# New Chaotic Crypto System <br> based on the Specific Generator and the Pickover's Attractor 

Adda ALI-PACHA, Naima HADJ-SAID and Mohamed Sadek ALI-PACHA<br>University of Sciences and the Technology of Oran USTO<br>Po Box 1505 El M'Naouer Oran 31000 ALGERIA<br>a.alipacha@gmail.com,nim_hadj@yahoo.fr, msadek.alipacha@gmail.com

Keywords: Stream Cipher, Chaos, Sensitivity to Initial Conditions, Pickover's Attractor, Ali-Pacha Generator.


#### Abstract

Several studies have shown that chaotic signals can become alterative of several usual cryptosystems; this is due to their random behavior and their high sensitivity to initial conditions. A chaotic signal is obtained from a deterministic system, it is possible to reconstruct algorithmically putting themselves in the same conditions as those who helped create it, and thus to recover the information. In this work, we will study the chaos through the Pickover's attractor and the possibility to use it in the stream cipher as a random stream generator to encrypt data to secure.


## 1 INTRODUCTION

There are many studies that have shown that chaotic systems can be an alternative in the future secure communications, because several of the cryptographic systems [1, 2, 3, 12] are based on the generation of pseudo-random sequences to hide messages. The chaotic attractors with its intrinsic characteristics such as high sensitivity to initial conditions and the randomness of the generated signals; became able to replace its pseudorandom generator in the stream cipher. With the restriction that the onset of chaos is only possible from an initial state (seed) and after a transitional regime, the attractor plays the role of generating random data if we can applying to it [7] a sampling then the quantification and finally the coding so that it meets the requirement of sensitivity to initial conditions.

In this chaotic cryptosystem we choose one of the temporal functions the dynamics characterizing the Pickover's attractor, thus, it gives us a chaotic flow (random) which will be added to the secure data.

## 2 CHAOS THEORY

The chaos is generally defined as a particular behavior of a nonlinear deterministic and dynamical system [4].

Mathematically, a dynamic system is defined from a set of variables that make up the state vector:

$$
\begin{equation*}
\mathbf{x}=\left\{x_{i} \in \mathrm{R}\right\}, \mathrm{i}=1 \ldots n \tag{1}
\end{equation*}
$$

Where $n$ represent the dimension of the vector.
These sets of variables are the property to completely characterize the instantaneous state of the dynamic system generic. Associating in the more a coordinate system, we obtain the state space that is also called the phase space $[5,6]$.

It is a space of two or three dimensions in which each coordinate is a state variable of the system considered. Conjunction with state space a dynamic system is also defined by an evolution law, generally this dynamic characterizes the evolution of the system state in time.

A dynamical system is a typical system that evolves over time or in continuous (continuous time) described by differential equations which, however, are discretized for the purpose of computing: They are simulated by a time step very small compared to the scale time of the study phenomenon. Either discretely or in discrete time, they are the iterated applications.

An iterated application is a reduced description (in terms of information) of the system dynamics:

- The state of the system is described by a sequence of state vectors $X_{n}$, obtained by a

Poincare section of the state space which belongs to the state vector $\mathrm{X}(\mathrm{t})$ : in practice, the vector $\mathrm{X}(\mathrm{t})$ is sampled at instants $\mathrm{t}_{\mathrm{n}}=\mathrm{n} \mathrm{T}$, which may for example be obtained by experimental measurements;

- The iterated application can then go from $X_{n}$ state to $X_{n+1}$ state, it can for example be constructed retrospectively from a large enough following vector $X_{n}$;

The interesting point is that the numerical simulation from the iterated application can bring back a resolution problem for a differential equation (non-linear) to problem significantly simpler, the equations with recurrences. In spite of reduction of information it requires on knowledge of the exact dynamics of the system, this simulation can nevertheless highlight a chaotic behavior, and the transition to chaos associated with it [4, 5, 6].

A dynamic system usually has one or more parameters called "control", which act on the characteristics of the transition function. Depending on the value of the control parameter, the same initial conditions lead to trajectories corresponding to qualitatively different dynamical regimes. Changing the control parameter may lead to a change in the nature of dynamical regimes developed in the system.

The notion of determinism comes from the fact that a system is completely characterized by its initial state and its dynamics. A necessary condition for the appearance of chaos is that the system is nonlinear. From an initial state $x_{0}$ (seed) and after a transitional regime, the trajectory of a dynamic system reaches a limited region of phase space. This asymptotic behavior obtained for $\mathrm{t}, \mathrm{k} \rightarrow \infty$ is one of the most important features [5] for any dynamic system.

## 3 PICKOVER'S ATTRACTOR

An example for an iterative application is the chaotic attractor of Pickover given by the system below.

$$
\left\{\begin{array}{c}
x_{k+1}=\sin \left(b \cdot y_{k}\right)+c \cdot \sin \left(b \cdot x_{k}\right)  \tag{2}\\
j \\
y_{k+1}=\sin \left(a \cdot x_{k}\right)+d \cdot \sin \left(a \cdot y_{k}\right)
\end{array}\right.
$$

For having chaotic behavior, Pickover chooses the values of the control parameters of the system as follows:

$$
\mathrm{a}=-0.96, \quad \mathrm{~b}=2.87, \quad \mathrm{c}=0.76, \quad \mathrm{~d}=0.74
$$

Figure 1 represents the Pickover's attractor for $\left(\mathrm{x}_{0}, \mathrm{y}_{0}\right)=(1,1)$


Fig. 1 : Pickover's Attractor with 50000 iterations
The evolution of $x(k)$ and $y(k)$ for this attractor and for the first 200 iterations are given by Figures 2 and 3 respectively.


Fig. 2 : Evolution of the signal $x(k)$


Fig.3: Evolution of the signal $y(k)$.
We note that the two signals $x(k)$ and $y(k)$ are evolving in a chaotic way according of $k$, this behavior is accompanied by a high sensitivity to initial conditions as shown in Figure 4 (two signals $x(k)$ and $x^{\prime}(k)$ are generated by two initial conditions of a difference of $10^{-10}$ ):


Fig.4: Sensitivity of the attractor Pickover to initial conditions

We note that a very small error on the knowledge of the initial state $\left(x_{0}, y_{0}\right)$ in the phase space will be rapidly amplified, and gives us two widely different signals. Quantitatively, the error growth is locally exponentially for strongly chaotic systems (sensitivity to initial conditions).

Note that the error on the initial conditions in this case is $10^{-15}$ and this is the smallest value for Matlab work with only 52 bits, but the system can be sensitive to values smaller than $10^{-15}$ according to the work environment.

