value calculation and the value of the current that crosses it at that given moment. To achieve this objective, the mathematical model takes into consideration the equations of the ASM phase current derived from the first two equations of the system (1)

\[ \frac{di_a}{d\tau} = \left[ (U_a - U_N) - R_{SM} I_a - L_i (r_{SM} I_a - w_i \psi_{aM}) \right] \frac{1}{L_{SM}} \]

\[ \frac{di_b}{d\tau} = \left[ (U_b - U_N) - R_{SM} I_b - L_i (r_{SM} I_b - T \psi_{bM} - w_i \psi_{bM}) \right] \frac{1}{L_{SM}} \]

\[ \frac{di_c}{d\tau} = \left[ (U_c - U_N) - R_{SM} I_c - L_i (r_{SM} I_c - T \psi_{cM} + w_i \psi_{bM}) \right] \frac{1}{L_{SM}} \]

Where:

\[ U_a = U_a \]

\[ U_b = \frac{-U_a}{2} + \frac{\sqrt{3}}{2} U_b \]

\[ U_c = \frac{-U_a}{2} + \frac{\sqrt{3}}{2} U_b \]

$U_a, U_b$ and $U_c$ :tensions of phases of AG

$U_N$ : Tension between the neutral points of the AG and ASM statoric coils.

The functioning algorithm of TCT in the lack of a neutral link line in the naval network is limited by three possible conduction regimes that are:

- Three-phased conduction: closed arm for all phases.
- Two-phased conduction: closed arm for any phase.
- Neutral conduction: open state of the arms for the three phases.

For the TCT chosen types, taking into account the previous simplifications on the asynchronous machines symmetry, $U_N$ value can be determined for any time moment. At the time of the functioning of the symmetrical electric machines, $U_N$ is different from zero only in the case of the two-phased conduction. For the thyristors that are used in the model and at the time of passing from TCT to any conduction state that precedes the two-phased one, the initial values of the currents in the phases of open thyristor are equal in value but opposed in phase, on the other hand, $U_N$ because in the AGD-ASM system, the symmetrical regime still exists at the following time moment. The same current will cross the two phases with open thyristor.
In these conditions, the instantaneous value of $U_N$ can be calculated using the system of equations (7) that leads to the following values:

The phase (a) closed:

$$U_N = \left( U_b + U_c + L_1 \frac{d\psi_{\alpha M}}{d\tau} \right) \frac{1}{2.2L_{\alpha M}}$$

The phase (b) closed:

$$U_N = \left( U_b + U_c - L_1 \left( \frac{1}{2} \frac{d\psi_{\alpha M}}{d\tau} - \frac{\sqrt{3}}{2} \frac{d\psi_{\beta M}}{d\tau} \right) \right) \frac{1}{2.2L_{\alpha M}}$$

The phase (c) closed:

$$U_N = \left( U_a + U_b - L_1 \left( \frac{1}{2} \frac{d\psi_{\alpha M}}{d\tau} - \frac{\sqrt{3}}{2} \frac{d\psi_{\beta M}}{d\tau} \right) \right) \frac{1}{2.2L_{\alpha M}}$$

The opening of the thyristors command signal is formed using the command law taken for the TCT.

3- Algorithm of calculation of the starting regime

The simulation by MATLAB software of the system ((1) - (5)), using the Runge-Kutta method, has allowed us to get the following results of the ASM starting powered by AGD.

![Fig. 1. Variations of the $I_s$ current in terms of the time for a direct ASM starting powered by AGD](image1)

![Fig. 2. Variations of the electromagnetic couple according to the time for a direct ASM starting powered by AGD](image2)

![Fig. 3. Variations of the angular speed $W_r$ according to the time for a direct ASM starting powered by AGD](image3)

![Fig. 4. Variations of the angular speed $W_r$ and the $U_t$ tension according to the time for an ASM starting with a TCT powered by AGD](image4)

![Fig. 5. Variations of the $I_s$ current in terms of time for an ASM starting with a TCT powered by AGD](image5)
Fig. 6. Variations of the electromagnetic couple Cem in terms of time for an ASM starting with a TCT powered by AGD

We notice on the two previous figures, the absence of any current or couple abrupt peaks. The resulting drop of tension that and mechanical shocks due to the brutal apparition of the couple. The starting time in this case can exceed the direct starting time by several times.

4 - Command of the starting regime by a Tension Converter with Thyristors

For an ASM starting, the limitation of the current peaks can be obtained not only by the reduction of the tension amplitude via the introduction of the auxiliary resistances in its statoric circuit, but also using other regulation methods of this tension value in the devices that allow the command of this starting regime, by means of varying the commutation of the allowed or blocked state of the semiconductor components (thyristor, power transistor, triac) [5]. [7]. [8]. The most efficient actionners of asynchronous mechanisms are the starting devices constructed on the basis of Tension Converters with Thyristor (TCT) commanded by a phase angle [8]. [9]. For an automatic command of the starting regimes of the asynchronous actionners mechanisms, several solutions exist currently. Amongst them we can mention the solution that uses gradators, where the power circuit includes in every phase two thyristors assembled head to foot; the variation of tension that powers the ASM is progressive and is obtained via varying the conduction time by phase angle of these thyristors during every half period (fig.7) [5]-[6].

This type of starting limits the call of current, the ensuing drop of tension and the mechanical shocks resulting from the brutal apparition of the couple. For the ADG energizing systems linked to an excitation device (DRY) whose action rapidity can be compared to the action rapidity of the command system by phase angle of the TCT key elements and which can have a positive effect if we introduce in the TCT automatic system a negative return loop between the starting angle of the thyristors’ triggers and the drop of tension between the AGD limits. The possibility to use a system with TCT for the ASM starting command from an AGD is represented on the figure (8). For the elements of commutation we use some phototyristors that can be commanded by a luminous impulse. To assure a galvanic insulation between the power circuit and the command circuit at the time of the of the installation functioning, the primaries of the impulsion transformers T1-T3 are joined with the AGD statoric coils, the secondary of these transformers are plugged with the Zener diodes DZ7-DZ12 stabilizing the phases tensions. If a drop of tension appears the phase changes and the length of the command luminous impulse that determines the value of the starting angle of the phototyristor also changes. The choice of the command tensions phase angle and the parameters of the phototyristors allow the creation of a negative return loop by a drop of tension. We can have a large regulation interval of the phototyristors starting angle by means of the formation of the phase command signal that has a tension which exceed the anode-cathode tension of these phototyristors in a varying interval between 30° and 120°. The relation of the phototyristors starting angle with the tension phase amplitude is determined by the equation (8)

$$\alpha = \frac{\pi}{3} + \arcsin\frac{U_{st}}{U_m}k_T$$

Where:

$$\alpha :$$ Pothothyristors starting angle  
$$U_{st} :$$ Stabilizer tension  
$$k_T :$$ Transformers transformation coefficient  
$$U_m :$$ Amplitude of the tension phase instantaneous value

Fig. 7. Block diagram of the ASM starting through a gradator
We take into consideration the variation character of the ASM power coefficient at the time of its starting. The chosen regulation interval of the starting angle is sufficient to command this starting because it ensures the conditions of TCT functioning in the case of three or two conductor arms.

On the figure (10) we represented the oscillograms of the regime in the case of an ASM starting through a TCT [9] [10] [11]. The curves represent the effective tension value, the frequency of the diesel generator actionner and the starting duration of ASM for different powers. The comparison between these features and the parameters of the ASM’s direct starting regimes (fig(1)-(2)) permits to point out that the use of the TCT lead to the decrease the drop of tension in the load, the starting time in this case can exceed the time of direct starting by several multiples.
5 Conclusion

We have developed a mathematical model of the common functioning regime dynamics of the AGD and of the mechanisms with asynchronous actionneur. This model takes into consideration the elements of the following system:

1- Diesel, asynchronous generating, asynchronous motor with auxiliary resistances in the statoric circuit with a resistance couple on the corresponding axis of the different mechanisms, tension regulator with thyristors, the calculation parameters of the transient regimes are closer of the experimental data with a precision around 13%.

2- The calculations of the parameters of the asynchronous machines starting regimes of 4A, AM and AO series, for a capacity of DRY equal to 1.45 pu and for a limitation of tension drop on the borders of AGD of 20% of $U_n$, give a steady direct starting of the asynchronous machines of a power of 20 to 25% of the nominal power of AG, without limitation of the tension drop value and for the same value of DRY capacity the AGD can assure a direct ASM starting with a power of 30 to 40% of the nominal power of AG.

3- The analysis of the possibilities of the starting regime command of the mechanisms with asynchronous actionner by a AGD with the use of a TCT with negative return loop between the angle of the thyristors triggers opened state and the drop of tension on the borders of AGD, shows that by this method we can limit the drop of tension on the borders of the load and can increase the unit of motors power started for a limited value of the DRY capacity. We noticed that the couple developed by ASM in the time of starting through a TCT has a smaller value compared to the starting through auxiliary resistances for the same value of tension drop that appeared on the AGD borders.

REFERENCES


Biological Growth in the Fractal
Space-time with Temporal Fractal Dimension

Marcin Molski

Abstract: In the biological systems the fractal structure of space in which cells interact and differentiate is essential for their self-organization and emergence of the hierarchical network of multiple cross-interacting cells, sensitive to external and internal conditions. Hence, the biological phenomena take place in the space whose dimensions are not represented only by integer numbers (1, 2, 3 etc.) of Euclidean space. In particular, malignant tumors and neuronal cells grow in a space with noninteger fractal dimension. Since, cellular systems grow not only in space but also in time, an idea has been developed that the growth curves representing neuronal differentiation or malignant tumor progression can be successfully fitted by the temporal fractal function \( y(t) \), which describes the time-evolution of the system, characterized by the temporal fractal dimension \( b_t \) and scaling factor \( a_t \). One may prove that in the case of biological systems whose growth is described by the Gompertz function, the temporal fractal dimension and scaling factor are time-dependent functions \( b_t(t) \) and \( a_t(t) \), which permit calculation their values at an arbitrary moment of time or their mean values at an arbitrary time-interval. The model proposed has been applied to determine the temporal fractal dimension of the tumor growth and synapse formation as qualitatively these processes are described by the same Gompertz function. The results obtained permit formulation of two interesting rules:

(i) each system of interacting cells within a growing system possesses its own, local intrasystemic fractal time, which differs from the linear \( b_t=1 \) scalar time of the extrasystemic observer;
(ii) fractal structure of space-time in which biological growth occurs, is lost during progression.

It will be proved that the fractal function \( y(t) \) is a special case solution of the quantal annihilation operator for the space-like, minimum-uncertainty coherent states of the time-dependent Kratzer-Fues oscillator. Such states propagate along the well defined time trajectory being coherent in space. Hence, the biological growth in the space-time with temporal fractal dimension is predicted to be coherent in space.

Keywords: Fractal space-time, Synapse formation, Tumorigenessis, Biological growth
1. Introduction

The morphometric computer-aided image analysis reveals that growth of biological systems occurs in the space-time with the spatial fractal dimension (also called Hausdorff dimension) defined by

\[ b_s = \lim_{\varepsilon \to 0} \frac{\ln n(\varepsilon)}{\ln(1/\varepsilon)} \]