## 4 CHAOTIC CRYPTO-SYSTEM PROPOSED:

The principle of operation of a chaotic cryptosystem proposed [7] is identical to the stream cipher. Encryption algorithms continuously convert the encrypted data one bit at a time [8]. This type of generator produces a stream of known length of numbers (streamkeys), logic zeros and logic ones: $\mathrm{K}_{1}, \mathrm{~K}_{2}, \mathrm{~K}_{3}, \ldots, \mathrm{~K}_{\mathrm{i}} \ldots$ with certain properties of chance, it is potentially difficult to identify the groups of numbers following a certain rule (group behavior). The output of such a generator is not completely random, but only they approached to the ideal properties of completely random sources. It is said random because this sequence is arbitrary. This stream is combined with exor function to the bit stream of the plaintext $m_{1}, m_{2}, m_{3}, \ldots, m_{i}$; to produce the bit stream of encrypted data.

$$
\begin{equation*}
\mathrm{C}_{\mathrm{i}}=\mathrm{m}_{\mathrm{i}} \oplus \mathrm{~K}_{\mathrm{i}} \tag{3}
\end{equation*}
$$

Sides of decipherment, the encrypted data bits are oxored, with a stream data identical to a process's cipher to retrieve bits of plaintext:

$$
\begin{equation*}
\mathrm{m}_{\mathrm{i}}=\mathrm{C}_{\mathrm{i}} \oplus \mathrm{~K}_{\mathrm{i}}=\left(\mathrm{m}_{\mathrm{i}} \oplus \mathrm{~K}_{\mathrm{i}}\right) \oplus \mathrm{K}_{\mathrm{i}}=\mathrm{m}_{\mathrm{i}} \tag{4}
\end{equation*}
$$

All the synchronous stream ciphers use the encryption keys (secret key) and generate the same stream data for encryption and decryption. This stream is generated independently of the message flow.

System security depends entirely on internal details of the pseudo-random number generator. In this chaotic cryptosystem, we choose a time function by the Law characterizing the dynamic attractor (Pickover in our case) so that it gives us a chaotic flow (random), which will be added [7] to the data secure.

It may be noted that the data we will elect are random, so ideal for a perfect encryption. Also there,
knowing the definition of chaos, chaotic flow that data has the following characteristics [7]:

1. Long period,
2. No rehearsals,
3. Local linear complexity,
4. Criteria for non-linear Boolean functions.

We will now use the chaotic generator made to encrypt a message; using the evolution of the signal $x(t)$ of the signal of the Pickover's attractor as the encrypting signal; as shown in Figure 5:


Fig. 5: Chaotic Crypto-system

### 4.1 Encryption Principle

Noted the M is the plaintext, is the message to encrypt: is a sequence of bits, a text file, a digitized voice recording or a digital video image anyway M , is nothing other than the binary information. This information is represented by the ASCII code for each character alphanumeric for text file and a pixel for images with BMP extension. In both forms we have a byte as the unit and which is represented by 8 bits. Consequently, the plaintext is a suite finished bytes, and each byte is in this form:

| $\mathrm{P}_{1}$ | $\mathrm{P}_{2}$ | $\mathrm{P}_{3}$ | $\mathrm{P}_{4}$ | $\mathrm{P}_{5}$ | $\mathrm{P}_{6}$ | $\mathrm{P}_{7}$ | $\mathrm{P}_{8}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

For a given integer k, we take the value of the byte $m_{k}$.

1) We permute the positions of the pixel $P_{i}$ or the character.

In mathematics, the notion of permutation expresses the idea of rearranging objects discernible. A permutation of $n$ distinct objects arranged in a certain order, corresponds to a change in the order of succession of $n$ objects. A permutation of $n$ elements (where n is a natural number) is a bijection from a finite set of cardinality n on itself.

Let $X$ be a finite set of $n$ elements. Even perform numbering, swap elements of X is equivalent to permute only the integers from 1 to $n$. The traditional notation of permutations is to place the
elements that will be swapped in the natural order on a first line, and the images mapped on a second line.

For example:

$$
\sigma=\left(\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8  \tag{5}\\
3 & 4 & 5 & 6 & 7 & 8 & 1 & 2
\end{array}\right)
$$

The $\sigma$ application is defined by: $\sigma(1)=3 ; \sigma(2)=4$; $\sigma(3)=5 ; \sigma(4)=6 ; \sigma(5)=7 ; \sigma(6)=8 ; \sigma(7)=1 ; \sigma(8)=2$.

After the permutation, we compute the pixel value permuted, it is noted $m p_{k}$.
2) In this step and for given initial conditions ( $x_{0}, y_{0}$, $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ ) for the attractor of Pickover. We Take the actual value of the sample $x_{k}$.

The reproduction in floating point describes a system for representing real numbers which supports a wide range of values. Generally, numbers are approximately represented by a fixed number of significant digits on a scale determined by an exponent. We use the IEEE 754 [9] for floatingpoint arithmetic. Both formats defined by the IEEE 754 are the 32 -bit for a single precision and the 64bit for a double precision. The single-precision floating number is stored in a 32 bit word: 1 sign bit, 8 bits for the exponent, and 23 for the mantissa. The exponent is shifted $2^{8}-1-1=127$ in this case. The exponent of a normalized number is therefore -126 to 127 . A normalized floating-point number has a value $v$ given by the formula [9] as follows:

$$
\begin{equation*}
v=s \times 2^{e} \times m \tag{6}
\end{equation*}
$$

- $s= \pm 1$ represents the sign (depending on the sign bit);
- $\quad e$ is the exponent before his shift to 127 ;
- $m=1+$ mantissa: represents the significant part (in binary), with $1 \leq m<2$ (mantissa is the decimal part of the significant part of between 0 and 1.).

3) In our case and for encrypting the actual value of the sample $\mathrm{x}_{\mathrm{k}}$, first, it is multiplied by the value of the amplifier AM of the Ali-Pacha's generator [7]:

$$
\begin{equation*}
\mathrm{x}_{\mathrm{k}}=\mathrm{AM}^{*} \mathrm{x}_{\mathrm{k}} \quad \mathrm{AM}>1 \tag{7}
\end{equation*}
$$

4) We choose the decimal part of this value, ie, one that takes its mantissa of 23 bits, and converts it to a number in base 10 , it is noted as:

$$
\begin{equation*}
\overline{x_{k}}=\left\lfloor x_{k}\right\rfloor \tag{8}
\end{equation*}
$$

Then we calculate:

$$
\begin{equation*}
y_{k}=\bmod \left(\widetilde{x_{k}}, 256\right) \tag{9}
\end{equation*}
$$

$y_{k}$ is also represented by 8 bits.
5) In what follows we calculate as follows $\mathrm{cp}_{\mathrm{k}}$ :

$$
\begin{equation*}
c p_{k}=\bmod \left(m p_{k}+y_{k}, 256\right) \tag{10}
\end{equation*}
$$

6) Apply the inverse permutation $\sigma^{-1}$ to $\mathrm{cp}_{\mathrm{k}}$.

Let $n$ distinct elements in a certain order. Apply a permutation $\sigma$ returns to change the order. Return to the original order is also done with a permutation; thereof is denoted $\sigma^{-1}$. More generally, this application $\sigma^{-1}$ is the inverse bijection of $\sigma$. if we applied $\sigma$ and then $\sigma^{-1}$, or $\sigma^{-1}$ and then $\sigma$, it is equivalent to applying the identity permutation. The permutation $\sigma^{-1}$ is called the reciprocal permutation or the inverse permutation of $\sigma$.

$$
\sigma^{-1}=\left(\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8  \tag{11}\\
7 & 8 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}\right)
$$

In other words, we compute the ciphertext associated the $m_{k}$ pixel as follows:

$$
\begin{equation*}
c_{k}=\sigma^{-1}\left(\mathrm{cp}_{\mathrm{k}}\right) \tag{12}
\end{equation*}
$$

### 4.2 Fields of Key Encryption

In this case, the choice of the encryption key [7] must be following these fields:

1. Choice of $a, b, c$, and $d$ The initial state $x_{0}$ and $y_{0}$, The value of the amplifier AM, The permutation $\sigma$.
2. The start time: $k$, where it begins the encryption process.