Here, \( n(\varepsilon) \) is the minimum number of hypercubes of dimension \( \varepsilon \) required to completely cover the biological, physical or mathematical object under consideration. The fractal dimension can be defined also by the self-similar power low scaling function

\[ y(x) = a_s x^{b_s} \quad x > 0 \]

in which \( y(x) \) denotes the number of self-similar objects in the sphere or circle of a radius \( x \), \( b_s \) and \( a_s \) stand for the spatial fractal dimension and the scaling factor, respectively. In the case of biological systems the fractal structure of space in which cells interact and differentiate is essential for their self-organization and emergence of the hierarchical network of multiple cross-interacting cells, sensitive to external and internal conditions. Hence, the biological phenomena take place in the space whose dimensions are not represented only by integer numbers (1,2,3 etc.) of Euclidean space. In particular tumors and synapses grow in a space with non-integer fractal dimension. Cellular systems grow not only in space but also in time. Recently, an idea has been developed that the curves describing the growth of biological systems can be successfully fitted by the temporal counterpart of the space fractal function [1,2]

\[ y(t) = a_t t^{b_t} \quad t > 0 \]

in which \( y(t) \) characterizes the time-evolution of the system, \( b_t \) is its temporal fractal dimension whereas \( a_t \) - a scaling factor. The main idea of the work is mapping the Gompertz function of growth [3]

\[ G(t) = G_0 e^{a t - e^{-a t}} \]

widely applied to fit the demographic, biological and medical data, onto the fractal function \( y(t) \). In this way we obtain the the-time dependent expressions \( b_t(t) \) and \( a_t(t) \), which permit calculation their values at an arbitrary moment of time or their mean values at an arbitrary time-interval. In the Gompertz function \( G_0 \) stands for the initial mass, volume, diameter or number of proliferating cells, \( a \) is retardation constant whereas \( b \) denotes the initial growth or regression rate constant.

2. The model

To find the explicit form of \( b_t(t) \) and \( a_t(t) \) the relation
and its first derivative

$$\frac{b}{a}t^h = e^{a(1-e^{-a})} - 1$$

are taken into consideration. The first of them satisfies the proper boundary conditions for \( t \to 0 \) for \( G_0=1 \) (one cell). Combining the above equations, we arrive at the analytical expressions

$$b_i(t) = bte^{-at} \frac{e^{a(1-e^{-a})}}{e^{a(1-e^{-a})} - 1}$$

$$a_i(t) = t^{-h} \left[ e^{a(1-e^{-a})} - 1 \right]$$

which define the temporal fractal function describing the growth of Gompertzian systems

$$y(t) = a_i(t) t^{b_i(t)}$$

By plotting one may easily prove that function \( y(t) \) is indistinguishable from the Gompertz function \( G(t) \), hence the mapping procedure is successful.

3. The results

The synapse formation can be characterized by the Gompertz growth curve obtained by the fitting the experimental data obtained by Jones-Villeneuve et al. [4]. The fit provided the following parameters: \( a=0.0739(89) \) [day], \( b=0.3395(378) \) [day] for constrained \( G_0=1 \) evaluated with the nonlinear regression coefficient \( R=0.9737 \). In the next step the parameters \( a \) and \( b \) have been used to calculate the time-dependent fractal dimension \( b_i(t) \) and scaling factor \( a_i(t) \) using the above specified formulae. Their plots are presented in Fig. 1.

![Plots of the time-dependent temporal fractal dimension and scaling factor for neuronal cells growth characterized by the Gompertz parameters.](image)

In the case of tumorigenesis we consider as an example the Flexner-Jobling rat’s tumor whose growth is described by the Gompertz function with
parameters: $a=0.0490(63)$ [day], $b=0.394(66)$ [day] determined by Laird [5].
They were used to generate plots of $b(t)$ and $a(t)$ presented in Fig. 2.

Fig. 2. Plots of the temporal fractal dimension $b(t)$ and the scaling factor $a(t)$ for Flexner-Jobling rat’s tumor whose growth is characterized by the parameters $a=0.0490(63)$ [day], $b=0.394(66)$ [day].

Analysis of the results obtained reveal that during neuronal differentiation and synapse formation, the temporal fractal dimension $b(t)$ increases from 1 for $t=0$ to a maximal value 1.80 for $t=11.97$ [day] and then decreases to zero. We find here an interesting correlation with the spatial fractal dimension calculated in vivo for retinal neurons; it takes value 1.68(15), whereas a diffusion-limited-aggregation model predicts 1.70(10) [6]. Those spatial dimensions are equal in the range specified standard errors to temporal fractal dimension 1.80 determined in this work. In the case of the brain’s neurons of two teleost species Pholidapus dybowskii and Oncorhynchus keta, the application of the box-counting method provided the fractal dimension equal to 1.72 for less specialized neurons, whereas highly specialized neurons displayed a relatively low dimension [7]. We conclude that the temporal fractal dimension can be applied as a numerical measure of the neuronal complexity emerging in the process of differentiation, which is correlated with the morphofunctional cell organization. In particular, the change from maximal value of the fractal dimension $b(t=11.97)=1.80$ to dimension attained at plateau $b(t=50)=0.43$ reflects the appearance of the highly specialized neurons evolving from the less specialized ones. The temporal fractal dimension of the Flexner-Jobling’s tumor growth increases from 1 for $t=0$ to a maximal value 2.98 for $t=20$ [day] and then decreases to zero. Both $b(t)$ and $a(t)$ determined for neuronal differentiation and tumour progression behave in the identical manner. We conclude that tumorigenesis has a lot in common with the neuronal differentiation and synapse formation, although the dynamics of these processes are different: the maximal values of the temporal fractal dimension are equal to 1.8 and 2.98, respectively.
4. The origin of y(t)

One may demonstrate that the fractal function \( y(t) \) is a special case solution of the annihilation operator

\[
\hat{A}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle
\]

for the space-like, minimum-uncertainty coherent states of the time-dependent counterpart of the Kratzer-Fues oscillator [8]. In the above equation

\[
\hat{A} = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \beta_0 - \frac{b}{x} \right) \left|\alpha\right\rangle = x^\alpha \exp[-\beta_0 x] \exp[\sqrt{2} \alpha x]
\]

and \( x = t/t_0 \) dimensionless temporal variable appearing in the time-dependent version of the Kratzer-Fues potential \( V(x) = D(1-x^2)^{\frac{1}{2}} \). In the limiting case \( \alpha \rightarrow 0, \beta_0 \rightarrow 0 \) the ground coherent state reduces to the fractal function \( y(t) \)

\[
\lim_{\alpha, \beta_0 \rightarrow 0} \left|\alpha\right\rangle = y(t) = at^b
\]

5. Conclusions

The results obtained permit formulation of two interesting rules governing the biological growth in the fractal space-time:

(i) each system of interacting cells within a growing system possesses its own, local intrasystemic fractal time, which differs from the linear \( b(t) = 1 \) scalar time of the extrasytemic observer;

(ii) fractal structure of space-time in which biological growth occurs, is lost during progression.