The increment (step) sampling [7] is equal to 1 for digital systems.
If we take 32 bits for $a, b, c$ and $d$ and $x_{0}, y_{0}, A M$, and 10 bits for, k , and 24 bits for the permutation $\sigma$. The length of the key is:

$$
(32 * 6)+32+10+24=258 \text { bits. }
$$

Practically, we chose the encryption key in our cryptosystem as follows:

- Making a choice for one on four control parameter (parameter selection $(00 \rightarrow \mathrm{a}, 01 \rightarrow \mathrm{~b}$, $10 \rightarrow \mathrm{c}, 11 \rightarrow \mathrm{~d}$ ), plus the value of this parameter. In other words, 2 bits for the selecting parameter and 32 bits for this value,
- We take the value of AM in the $(1<\mathrm{AM}<2)$; and it will be encoded on 23 bits.
- We keep the other parameters unchanged, we have the key length as follows :

$$
2+(32 * 3)+23+10+24=155 \text { bits. }
$$

## 5 INTERPRETATIONS RESULTS

If we take as a key data encryption :

- $\sigma=[7,8,1,2,3,4,5,6]$,
- Choice of Parameter : 00 et $a=-0.959$
- $\mathrm{x}_{0}=1.000005, \mathrm{y}_{0}=1$
- $\mathrm{AM}=1.60719$
- The time where encryption begins $\mathrm{k}=97$.

1) We have secured the Lena's image and the Emir Abdelkader's image in our cryptosystem. Histograms of plaintext images and encrypted images, showing that the proposed cryptosystem works correctly.


Fig. 6: Plaintext image and encrypted image of l'Emir Abdelkader
2) If we take the Plain Text in French as follows: «La cryptologie a connu une rapide évolution à notre époque surtout en ce qui concerne ces deux facteurs». This meant "Cryptology experienced a rapid evolution in our time especially in regard to these two factors."

Its cryptogram is: « ~m4hcwfm`\} etgq(a!k|`|! !ufp`jcepol/9dcx ■dyv\{.*|abjj1àcadlu0ya ■gdfy4)!qe2np (qt`4n|gruk\{k3jw|8rkgr3jrifk`k| ».
The cipher text is incomprehensible.

## 6 VALIDATION OF THE CRYPTOGRAPHIC SYSTEM

We work with the images clear and encrypted of Lena.

### 6.1 Histogram of Images: Information Entropy

For a monochrome image, that is to say with a single component, the histogram is defined as a discrete function that maps to each value intensity, the number of pixels of this value. The determination of the histogram is carried out by counting the number of pixel intensity for each of the image. The histogram can then be seen as probability density.

The entropy was founded by Shannon in 1948 [10, 11] and is given in the following equation:

$$
\begin{equation*}
H(m)=\sum_{i=0}^{2^{N}-1} P_{m_{i}} \cdot \log _{2}\left(\frac{1}{P_{m_{i}}}\right) \tag{12}
\end{equation*}
$$

Where $P\left(m_{i}\right)$ represents the probability of symbol $m_{i}$. The entropy $H(m)$ is expressed in bits. The entropy of the plaintext of Lena (figure 7) is equal to 7.4455 .


Fig. 7 : Image of Lena and its Histogram


Fig. 8 : Encrypted Image of Lena and its histogram
From figure 8: the encrypted image has a uniform histogram, which means that the gray levels have the same number of occurrences and hence the entropy is the maximum. Therefore, a gray scale image, where each pixel is represented by 8 bits, must have entropy for the encrypted image, the closest possible 8 bits/pixel. The encrypted Lena image is equal to $7.9987 \approx 8 \mathrm{bits} /$ pixel. The obtained value is very close to the theoretical one.

Referring to the results, we can clearly see that the plaintext image $(\mathrm{H}(\mathrm{m})=7.4455)$ differs significantly from her corresponding encrypted. Moreover, the histogram of the encrypted image (H $(\mathrm{m})=7.9987$ ) is quite uniform which makes it difficult the statistical extraction of pixels of the plaintext image.

Histograms are resistant to a number of transformations on the image [12]. They are invariant to translations and rotations, as well as to a lesser extent to changes in perspective and changes of scale.

### 6.2 Correlation of the Adjacent Pixels

In probability and in statistics, to study the correlation between two random variables or numerical statistics is to study the strength of the
bond that can exist between these variables. The searched link is an affine relationship, it is the linear regression. For example, we calculate the correlation coefficient between two sets of the same length (typical case: a regression). Assume we have the following table of values: $\mathrm{X}\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}}\right)$ and $\mathrm{Y}\left(\mathrm{y}_{1}, \ldots\right.$, $y_{n}$ ) of each of the two series. A measure of this correlation is obtained by calculating the linear correlation coefficient of Bravais-Pearson [12]. For the correlation coefficient linking these two sets, we apply the following formula:

$$
\begin{equation*}
\operatorname{Coef}(X, Y)=\frac{\operatorname{cov}(X, Y)}{\sqrt{D(X)} \cdot \sqrt{D Y}} \tag{13}
\end{equation*}
$$

Covariance between x and y is given as follows:

$$
\begin{equation*}
\operatorname{cov}(X, Y)=\frac{1}{N} \sum_{i=1}^{N}\left(\left(X_{i}-E(X)\right) \cdot\left(Y_{i}-E(Y)\right)\right) \tag{11}
\end{equation*}
$$

The average of X is :

$$
\begin{equation*}
E(X)=\frac{1}{N} \sum_{i=1}^{N} X_{i} \tag{15a}
\end{equation*}
$$

The average of Y is :

$$
\begin{equation*}
E(Y)=\frac{1}{N} \sum_{i=1}^{N} Y_{i} \tag{15b}
\end{equation*}
$$

The standard deviation of X is :

$$
\begin{equation*}
D(X)=\frac{1}{N} \sum_{i=1}^{N}\left(X_{i}-E(X)\right)^{2} \tag{16a}
\end{equation*}
$$

The standard deviation of Y is :

$$
\begin{equation*}
D(Y)=\frac{1}{N} \sum_{i=1}^{N}\left(Y_{i}-E(Y)\right)^{2} \tag{16b}
\end{equation*}
$$

The correlation coefficient is between -1 and 1 . Intermediate values provide information on the degree of linear dependence between two variables. Closer the coefficient is close to extreme values -1 and 1 , the closer the correlation between variables is strong we simply use the term "highly correlated" to describe the two variables. A correlation equal to 0 means that the variables are not correlated.

To test the correlation coefficient, we selected randomly $\mathbf{1 5 0 0}$ pairs of two adjacent pixels in both encrypted and clear picture.

Both Figures 8 and 9 show the correlation between two horizontally adjacent pixels of the image clear and encrypted. We see that the neighboring pixels in the image have a clear correlation (coeff $=\mathbf{0 . 9 5 2 4 7}$ ), while in the encrypted will have one little correlation (coeff $=\mathbf{0 . 0 0 3 7}$ ). This low correlation between two neighboring pixels in the encrypted image makes the attack of our cryptosystem difficult.


Fig. 9 : Correlation between the horizontally adjacent pixels of the plaintext image


Fig. 10 : Correlation between the horizontally adjacent pixels of the encrypted image
Also, it is clear that in the image clear, several lines can be adjusted to scatter but among all these lines can be retained which has a remarkable property giving rise to the right of the form $(y=a * x+b)$ representing a linear correlation.

### 6.3 Confusion and Diffusion

The proposed cryptosystem satisfies both concepts (Confusion and Diffusion) that have been identified by Claude Shannon in his paper Communication Theory of Secrecy Systems published in 1949.

- Confusion: Introduced in step 3 of the encryption (eq. 8) consisting in making a modular addition or exclusive.
- Permutation: introduit dans l'étape 1 (eq. 5) et à l'étape 4 (éq. 9) dans le processus de chiffrement, et consistent à faire la permutation $\sigma$ alors la permutation $\sigma^{-1}$ respectivement.


## 7 CONCLUSION

The theory of nonlinear dynamical systems is far from being the panacea that some researchers imagined in its infancy. It nonetheless its interesting concepts can be applied to problems targeted by the

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey
use of mathematical methods carefully chosen and adapted to the systems under study. Chaos theory teaches us the contrary it does not lie in the opposite of the order, but also contains its own order as long as he is allowed to occur. A field as complex as the cryptology, can benefit from the addition of such methods of investigation if their forces, and especially their limitations are understood.

In this work we have implemented in Matlab 6.0.1 a chaotic cryptosystem based on the Ali-Pacha generator and on the Pickover's attractor in order to drown the data to encrypt. Tests done on the proposed algorithm shows good behavior of the algorithm as regards to the aspect of confidentiality.

## REFERENCES

[1] N.K. Pareek, Vinod Patidar, K.K. Sud, "Image encryption using chaotic logistic map", Image and Vision Computing 24, pp. 926-934. 2006.
[2] S. Behnia, A. Akshani, S. Ahadpour, H. Mahmodi, A. Akhavan. "A fast chaotic encryption scheme based on piecewise nonlinear chaotic maps". Physics Letters A (2007) 391-396. 2007
[3] S. BEHNIA, A. AKHSHANI, H. MAHMODI and A. AKHAVAN, A novel algorithm for image encryption based on mixture of chaotic maps, Chaos, Solitons \& Fractals, vol. 35, n ${ }^{\circ} .2$, pp. 408-419, 2008.
[4] R. Devaney, "An Introduction to Chaotic Dynamical Systems", 2nd ed. Redwood City, CA: AddisonWesley, 1989.
[5] R. Devaney, "A First Course in Chaotic Dynamical Systems", Now published by Westview Press, 1992.
[6] L. KOCAREV, Chaos-based cryptography: a brief overview, IEEE Circuits and Systems Magazine, vol. 1, $\mathrm{n}^{\circ} .3$, pp. 6-21, 2001.
[7] A. Ali-Pacha, N. Hadj-Said, A. M'Hamed and A.Belgoraf "Lorenz's attractor applied to the stream cipher (Ali-Pacha generator)", Chaos, Solitons \& Fractals, Volume 33, Issue 5, pp 1762-1766. 2007.
[8] B. Schneier," Applied Cryptography-Protocols, Algorithms and Source Code in C", John Wiley \& Sounds, Inc, New York, Second Edition, 1996.
[9] David Goldberg (March 1991),"What Every Computer Scientist Should Know About Floating-Point Arithmetic", ACM Computing Surveys 23 (1): 5-48, doi: 10.1145/ 103162.103163. Retrieved 28 April 2008.
[10] C. E. SHANNON, A mathematical theory of communication, Bell System Technical Journal, vol. 27, nº. 7, pp. 379-423, July 1948.
[11] C. E. SHANNON, A mathematical theory of communication, Bell System Technical Journal, vol. 27, $\mathrm{n}^{\circ} .10$, pp. 623-656, October 1948.
[12] Guanrong Chen, Yaobin Mao, Charles K. Chui, "A symmetric image encryption scheme based on 3D chaotic cat maps", doi:10.1016/j.chaos.2003.12.022, Chaos, Solitons and Fractals 21, pp: 749-761, 2004.

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

# Synchronized Attractors and Phase Entrained Chaos 

M. Abdul ${ }^{1, *}$ and F. Saif ${ }^{1}$<br>${ }^{1}$ Department of Electronics Quaid-i-Azam University, 45320 Islamabad, Pakistan ${ }^{\dagger}$


#### Abstract

We find that the coupled logistic equations, symmetrical in nature, produce identical, synchronized chaotic attractors that are orthogonal to each other. Chaotic attractors are in pairs for certain range of controlling parameters. As we change coupling strength within a certain range, periodicity, quasiperiodicity and chaoticity of chaotic attractors appear in a sequence. Beyond the range of controlling parameters and critical coupling strength synchronization breaks, indicating onset of spatiotemporal chaos, that displays symmetry breaking. Quantitative measures of transition from synchronization chaos are provided by mean of period bifurcation.


PACS numbers: 05.45.Gg, 05.45.Pq, 42.60.Jf

Synchronization phenomena in coupled dynamical systems have been studied especially in laser, electronics circuits, heart and mind, chemical and biological system, secure communication. Complete synchronization mean coincidence of states and chaotic attractors in phase space of interacting systems, $x(n)=y(n)[1,4]$; it display only if interacting systems are identical, if the parameters of coupled oscillators slightly mismatch, the state are close $\|x(n)-y(n)\| \approx 0$ but remain different [2]. The phase synchronization and entrained have explained in $[3,5,6]$ also found in our dynamical system, whereas their (attractors) amplitudes remain chaotic and correlated [3]. In our dynamical coupled oscillators we have observed that, increasing coupling strength which is common between them, first the transition from non-synchronous state to phase synchronization state but periodicity remain invariant beyond certain value of coupling. For larger values of parameters including coupling strength new regimes which we can say lag synchronization is found, cause of crises. It is a requirement of self-consistency that a quasistationary field should be maintained by the induced polarization that leads to the equations which determine the amplitudes and frequencies of multimode oscillation as a functions of various parameters characterizing the field growth. The study of real spatially extended systems that are accurately described by a finite set of coupled ordinary differential equations provides insight into the nature of spatiotemporal chaos $[7,8]$.

In this paper we consider spontaneous emergence of synchronized chaotic attractors in a spatially distributed two-mode nonlinear system. In general in the absence of coupling, each behaves like a single mode. We find that there is a certain range of coupling strength for which synchronized chaos exists. Beyond this range of coupling strength synchronization breaks down, and the system enters a regime of turbulence chaos [7, 9]. Hence, we report that in two-mode or coupled identical dynamical system synchronization oscillates between quasi-periodic, intermittency state, period bifurcation and chaoticity, by

[^0]fixing controlling parameters and changing the coupling strength. Here, we also discuss period fusing and emerging because of crises where strange attractors fluctuate corresponding to the change of parameters. If we have coupled logistic equations [10], i.e.,
\[

$$
\begin{align*}
& x_{n+1}=x_{n}+2 \lambda_{1} x_{n}\left(1-x_{n}\right)-\gamma x_{n} y_{n}=f_{\lambda_{1}, \gamma}^{(1)}\left(x_{n}, y_{n}\right) \text {, } \\
& y_{n+1}=y_{n}+2 \lambda_{2} y_{n}\left(1-y_{n}\right)-\gamma x_{n} y_{n}=f_{\lambda_{2}, \gamma}^{(2)}\left(x_{n}, y_{n}\right) \text {, } \tag{1}
\end{align*}
$$
\]

where, $\lambda_{i}$ and $\gamma=2 \xi \lambda_{i},(i=1,2)$ are the characteristic parameters, therefore, the system is controlled by mean of three parameters. A mapping of bilinear and linear coupling terms have been shown to exhibit complicated dynamical behavior including quasi periodicity, phase locking, intermittency, period adding, long-lived chaotic transitions and then periodicity [11, 12]. Following we explain necessary terms and their symbols used in our later discussion.