The possibility of mapping the Gompertz function, describing the biological growth onto the temporal fractal power law scaling function confirms a thesis that biological growth is a self-similar, alometric and coherent process of a holistic nature [9]. It means that all spatially separated subelements (cells) of the whole system, are interrelated via long-range (slowly decaying) interactions, which seem to be an essential ingredient of the self-organized systems. Such interactions can be mediated e.g. through diffusive substances (growth factors), which interact with specific receptors on the surface of the cells, affecting and controlling proliferation. It has been proved [9] that the Gompertz function represents the coherent state of the growth which is a macroscopic analog of the quantal minimum-uncertainty coherent state of the Morse oscillator. Such states are space-like (nonlocal) and propagate along the well-defined time trajectory being coherent in space. The mapping procedure transfers this peculiar property of the Gompertz function onto the fractal function \( y(t) \). Hence, the biological growth in the fractal space-time with temporal fractal dimension is predicted to be coherent in space.
References
Adaptive control of the singularly perturbed chaotic systems based on the scale time estimation by keeping chaotic property

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Abstract
In this paper, a new approach to the problem of stabilizing a chaotic system is presented. In this regard, stabilization is done by sustaining chaotic properties of the system. Sustaining the chaotic properties has been mentioned to be of importance in some areas such as biological systems. The problem of stabilizing a chaotic singularly perturbed system will be addressed and a solution will be proposed based on the OGY (Ott, Grebogi and Yorke) methodology. For the OGY control, Poincare section of the system is defined on its slow manifold. The multi-time scale property of the singularly perturbed system is exploited to control the Poincare map with the slow scale time. Slow scale time is adaptively estimated using a parameter estimation technique. Control with slow time scale circumvents the need to observe the states. With this strategy, the system remains chaotic and chaos identification is possible with online calculation of lyapunov exponents. Using this strategy on ecological system improves their importance in some other areas such as biological systems.

Keywords: OGY, lyapunov exponent, slow manifold, adaptive, singular perturbation, scale time

1. Introduction
Nonlinear singularly perturbed models are known by dependence of the system properties on the perturbation parameter [5]. Multi time scale characteristic is an important property of this class of nonlinear systems. For this class of systems a two-stage procedure for design composite controller is presented in [9]. On the other hand, chaotic behavior is an important characteristic of a class of nonlinear systems. Many researchers have shown interest in the analysis and control of the chaotic systems. Among the proposed approaches is the control of the Poincare Map (the OGY-Method) [8].

In this paper, the OGY method is applied to the singularly perturbed chaotic systems. The proposed control strategy exploits the chaotic property of the system and a discrete system model on the Poincare map is defined. This Poincare map lies on the slow manifold of the system. It is shown that by using the two time scale property of the system, an OGY control with slow time scale on the slow manifold of system, could be defined.

This strategy of control results in keeping chaotic property of the system and then online identification of chaos with calculation of lyapunov exponents is possible. As an adaptive parameter estimation technique is used to estimate perturbation parameter and the slow time scale of the singularly perturbed system. Population models are examples of systems where sustaining the dynamical property of controlled systems is important for survival of them. Chaotic model of food chains were initially found in [2,4]. Recently, chaotic impulsive differential equations are used in biological control [6,11-13]. Multi time scale approach was first used in [7] for food chain models. Method is implemented on a prey-predator type of population model.

The paper is organized as follows. In section 2 the slow-fast manifold separation based on the slow and fast states for singularly perturbed systems is introduced. In section 3 an adaptive estimation technique for the estimation of perturbation parameter is proposed. In section 4 chaotic property of the system is exploited and the OGY control is implemented for the stabilizing problem. Then singularly perturbed property is exploited and slow manifold of the system is selected as the Poincare section. Then a new control based on the slow scale time estimation is introduced. Section 5 presents the results of employing the proposed method on the ecological prey-predator system.

2. Problem Formulations
In this paper, chaotic singularly perturbed systems of the following form are considered,

\[ \dot{x} = f(x, y) \]
\[ \dot{y} = g(x, y) (t) \]
Where, \( x \in \mathbb{R} \), \( y \in \mathbb{R}^{n-1} \) and \( \varepsilon \) is a small parameter. \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( g : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) are both smooth functions and the system is chaotic.

The slow manifold of (1) is defined with

\[
\begin{align*}
0 &= f(x, y) \\
\dot{y} &= g(x, y) 
\end{align*}
\]

This \( S \) manifold \( S : \{ f = 0 \} \) is smooth and results in separation of time scales as \( x \) the fast, and \( y \) as the slow variable. It is easily seen that,

\[
\frac{\dot{y}}{\dot{x}} = \frac{g(x, y)}{ef(x, y)} \rightarrow \frac{\dot{y}}{\dot{x}} \propto \frac{1}{\varepsilon} \tag{3}
\]

By taking

\[
\tau = \frac{t}{\varepsilon} \tag{4}
\]

the second scale time of system is \( T = \frac{1}{\varepsilon} \).