Fixed points: A fixed point $x$ is a point in the space defined by function $f$, so that $f_{n}(x)=x, \forall n$.

Periodic motion: A periodic motion ( P ) of a system is defined as $f_{n}(x)=f_{n+1}(x)$.

Quasi-periodic motion (QP) of a dynamical system is defined as $f: R \rightarrow R^{n}$, dynamical function can be represented in the form $f=H\left(\omega_{1}, \ldots \ldots, \omega_{n}\right)$, where $H$ is periodic with period $2 \pi$ in each argument, and the real numbers $\left(\omega_{1}, \ldots, \omega_{n}\right)$ describe the finite set of base frequencies [13, 14].

Phase Locking: The process of phase locking exists whenever the chaotic actions of the individual subsystems shift to the ordered actions of the collective system [16, 17]. Sometimes phase entrainment is called phase locking or synchronization.

Chaotic motion: In the chaotic system, there occurs sensitive dependence to initial conditions, i. e, chaotic trajectories locally diverge away from each other and small changes in starting conditions build up exponentially fast into large deviations in the evolution [18].

There occurs a fascinating behavior of the coupled equations (1) for various values of $\lambda_{i}$ and $\gamma$. The key to understand the structure of these equations in $x y$-space is a careful analysis of fixed points of the mapping functions as well as their iterates. Since the function $f^{(1)}$ and
$f^{(2)}$ are symmetrical, we expect symmetrical behavior in $x$ and $y$. The two fixed points [19], corresponding to Eq. (1) are,

$$
\begin{equation*}
x_{1}^{(1)}=0, \quad \text { and } \quad x_{2}^{(1)}=\frac{2 \gamma-2 \lambda_{1}}{\left(\gamma^{2} / \lambda_{2}-4 \lambda_{1}\right)} \tag{2}
\end{equation*}
$$

These are stable if they follow the condition,

$$
\begin{equation*}
\left\|\left(\frac{d x_{n+1}}{d x_{n}}-\frac{d x_{n+1}}{d y_{n}}\right)\right\|<1 \tag{3}
\end{equation*}
$$

In case $\lambda_{1}=\lambda_{2}$ these fixed points become,

$$
\begin{equation*}
x_{1}^{(1)}=0, \quad \text { and } \quad x_{2}^{(1)}=\frac{1}{\frac{\gamma}{2 \lambda}-1}, \tag{4}
\end{equation*}
$$

which overlap for $\lambda_{i}=0$. By fixing $\lambda$ and varying $\gamma$ we have found evidence for a boundary crises in our system, like or similar to what was found in Henon's maps by Grebogi, Ott and Yorke [12, 20]. A boundary crises occurs in our case through the collision of a chaotic attractor with the basin boundaries that separate it from the several other coexistent periodic attractors, in addition with another chaotic attractor [21, 22]. An increase of $\gamma$ beyond its critical value for the onset of crises, results in disappearance of the chaotic attractors and its basin while the basins of remaining attractors undergo a sudden expansion.


FIG. 1: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.19$ and $\gamma=0.660$. We note that the system displays a 4 P to 2 P behavior through period bifurcation by means of crises. In the plot on right hand side, we present $y$ axis of the phase space as a function of number of iterations $n$.

We report chaotic behavior in our system corresponding to various values of the parameters. We explain the behavior and classify our discussion in two cases: In the first case we fix $\lambda_{i}$ and vary $\gamma$ over a range $0.001 \leq \gamma \leq 0.999$ for $x_{0}=0.1$ and $y_{0}=0.11[23,24]$. At $\lambda_{i}=0.25$ the trajectory in phase space $(x, y)$ converges to a fixed point. The asymptotic character of the solution is typical 1 P . For $\lambda_{i}=1.0$, system shows oscillatory behavior between QP and 2 P character upto $\gamma=0.999$. There is still 2 P character for $\lambda_{i}=1.190$ and in the interval $\gamma \in[0.001,0.21]$. At the upper range of this interval, it show QP 2 Torus, 4P and 2P respectively at $\gamma=0.22,0.222,0.6851$. Periodic bifurcation


FIG. 2: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.25$, and (a) $\gamma=0.1845$, (b) $\gamma=0.2$. Isolated points in (a) and (b) are part of transient evolution. The plots on right hand side shows behavior of our system, which is going from 4 P state to chaos through QP in which each attractor shows nine wing pattern as in Fig. 2(a) and fluctuates up to certain values of coupling strength. The attractors expand as a function of increasing coupling strength, $\gamma$, as shown in Fig. 2(b), corresponding to frequency locking.
phenomena at $\gamma=0.660$ is also observed, as shown in the Fig. 1. There is no chaotic behaviors seen but our coupled system oscillates between 2 P to 4 P through quasi-periodicity. It is clear that if coupling strength is increased periodicity fuses and comes out of the crises, finally we get period 2 at $\gamma=0.999$.

For $\lambda_{i}=1.25$ and $\gamma=0.001$ the four chaotic attractors grow as coupling strength increases. They are synchronized (mirror image of each other) as well as orthogonal, and the trajectory in phase space converges to 4 P . When $\gamma=0.1750$ transition from 4 P to 8 P takes place, which shows quasi periodicity in which each attractor displays curious nine wing pattern before settling to the asymptotic 8 P state. They also fluctuate and expand at $\gamma=0.179$ which generates mirror image of laser attractors. At $\gamma=0.1845$, transient solution again show curious nine wing pattern before going to periodic regime. At $\gamma=0.2$ the lasing system shows chaotic behavior through QP and laser attractors in phase space have exact mirror image of each other, as shown in Fig. $2(a, b)$. In laser chaos there exists inner crises, a boundary crises where a strange attractor collides with unstable fixed points on the boundary of basin of attraction, causing disappearance of both [20? ]. But in our case when they collide unstable fixed points disappear and shape of the strange attractor remains invariant that becomes unstable as parameters increases. However, at $\gamma=0.269$ each period (line) split into eight to four to two and then


FIG. 3: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.25$, and (a) $\gamma=0.2690$, (b) $\gamma=0.4450$. Our dynamical system is going from unstable state to stable state through QP and trajectory of basin boundary remains invariant after crises. The plots on the right hand side show that (a) there occurs 32 P state and (b) QP state after crises.
finally fused in 4P and chaotic attractors remain mirror images to each other, as shown in the Fig. 3(a,b). It has been observed that our coupled nonlinear system got synchronized to each other during these intervals, such that $\gamma \in[0.26,0.27]$ and $[0.29,0.3][11,25]$. Chaotic attractors are mirror images of each other in the interval, i. e, $[0.319,0.445]$. While the trajectory changes at $\gamma=0.384$ and increases above this value as we get different form of trajectory, i. e, chaotic attractors are not mirror images [25]. The coalesce $(8 C \rightarrow 4 C \rightarrow 2 C)$ chaos to chaos is analogous to the band emerging in a logistic map [11]. Therefore, periodicity of coupled lasing system oscillates ( 4 P to 32 P , each line of period 4 bifurcate then multi-bifurcation behavior and then converges to 4 P at $\gamma=0.999) 4 \mathrm{P}$ to 8 P to 16 to 32 P through QP and chaotic and then comes at 4P at last value of $\gamma$, and attractors rotate clockwise direction.