3. Adaptive Estimation of \( \varepsilon \)

In [10] an estimation method for constant terms using the least-squares approach is proposed. Here the method is used here for \( \varepsilon \) estimation. The estimated \( \varepsilon \) is found to minimize the total prediction error as

\[
J = \int_0^T e^2(t) dt
\]

Where the prediction error \( e(t) \) is defined as

\[
e(t) = \hat{\varepsilon}g(x, y)\dot{x} - f(x, y)\dot{y}
\]

This total error minimization can average out the effects of measurement noise. The resulting estimation is [4]

\[
\hat{\varepsilon} = \frac{\int_0^T xg(x, y)\dot{y}f(x, y)dt}{\int_0^T x^2g(x, y)^2dt} \tag{5}
\]

To reduce the size of manipulations we defined window, then (5) changes to (6)

\[
\hat{\varepsilon} = \frac{\int_{-w}^t xg(x, y)\dot{y}f(x, y)dt}{\int_{-w}^t x^2g(x, y)^2dt} \tag{6}
\]

4. OGY Control Based On Second Time Scale Estimation

In this part a new control strategy is proposed such that controlled system remains chaotic. This strategy exploits OGY method to design control and then uses two time scale property of the system to improve the designed control such that system remains chaotic.
4.1 Fast Direction Properties

Consider the chaotic singularly perturbed system (1). As $\varepsilon$ is a small parameter, an approximation of the fastest eigenvalue of Jacobian matrix (7) is $\frac{1}{\varepsilon} \times \frac{\partial g}{\partial \varepsilon}$ . Since the chaotic systems are dissipative and the absolute value of the sum of negative exponents is bigger than the sum of the positive Lyapunov exponents, this big value is almost negative. It means that calculation of Lyapunov exponents in fast direction is not necessary in chaos identification. And for system with one perturbation term the fast direction is a stable direction.

$$
J = \begin{bmatrix}
\frac{1}{\varepsilon} \frac{\partial f}{\partial x} & \ldots & \ldots & \frac{1}{\varepsilon} \frac{\partial f}{\partial y_{n-1}} \\
\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y_1} & \ldots & \ldots & \frac{\partial g_1}{\partial y_{n-1}} \\
\vdots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial g_{n-1}}{\partial x} & \frac{\partial g_{n-1}}{\partial y_1} & \ldots & \ldots & \frac{\partial g_{n-1}}{\partial y_{n-1}}
\end{bmatrix}
$$

(7)

4.2 OGY Control On The Slow Manifold

In the OGY control design, a manifold is defined such that the discrete model of the system will be obtained by the intersections of this manifold with system trajectories. Then, the control of this discrete model on this manifold will result in the control of the system. It is obvious that the manifold approach will result in a more accurate control of system if it contains all unstable modes of the system. Stable modes lead the system dynamics toward a desired point. One of the modes should be eliminated to have a manifold with unit co-dimension. Eliminating the fastest stable mode and letting it to be free leads to a more accurate control (compared to the elimination of other modes).

Considering Jacobian matrix (7) Where $x_{eq}$ is the value on the fixed point of the system while the fixed point is calculated as:

$$
\begin{align*}
\dot{x}(x_{eq}, y_{eq}) &= 0 \\
g(x_{eq}, y_{eq}) &= 0
\end{align*}
$$

(8)

Then, the discrete model will be:

$$
y_{k+1} = p(y_k, u_k), x = x_{eq}
$$

(9)

The OGY control, $u_k$ proposed with the following strategy on slow manifold will be:

$$
u_k = \begin{cases} 
K(y_k) & \text{if } |y_k - y_{eq}| \leq \Delta \\
0, & \text{otherwise}
\end{cases}
$$

(10)

Where $\Delta$ is the dead zone in traditional OGY method and $K(y_k)$ is a control for slow states of discrete model (9) designed with a suitable method for example, proportional feedback.
4.3 New OGY Control Based On Slow Time Scale Estimation

In the OGY method, control of the Poincare map is equivalent to the control of the chaotic system (1). According to two time scale property of the system, to control this Poincare section on the slow manifold, it is sufficient to control it with the slow scale time, because the states on the slow manifold have slower motions than total dynamical system. Hence, control is designed by following strategy: Control starts with the OGY control and as soon as the first section with the Poincare map is detected, system could be controlled with the slow scale time.

Suppose that $T$ is the estimation of slow scale time of the system. And $k_0$ is the time of first section or first pulse, the system can be controlled by inserting control action (10) only in the following instants:

$$k = k_0 + nT, n = 0,1,2,\ldots$$

Where, according to (4) $T$ is

$$T = \left[ \frac{1}{\varepsilon} \right] \quad (11)$$

and $[\ldots]$ is a bracket symbol.

an approximation of the fastest mode will be $\frac{1}{\varepsilon} \times \frac{\partial g}{\partial z}$. Then the accurate manifold is defined by the equation;

$0 = f(x, y)$ which is the slow manifold of the system. In other expression with this strategy, the Poincare section in this problem lies on the slow manifold (2). For stabilizing problem in fixed point Poincare section becomes:

$$S = \left\{ y : x = x_{eq} \right\} \quad (12)$$

The main idea of this new control method is keeping the system on its chaotic state without resisting to be settled down in the desired rejoin.

The control strategy can be summarized as; when the chaotic system states enter the dead zone, by insertion of the control pulse, the states settle more in the neighborhood of the slow manifold. Afterward system works in open loop and remains by its dynamic in the slow manifold. This slow manifold contains all unstable modes that are also all slow. If unstable modes try to abduct trajectory from the desired point, it needs a time. Since smaller $\varepsilon$ result in bigger fast stable modes then the time that system remains on slow manifold increases too, an approximation of this time is slow time scale of the system.

During this time after the application of the control pulse, states of the system remain in the neighborhood of the desired trajectory. Then, after this time before the exit from the desired region, the loop is closed again and insertion of an enough effective control pulse returns the trajectories closer in the slow manifold. Then system becomes open loop again and so on.

**Result 1:** control of Poincare map and control of system (1) are equivalent. With $T$ period the system is controlled. Then all needed information to control the Poincare map exist at $k = k_0 + nT$. Then $T$ is the sufficient census time (sufficient period of observation) for system (1).

**Result 2:** By this method between the pulses system is open loop. Then system remains chaotic and online calculation of the Lyapunov exponents result in positive maximum lyapaunov exponent. By defining $\Gamma$ as

$$\Gamma = u(\lambda_{max}) = \begin{cases} 0, & \lambda_{max} \leq 0 \\ 1, & \lambda_{max} > 0 \end{cases} \quad (13)$$

When $\Gamma = 1$ the system identified as chaotic and control rule (10) could be inserted adaptively.
4.4 Algorithm

According to the above discussions a new algorithm to adaptive OGY control for the one term singularly perturbed systems is proposed as follows:

**Step 0:** By the slow manifold (12) construct the Poincare map (9) and design \( K(y_k) \) appropriately to control this discrete model.

**Step 1:** At the first time \( t \) that condition (10) is satisfied, insert the impulse control \( u_k \). \( \text{pulses} = 1 \)

**Step 2:** During \( t \) to \( t + w \) estimate \( \varepsilon \) using (5) to estimate the slow scale time \( (T) \) with (11).

\[
\text{census} = \text{census} + w
\]

**Step 3:** do no act till \( t = t + T \) If condition (10) is satisfied insert control \( u_k \).

\[
\text{pulses} = \text{pulses} + 1
\]

**Step 4:** back to 2.

5. Simulation Results

In this section, the planned algorithm of section 4 is implemented on the Rosenzweig–MacArthur model. The system is model of food chains of prey-predator type. Chaotic property of the system in some range of parameters is proved in [1,3]. This model includes three states: a prey \((x)\), a predator \((y)\) and a top-predator \((z)\), with the following equations:

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - x - \frac{y}{\beta_1 + x}) \\
\frac{dy}{dt} &= y(\frac{x}{\beta_2 + x} - \frac{\delta_1 - \frac{z}{\beta_2 + y}}{\beta_2 + y}) \\
\frac{dz}{dt} &= \xi(\frac{y}{\beta_2 + y} - \delta_2)
\end{align*}
\]

(14)

Where

\[
\beta_1 = 0.3, \beta_2 = 0.1, \delta_1 = 0.1, \delta_2 = 0.62, \xi = 0.3
\]

Problem of stabilizing equilibrium point of saddle type is addressed. The Poincare section is on the following slow manifold

\[
S = \{ (y, z) : x = x_{eq} \}
\]

Extinction of species is not desired. While, the equilibrium point with positive and nonzero terms are desired (of biological significance). Desired fixed point is \((0.8593,0.1632,0.1678)\).

To design OGY and new method of control, Poincare section is linearized, and proportional feedback is used to control it. For efficiency of the method, close loop poles selected enough faster than the fastest stable pole.

Figures (1) shows the result of stabilizing with OGY control and new method. It indicates that the stabilizing with new method converges to results of OGY method. New method has lower accuracy only in the early times. But the
numbers of inserted pulses decreased considerably in comparison to OGY method of control (approximately proportional to $\frac{1}{\epsilon}$).

Figures (2) shows the lyapunov exponents under new method. It indicate that maximum lyapunov exponent is positive and the condition $\Gamma = 1$ for insertion the control rule (10) is satisfied and positive lyapunov exponent are in slow directions.

Figure (3) shows slow variations of states in neighbourhoud of slow manifold and effect of control pulses on the states under new method. It indicates that in interval between the pulses, the states have slow variations.

![Figure (1) Comparison of the states errors by OGY control and new method (for $\epsilon = 0.01$).]
Figure (2) Lyapunov exponents of the controlled system by new method (for $\varepsilon = 0.01$).
This variations are such slow that dynamic of this open loop situation remains in the neighborhood of the desired trajectory. Each time insertion of the control pulses approaches systems more to the desired trajectory.

6. Conclusions
The simulation results on ecological model satisfying the efficiency of the new method. In proposed OGY control on slow manifold, instead of trying to drive the system trajectory to a stable rejoin, system is guided to a dynamical unstable slow manifold. Since that instability is slow, by applying the control pulses in proper times, states of system remains in neighborhood of the desired point. One of the advantages of this control strategy is that it removes necessity of observation of states for all samples. This is very important for situations that census has high expenditure (for example in biological populations) or for situation that dispatch of control action has higher expenditure (for example in pesticide). Also, maximum Lyapunov exponents remain positive and then it is useful for online adaptive identification of chaotic property of the system.

References


Analysis of Two Time Scale Property of Singularly Perturbed System on Chaotic Attractor

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Abstract
The idea that chaos could be a useful tool for analyze nonlinear systems considered in this paper and for the first time the two time scale property of singularly perturbed systems is analyzed on chaotic attractor. The general idea introduced here is that the chaotic systems have orderly strange attractors in phase space and this orderly of the chaotic systems in subscription with other classes of systems can be used in analyses. Here the singularly perturbed systems are subscripted with chaotic systems.

Two time scale property of system is addressed. Orderly of the chaotic attractor is used to analyze two time scale behavior in phase plane.

Keywords: chaos, singular perturbation, strange attractor, phase space

1. Introduction
Phase space analysis is common method in analysis of nonlinear systems [3]. Chaotic systems are class of nonlinear systems that are known by dependance of system dynamics on initial values. Since for first time in 1963 chaotic property introduced by Lornz, many researchers have shown interest in the analysis of them. On other hand nonlinear Singular perturbation models are known by dependence of the system properties on the perturbation parameter [3]. Multiple time scale characteristic is an important property of this class of systems. In this paper for the first time the two time scale property of singularly perturbed systems is analyzed on chaotic attractor. The general idea introduced here is that the chaotic systems have orderly strange attractors in phase space and this orderly of the chaotic systems in subscription with other classes of systems can be used in analyses. Here the singularly perturbed systems are subscripted with chaotic systems.

Two time scale property of system is addressed. Orderly of the chaotic attractor is used to analyze two time scale behavior in phase plane. Linearization method only gives the information around the point that system is linearized but phase space analysis gives all information about all points of the system. Mathematical models of ecological systems are examples of chaotic singularly perturbed systems that analysis done on them here.

The paper is organized as follows. In section 2 the two time scale property of the singularly perturbed systems is introduced. In section 3 linearization method introduced to analyze the time scale. Section 4 presents the results of employing the linearization method on the three ecological prey-predator systems.

In section 5 the two time scale behavior of singularly perturbed system on the chaotic attractor is analyzed. Section 6 contains the conclusion of this paper.