For $\lambda_{i}=1.26$ and $\gamma \in[0.001,0.15]$ it has same behavior as for $\lambda_{i}=1.25$ for all value of $\gamma$. From $\gamma=0.152$ to 0.2271 , the trajectory converges to a periodic solution with the period equal to 4 but strange attractors are not mirror image of each other. However for $\gamma=0.2275$ to 0.2279 system remains chaotic and then converges to exactly 20 iteration ( 20 P and 20 attractors) at $\gamma=0.2285$. Chaoticity and periodicity (20P) oscillate in the interval [0.2285,0.2395] and some other complicated phenomena also happened [? ]. At $\gamma=0.2398$, there exists transient state in which phase space trajectory and periodicity change because of crises as shown in Fig. 4(b). When $\gamma$ is increased in the interval [0.295, 0.445 ] having 20 P , chaotic behavior through inter-


FIG. 4: Phase space ( $x_{n}, y_{n}$ ) is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.26$, and (a) $\gamma=0.09$, (b) $\gamma=0.2398$. Isolated points in (a) and (b) are part of transient evolution. The plots on the right hand side show behavior of our system that is going from stable state to stability through chaotic and QP states. Here trajectory and periodicity of our system dramatically change because of crises. Here exterior crises is observed between two trajectories.


FIG. 5: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1} \lambda_{2}=1.26$ and $\gamma=0.4490$. Isolated points are part of transient evolution, intermittency state corresponds to strange attractors are synchronized as well as mirror images of each other. The plot on the right side in phase space shows intermittency state in $(y, t)$ plane.
mittency which occurs at $\gamma=0.449$, as shown in Fig. 5 and QP at $\gamma=0.510$ and then converges to 4 P at all above values of $\gamma=0.575$. Therefore, for this fixed value of controlling parameters and changing coupling strength our system remains chaotic in between 4P.

For $\lambda_{i}=1.27$ and $\gamma \in[0.001,0.011]$ the trajectory in the phase space converges to four strange attractors corresponding to it has asymptotic character of solution is 4 P as shown in the Fig. 6. At the upper range of this interval our system shows chaotic behavior through nodal period, QP and 20P. Lasing system shows chaotic behavior for small interval of time and comes back to


FIG. 6: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.27$, and (a) $\gamma=0.045$, (b) $\gamma=0.6099$. As we mentioned earlier, isolated points in (a) and (b) are part of transient evolution. The plots on the right side of (a) shows stability and (b) shows intermittency state of the system. Therefore, our dynamical system is going from stable state to unstable state through QP because of crises.
its initial state when the coupling strength reaches its maximum value for small values of parameters.




FIG. 7: Phase space $\left(x_{n}, y_{n}\right)$ is plotted by solving Eq. (1) for $\lambda_{1}=\lambda_{2}=1.28$, and (a) $\gamma=0.0321$; (b) $\gamma=0.0445$. The plots on the right hand side show our system is going from unstable state to stable state through chaotic and QP states.

For $\lambda_{i}=1.28$ and in this interval $\gamma \in[0.001,0.017]$ the trajectory converges to the 8 QP or torus corresponding to periodicity it has as 8 P asymptotic character of so-
lution. Here, we observe QP (from 8P to 16P) and then $(16 \mathrm{P}$ to 8 P$)$ in the interval $[0.018,0.0321]$ and then chaotic behavior at $\gamma=0.0445$ through intermittency state at $\gamma=0.0429$ [26]. From Fig. 7(a), there are 16 chaotic attractors which are not mirror images at $\gamma=0.0321$ and correspond to intermittence state and show QP after crises, i.e, 16 C coalesce into 4 T at $\gamma=0.0445$, which are mirror image as well as synchronized [12, 22]. Moreover, initially chaos come through QP and intermittency state when system transit from 8 P to 16 P and after that our system shows chaoticity, but when it returns to its period 16 from chaoticity (infinite periods), no QP and intermittency state is observed up to four decimal places and then periods 8. It is also seen that periodicity, intermittency and chaoticity oscillate in the interval [0.365,0.37]. We observe a transitions from 8 P to 4 P because of crises at $\gamma=0.6777$ and then come back to it initial state of 8 P at final value of $\gamma=0.9999$ through QP and chaotic states. At the upper range of this interval $\lambda_{i} \in[1.29,1.32]$ our system transits permanently to chaotic regime for all values of coupling strength.

Thus, we can say that there exists such type of transition in which we observe "cycle $\rightarrow$ (doubling).... $\rightarrow$ longer cycle $\rightarrow$ Hopf bifurcation $\rightarrow$ torus $\rightarrow$ various frequency locking $\rightarrow$ chaos $\rightarrow$ evolution of chaos $\rightarrow$ chaos (fusion) and then hyper-chaos in our system [11]. If one analyzes the two dimensional plots $\left(x_{n}, y_{n}\right)$ as Poincare surfaces of section for the continuous system, the sequence can be described as: The 2 P corresponds to a stable limit cycle. As the $\gamma$ increasing further, the limit cycle become unstable and bifurcates into a four-loop limit cycle and then evolve into a eight-loop torus through a Hopf bifurcation. The torus represents quasi-periodic behavior of our system and is responsible for the four invariant orbit on the Poincare surfaces of section. The four intermittency periodic behavior is obtained when the four characteristic frequencies on the torus are in ratio of two small integers [24]. Higher bifurcation of the torus occurs as the system moves out of quasi-periodic region, by increasing $\gamma$ (Ruelle-Takens-Newhouse scenario) [24].

When $\lambda_{1}$ and $\lambda_{2}$ are not equal to each other then irregular behaviors displayed by means of period doubling bifurcation [24]. But in our dynamical system we have in region for which, $\lambda_{1}=1.5=\lambda_{2}$ start at same values for $\gamma=0.051$. If $\gamma$ is fixed and $\lambda_{i}$ is varied from 1.5 to 4 , we have detected period doubling bifurcation which leads to 64 P solutions and then to chaos. Thus, from the above two cases we conclude that when, $\lambda_{1}=\lambda_{2}$ and variation exists in $\gamma$, chaos emerges through quasi-periodicity, however, when $\lambda_{1} \neq \lambda_{2}$, chaos emerges by mean of period doubling bifurcation sequence.

## I. CONCLUSION

In this paper, we here provided detailed study of transition from stability to chaos and torus to chaos in two-dimensional mapping. It is seen that transition


FIG. 8: Bifurcation diagram for the coupled lasers Eq. (1) with $\lambda_{1}=\lambda_{2}=1.5$ range from 2.5 to 2.9 and $\gamma=0.051$. For each value of $\lambda_{i}$ we used the final point of the previous $\lambda_{i}$ value and 1600 iterates are plotted. This shows the period-doubling sequence as well as QP and chaotic regions.
from periodicity (stability) and quasi-periodicity (torus) to chaos occurs with frequency locking. Through our numerical calculations for two-mode ring lasers, we have concluded following points: (i) Torus appears by way of Hopf bifurcation; (ii) Shape of strange attractors changes as controlling parameter changes or torus is distorted as $\gamma$ change. At certain values they expand and after that reduce in size; (iii) Chaos appears through a perioddoubling bifurcation of some frequency-locked cycle at
some value of the bifurcation parameter; (iv) Our dynamical system oscillate from 2 P to 4 P and then 4 P to 2 P through quasi-periodicity and intermittency state, at $\lambda \in[0.25,1.23]$ and for all values of $\gamma \in[0.001,0.999]$; (v) Above this value of $\lambda_{i}$ our system shows random behaviors through QP and some other complicated states. At some place we get periodicity and then chaoticity and then again periodicity not through QP but direct change of the state; (vi) Synchronization destroyed and reinforced due to crises, corresponding to change of the coupling parameter.