2. Two Time Scale Singularly Perturbed Systems
In this paper, chaotic singularly perturbed systems of the following form are considered,

\[ \dot{x} = f(x, y) \]
\[ \dot{y} = g(x, y) \]

Where, \( x \in \mathbb{R} \), \( y \in \mathbb{R}^{n-1} \) and \( \varepsilon \) is a small parameter. \( f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}^{n-1} \) are both smooth functions and the system is chaotic.

The slow manifold of (1) is defined with

\[ 0 = f(x, y) \]
\[ \dot{y} = g(x, y) \]

This \( S \) manifold \( S : \{ f = 0 \} \) is smooth and results in separation of time scales as \( x \) the fast, and \( y \) as the slow variable. It is easily seen that,

\[ \dot{\frac{y}{x}} = \frac{g(x, y)}{\mathcal{E}f(x, y)} \xrightarrow{\varepsilon \to 0} \frac{\dot{y}}{\dot{x}} \propto \frac{1}{\varepsilon} \]

By taking

\[ \tau = \frac{t}{\varepsilon} \]
as the slow time and \( t \) as the fast time, rescaling gives

\[
\frac{dx}{d\tau} = x' = f(x, y)
\]

\[
\frac{dy}{d\tau} = y' = g(x, y)
\]

The fast manifold yields:

\[
x' = f(x, y)
\]

\[
y' = 0
\]

3. Analysis of Two Time Scale Behavior with Linearization around Slow Manifold

In this section system (1) is linearized around its fixed point. Then slow manifold produced with (2). Then eigenvalues of jacobian matrix for full system and reduced system (slow manifold) used to analyze the speed of states. The equations

\[
\dot{x} = 0
\]

\[
\dot{y} = 0
\]

or equivalently the equations

\[
0 = f(x, y)
\]

\[
0 = g(x, y)
\]

give the fixed points \((x_{eq}, y_{eq})\) of the system (1). And according to (2) the slow manifold yields with

\[
S = \{ (y) : x = x_{eq} \}
\]

Linearization of full system around the fixed point result in following jacobian matrix

\[
J = \begin{bmatrix}
\frac{1}{\epsilon} \frac{\partial f}{\partial x} & \cdots & \frac{1}{\epsilon} \frac{\partial f}{\partial y_{n-1}} \\
\frac{\partial g_1}{\partial x} & \cdots & \frac{\partial g_1}{\partial y_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{n-1}}{\partial x} & \cdots & \frac{\partial g_{n-1}}{\partial y_{n-1}}
\end{bmatrix}
\]

(8)

and linearization of reduced system result in following jacobian matrix

\[
J = \begin{bmatrix}
\frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{n-1}}{\partial y_1} & \cdots & \frac{\partial g_{n-1}}{\partial y_{n-1}}
\end{bmatrix}
\]

(9)

It is obvious that eigenvalues with nonzero real parts of this matrixes (8),(9) show the speeds of states around the fixed pointes.
4. The Linearization Method on Three Ecological Models

Here the linearization method is implemented on three models of food chains of prey-predator type. The Rosenzweig–MacArthur, the Hastings–Powell, and the Volterra–Gause model are investigated here. All are singularly perturbed and the chaotic property of them in some range of parameters is proved in [1-2]. The models include three states: a prey ($x$), a predator ($y$) and a top-predator ($z$).

4.1. The Rosenzweig–Mac Arthur Model

\[ \varepsilon \frac{dx}{dt} = x(1 - x - \frac{y}{\beta_1 + x}) \]

\[ \frac{dy}{dt} = y(-\frac{x}{\beta_2 + x} - \delta_1 - \frac{z}{\beta_2 + y}) \] \hspace{1cm} (10)

\[ \frac{dz}{dt} = \xi z(-\frac{y}{\beta_2 + y} - \delta_2) \]

Where

$\beta_1 = 0.3, \beta_2 = 0.1, \delta_1 = 0.1, \delta_2 = 0.62, \xi = 0.3$

Fixed point $(0.8593, 0.1632, 0.1678)$ is on the slow manifold. Eigenvalues of Jacobian matrix around this point for full system and reduced system for $\varepsilon = 0.1$ are

\[ \lambda(\varepsilon = 0.1) = \begin{bmatrix} -7.516 \\ 0.182 + 0.112i \\ 0.182 - 0.112i \end{bmatrix} \]

\[ \lambda_{\text{reduced}}(\varepsilon = 0.1) = \begin{bmatrix} 0.199 + 0.0759i \\ 0.199 - 0.0759i \end{bmatrix} \]

For $\varepsilon = 0.01$ eigenvalues change to

\[ \lambda(\varepsilon = 0.01) = \begin{bmatrix} -75.467 \\ 0.181 + 0.112i \\ 0.181 - 0.112i \end{bmatrix} \]

\[ \lambda_{\text{reduced}}(\varepsilon = 0.01) = \begin{bmatrix} 0.199 + 0.076i \\ 0.199 - 0.076i \end{bmatrix} \]

4.2. The Volterra–Gause Model
\[ \varepsilon \frac{dx}{dt} = x(1 - x) - \sqrt{xy} \]
\[ \frac{dy}{dt} = -\delta_1 y + \sqrt{xy} - \sqrt{yz} \]  \hspace{1cm} (11)
\[ \frac{dz}{dt} = \xi z(\sqrt{y} - \delta_2)z \]

Where
\[ \delta_1 = 0.577, \delta_2 = 0.376, \xi = 1.428 \]

Fixed point \((0.8463235, 0.141376, 0.1289524)\) is on the slow manifold. Eigenvalues of jacobian matrix around this point for full system and reduced system for \(\varepsilon = 0.1\) are

\[
\lambda(\varepsilon = 0.1) = \begin{bmatrix}
-7.604 \\
0.040 + 0.302i \\
0.040 - 0.302i
\end{bmatrix}
\]

\[
\lambda_{\text{reduced}}(\varepsilon = 0.1) = \begin{bmatrix}
0.086 + 0.291i \\
0.086 - 0.291i
\end{bmatrix}
\]