There are two pairs of laser attractors in phase space, which are totally different from each other at certain values of parameters, whereas synchronized as well as mirror images of each other at other values of characteristic parameters. For $\lambda_{i}=1.26$ and $\gamma \in[0.001,0.15]$ they display approximately same behavior as mentioned above, the difference is that it remains chaotic at $\gamma=0.2279$ and obtain asymptotic solution of 20P at $\gamma=0.2285$. Therefore, our dynamical system shows periodicity, quasiperiodicity, intermittency state to chaotic, to periodic state and then show permanently chaotic behavior at $\lambda_{i}=1.3$ and all higher values, and for all positive values of $\gamma \in[0.001,0.999]$. In coupled laser logistic equations, periodicity changes during frequency locking because of interior crises.
[1] H. Fujisaka, and T Yamada, Prog. Theor. Phy. 69, 32 (1983).
[2] A. S. Pikovsky, Z, Phy. B 55, 149. (1984).
[3] M. Abdul, and Farhan Saif, To be publish , (2012).
[4] L. M Pecora, and T. L Carroll, Phy. Rev. Lett. 64, 821 (1990); M. de Sousa Vieira, A. J. Lichtenberg, and M. A. Lieberman, Int. J. Bifurcation Chaos Appl. Sci. Eng. 1, 691. (1991).
[5] M. Rosenblum A. Pikovsky, and J. Kurths, Phy. Rev. Lett. 76, 1804 (1996).
[6] M. Rosenblum A. Pikovsky, and J. Kurths, Europhy. Lett. 34, 165 (1996).
[7] Shuguang Guan, G. W. Wei, and C. H Lai, Phys. Rev. E. 69, 066214 (2004).
[8] Herbert G. Winful and Lutfur Rahman, Phys. Rev. Lett. 65, 13 (1990).
[9] Sudeshna Sinha, Phy. Rev. E 66, 016209 (2008).
10] M. Abdul, and Farhan. Saif, AMIS 6, 73-82 (2011).
[11] Kunihko. Kaneko , Progress of Theoretical Physics 69, 1427 (1983).
[12] G. Tanaka, M. A. F. Sanjun, and K. Aihara, Phys. Rev. E. 71, 016219 (2005).
[13] Wiggins. S, Global Bifurcations and Chaos: Analytical Methods. (New York: Springer-Verlag, New York, 1988).
[14] Parker. T. S and L. O. Chua, Practical Numerical Algorithms for Chaotic System. (New York: Springer-Verlag, New York, 1989).
[15] Falconer K. J, The Geometry of Fractal Sets. (Cam-
bridge University Press, Cambridge, 1985)
[16] Garfinkel A, Self-organizing systems(Plenum Press, New York, 1987).
[17] Briggs. J, and Peat. F. D, The turbulent mirror (Harper and Row Press, New York, 1989).
[18] Rossler. O. E. The chaotic hierarchy. In A Chaotic Hierarchy. (, Singapore, 1991).
[19] R. Blumel, and W. P. Reinhardt, Chaos in Atomic Physics (Cambridge University Press, Cambridge, 1997).
[20] C. Grebogi, E. Ott and J. A. Yorke, Phys. Rev. Lett. 48, 1507 (1982).
[21] Xingyuan. Wang and Chao Luo, Inter. Jour. of Mode. Phys. B. 22, 427 (2008).
[22] V. Astakhov, A. Shabunin, T. Kapitaniak and V. Anishchenko, Phys. Rev. L. 79, 6 (1997).
[23] Yan. Gu, Mingwhei. Tung, Jian-Min. Yuan, Da Hsuan. Feng and Lorenzo. M. Narducci, Phys. Rev. Lett. 52, 9 (1984).
[24] J. M. Yuan, M. Whei. Tung, D. H. Feng, and L. M. Narducci, Phys. Rev. A. 28, 1662 (1983).
[25] Arne Jakobsen, Physica. D 237, 3382 (2008).
[26] Zhanybai T. Zhusubaliyev, and Erik Mosekilde, Phys. Lett. A. 373, 946 (2011).
[27] M. M. Tehrani and L. Mandel, Optical Communication 16, 16 (1976).
[28] C. Grebogi, E. Ott and J. A. Yorke, Phys. Rev. Lett. 50, 935 (1983).

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

# Sensing through dynamic Brillouin gratings sustained by chaotic lasers 

M. Santagiustina ${ }^{1}$ and L. Ursini ${ }^{1}$<br>Department of Information Engineering, University of Padova, 35131 Padova, Italy (E-mail: marco.santagiustina@unipd.it)


#### Abstract

A method, based on the thumbtack correlation properties of chaotic laser signal, is presented to induce permanently sustained, localized dynamic Brillouin gratings in polarization maintaining optical fibers. A numerical analysis of two possible experimental setups, all-optical and electro-optical, is performed. The possibility to apply the permanent grating in sensing is explored. Keywords: Chaos, Nonlinear Optics, Sensing.


## 1 Introduction

Dynamic Brillouin gratings (DBG) in polarization maintaining fibers (PMFs) [1] are a powerful technique to realize fiber sensing [2-5] and to obtain unconventional signal processing [6] or delay lines [7].

In stimulated Brillouin scattering (SBS) the interaction through electrostriction of two counter-propagating optical waves at frequency $\nu_{w 1}$ and $\nu_{w 2}=$ $\nu_{w 1}-\nu_{B}$, where $\nu_{B}$ is the Brillouin shift, generates an acoustic wave that longitudinally modulates the fiber refractive index, thus creating the grating (DBG writing process). The DBG decays on a time scale of several ns and moves at the sound speed; so, over its short lifetime, it can be considered static. Another light beam injected into the fiber can be backscattered by the DBG (reading process). In PMFs, the DBG writing and reading processes are decoupled by launching write and read signals on orthogonal states of polarization aligned to the fiber birefringence axes [1]. In fact, the acoustic wave equally scatters all light polarizations owing to its longitudinal nature.

The possibility to localize the DBG at a desired position is a particularly interesting feature for sensing applications. This has been achieved through pulse collision, frequency modulation or finally stress induced changes of the SBS frequency shift $[4,5]$. However, all those techniques present drawbacks. Pulsegenerated DBGs require high peak powers (of the order of few hundreds Watts) and the DBG is refreshed periodically, so its amplitude oscillates with time. Frequency modulation techniques at practical frequencies actually lead to create multiple DBGs.

More recently two techniques have been introduced to localize the DBGs at predefined positions within the fiber. They exploit the auto-correlation properties of chaotic [9] or pseudo-random [10] signals.

The application of pseudo-random modulation in sensing has been pointed out in [11]. Here, the method based on chaotic laser (CL) emision is first studied in detail by comparing two possible experimental setups, and then the characteristics of sensing with CLs are assessed.