For \(\varepsilon = 0.01\) eigenvalues change to

\[
\lambda(\varepsilon = 0.01) = \begin{bmatrix}
-76.857 \\
0.040 + 0.301i \\
0.040 - 0.301i
\end{bmatrix}
\]

\[
\lambda_{\text{reduced}}(\varepsilon = 0.01) = \begin{bmatrix}
0.086 + 0.291i \\
0.086 - 0.291i
\end{bmatrix}
\]

### 4.3. The Hastings–Powell Model

\[ \varepsilon \frac{dx}{dt} = x(1 - x) - \frac{a_1 xy}{1 + \beta_1 x} \]
\[ \frac{dy}{dt} = y\left(\frac{a_1 x}{1 + \beta_1 x} - \delta_1 \right) - \frac{a_2 yz}{1 + \beta_2 y} \]  \hspace{1cm} (12)
\[ \frac{dz}{dt} = \xi z \left(\frac{a_2 y}{1 + \beta_2 y} - \delta_2 \right) \]

Where
\[ a_1 = 5, a_2 = 0.1, \delta_1 = 0.4, \delta_2 = 0.01, \beta_1 = 3, \beta_2 = 2, \xi = 0.2 \]

Fixed point \((0.8192, 0.1259, 8.08)\) is on the slow manifold. Eigenvalues of Jacobian matrix around this point for full system and reduced system for \(\varepsilon = 0.1\) are
\[ \lambda(\varepsilon = 0.1) = \begin{bmatrix} -6.818 \\ 0.034 + 0.011i \\ 0.034 - 0.101i \end{bmatrix} \]

\[ \lambda_{\text{reduced}}(\varepsilon = 0.1) = \begin{bmatrix} 0.148 \\ 0.008 \end{bmatrix} \]

For \( \varepsilon = 0.01 \) eigenvalues change to

\[ \lambda(\varepsilon = 0.01) = \begin{bmatrix} -68.978 \\ 0.034 + 0.011i \\ 0.034 - 10.01i \end{bmatrix} \]

\[ \lambda_{\text{reduced}}(\varepsilon = 0.01) = \begin{bmatrix} 0.148 \\ 0.008 \end{bmatrix} \]

It is obvious that for three systems fast mode is in perturbed direction. Fast modes are stable and very big in comparison to other poles. With decrement of \( \varepsilon \) stable fast mode becomes faster and slow modes approximately remain unchanged. Then two time scale behavior in such systems means that with decrement of \( \varepsilon \) value fast states become faster and slow states speed is approximately unchanged. The eigenvalues of the reduced system (slow manifold) also remain unchanged with \( \varepsilon \) value changes.

5. Phase Space Analysis on Chaotic Attractor

Phase space analysis is common method in analysis of nonlinear systems. For nonlinear systems the phase portrait of a solution is a plot in phase space of the orbit evolution [4]. One of the most important properties of chaotic systems is that they have strange attractors; that has an apparent qualitative and bounded shape for each systems in range of parameters that system is chaotic and initial conditions that arisen from basin of attraction. We named here this property as orderly of the chaotic systems.

Strange attractor can be shown with plot of trajectories in phase portrait. Here the property of chaotic systems that "qualitative shape of system attractor is unchanged and bounded", or in other expression the orderly of the strange attractor of chaotic systems in phase portrait, is used to analyze the two time scale behavior of singularly perturbed chaotic systems in phase space.

According to (3) by \( \varepsilon \) variation, speeds of systems states meet different scale times proportional to \( \frac{1}{\varepsilon} \), theoretically. Figures (1) shows the strange attractor of three above ecological systems for two different \( \varepsilon \) values in phase space. According to figure (1) by changing the \( \varepsilon \) value the qualitative shape of attractor is approximately unchanged.

Figure (2) shows the two dimensional plot of attractors for fast states \((y,z)\). According to figure (2) the qualitative shape and quantitative domain of variation of attractor for slow states is approximately independent on variation of \( \varepsilon \) value and there is no sensible variation in slow states.

Figure (3) shows the two dimensional plot of the same attractors for the fast state \((x)\) and one of the slow states\((y)\). It shows that the speed of states increase in fast direction.
Figure (1) chaotic strange attractor of three food chain models (for $\varepsilon = 0.1$ in left and for $\varepsilon = 0.01$ in right).
Figure (2) 2-Dimensional perspective of chaotic attractor of three food chains models for slow states (for $\varepsilon = 0.1$ in left and for $\varepsilon = 0.01$ in right).
Figure (3) 2-Dimensional perspective of chaotic attractor of three food chains models for fast state $x$ (respect to one of the slow states $z$ (for $\varepsilon = 0.1$ in left and for $\varepsilon = 0.01$ in right).

According to these figures, the quantitative domain of variation of attractor in the direction of fast state increased by the decrement of $\varepsilon$ value, and for slow states is approximately no sensible variation. Then, analyze of the two scale time behavior of singularly perturbed systems on chaotic attractor shows that be $\varepsilon$ decrement the slow states speed is approximately unchanged but the speed of fast states increase. This result is for all trajectories of the system not only about the around the fixed point on slow manifold.

6. Conclusions
In linearization method the eigenvalues with nonzero real parts introduced to analyze the multi time scale property of system around the point that system linearized. Results of implementation of this method on three ecological models show that the eigenvalues of jacobian matrix in fast direction are very bigger than slow directions. To analyze the system behavior on all points the phase portrait method is used. Because the system is chaotic its strange attractor in phase portrait is bounded and has a regular qualitative shape. Using phase portrait method satisfied the results of linearization method but applicable for all points of the system on the strange attractor. Both method show that by decrement of $\varepsilon$ value the speed of fast state increases but the speed of slow states are approximately unchanged. The orderly of the chaotic system on strange attractor used to analyze the two scale time property of the singularly perturbed class of nonlinear systems. Using chaotic property in subscription with other classes of nonlinear systems may be extendable to analyze them.

References