Fig. 1. All-optical (AO) and electro-optical (EO) setup. LD: laser diode; CL: chaotic laser; MOD: modulator; A: optical amplifier; PG: pulse generator; PD: photodiode; PBF: passband filter; PBC: polarization beam combiner; FBG: fiber Bragg grating; PMF: polarization maintaining fiber.

The localization is based on the the peculiar features of CLs [12], so far applied mainly in optical cryptography [13-17], in particular, on the aperiodic, ultrawideband, thumbtack autocorrelated, time evolution of CLs emission, previously exploited in RADAR, LIDAR and OTDR systems [18-20].

## 2 Theoretical model

Two possible setups are proposed and numerically investigated; their simplified diagrams are sketched in Fig. 1. The first option is an all-optical (AO) scheme: a semiconductor laser (CL) at frequency $\nu_{w 1}$ is induced to chaotic emission [13,16]; part of the waveform is launched into the slow axis of a PMF and part is modulated and filtered to obtain a replica of the chaotic waveform, downshifted at frequency $\nu_{w 2}$. The second option is an electro-optical (EO) scheme: the photodetected CL emission drives two optical modulators, that modulate the phase of a laser signal at frequency $\nu_{w 1}$ and a sideband at frequency $\nu_{w 2}$. The biases of the modulators are adjusted such that the applied phase shift is zero at the mean value of the driving electrical signal. Phase modulators are used because the scheme proposed exploits the constructive-destructive convolution that is operated by the SBS interaction, as it will be shown below.

The aim, in both schemes, is to generate very similar (ideally identical) counter-propagating optical waves, which do not repeat in time to avoid multiple DBG formation. The AO scheme offers the advantage of requiring a reduced equipment set however the EO scheme yields better performance as shown below.

The CL is routed to chaotic emission by an external optical feedback and this situation is theoretically described by the Lang-Kobayashi (LK) dynamical equations [12,13,16]:

$$
\begin{align*}
& \frac{d E}{d t}=(1-j \alpha)\left[G(t)-\frac{1}{\tau_{p}}\right] \frac{E(t)}{2}+\gamma E(t-\Delta t) \exp \left(j \omega_{0} \Delta t\right)+\sqrt{R} F(t)  \tag{1}\\
& \frac{d N}{d t}=\frac{I}{e}-\frac{N(t)}{\tau_{c}}-G(t)|E(t)|^{2} \tag{2}
\end{align*}
$$

where $E(t)$ is the intracavity complex envelope of the laser electric field, normalized such that $|E|^{2}$ is the photon number, $N(t)$ is the carrier number and $G(t)=g\left[N(t)-N_{0}\right] /\left[1+\epsilon|E(t)|^{2}\right.$. Moreover in Eqs. 1 and 2: $\alpha$ is the linewidth enhancement factor, $G(t)$ is the saturated gain, $g$ is the differential gain, $N_{0}$ is the carrier number at transparency, $\epsilon$ is the gain suppression coefficient, $\tau_{p}$ is the photon lifetime, $\gamma$ is the coupling rate of the optical feedback, $\omega_{0}$ is the optical carrier angular frequency, $\Delta t$ is the external cavity delay, $I$ is the pump current, and $e$ is the electron charge, $\tau_{c}$ is the carrier lifetime. The spontaneous emission noise is included with the Langevin noise term $F(t)$, i.e. a complex white noise of zero mean and correlation function $<F(t) F^{*}\left(t^{\prime}\right)>=\delta\left(t-t^{\prime}\right)$, where angle brackets represent ensemble averages, $\delta(t)$ is the Dirac delta function, and the star denotes complex conjugation. The spontaneous emission rate is $R=g N(t)$.

The LD is also modeled by the LK equations without the optical feedback. The simulations are performed by numerically integrating LK equations through a Runge-Kutta method [16]. The values of the CL parameters are defined as in [21], except for the external-cavity delay fixed at $\Delta t=330 \mathrm{ps}$.

The process of DBG creation and readout in a PMF is governed by the following set of equations [6]:

$$
\begin{align*}
\partial_{z} A_{w 1}+\beta_{1 s} \partial_{t} A_{w 1} & =-\eta g_{B} Q A_{w 2},  \tag{3}\\
-\partial_{z} A_{w 2}+\beta_{1 s} \partial_{t} A_{w 2} & =\eta g_{B} Q^{*} A_{w 1},  \tag{4}\\
\partial_{z} A_{\text {out }}+\beta_{1 f} \partial_{t} A_{\text {out }} & =-\eta g_{B} Q A_{r},  \tag{5}\\
-\partial_{z} A_{r}+\beta_{1 f} \partial_{t} A_{r} & =\eta g_{B} Q^{*} A_{\text {out }},  \tag{6}\\
2 \tau_{B} \partial_{t} Q+Q & =A_{w 1} A_{w 2}^{*}+A_{\text {out }} A_{r}^{*}, \tag{7}
\end{align*}
$$

where $A_{w 1}, A_{w 2}, A_{r}, A_{o u t}$ are the slowly varying envelopes of the optical waves, whose carrier frequencies must satisfy SBS matching conditions in PMFs, i.e. $\nu_{w 1}=\nu_{w 2}+\nu_{B}, \nu_{r}=\left(1-\Delta n / n_{s}\right) \nu_{w 1}$ and $\nu_{o u t}=\nu_{r}+\nu_{B}$, where $\Delta n$ is the effective index difference between the slow and fast axes due to the fiber birefringence [1]. The optical waves, as well as the normalized acoustic wave, $Q$, are functions of time $t$ and space $z$. The parameters used in simulations are: fiber length $L=1 m, \lambda_{w 1}=1550 \mathrm{~nm}$, fiber birefringence $\Delta n=5 \cdot 10^{-4}$, SBS frequency shift $\nu_{B}=10.93 \mathrm{GHz}$, SBS gain coefficient $g_{B}=5 \cdot 10^{-11} \mathrm{~m} / \mathrm{W}$, acoustic wave lifetime $\tau_{B}=5 \mathrm{~ns}$. The coefficient $\eta=2 \cdot 10^{-3} \Omega^{-1}$ is an amplitude normalization factor, while $\beta_{1 i}(i=s, f)$ are the group delays per unit length for the slow and fast axis. The mean power of input writing chaotic waveforms is set to 100 mW . Eqs. 3-7 are integrated through a split-step method.


Fig. 2. Left: DBG peak amplitude as a function of time for the AO setup (blue curve) and the EO setup (black curve). The dashed, red curves are the theoretical value $\bar{Q}_{0}(t)$. Right: DBG time-space contour plot, for the AO (right) and EO (left) scheme. The DBG amplitude is normalized to its maximum value.

## 3 Results and analysis

The generation of DBGs can be studied by setting $A_{r}=A_{\text {out }}=0$. In the ideal condition (identical counter-propagating signals) there exists a unique position within the fiber $\left(z_{0}\right)$ at which the condition $A_{w 1}\left(z_{0}, t\right)=A_{w 2}\left(z_{0}, t\right)=E(t-$ $\left.\beta_{1 s} z_{0}\right)$ is satified at all times, i.e. there exists a unique position where waveforms are perfectly correlated. The acoustic wave at $Q\left(z_{0}\right)=Q_{0}$ is therefore governed by eq. 7 which then reads:

$$
\begin{equation*}
\partial_{\tau} Q_{0}(\tau)=-Q_{0}(\tau)+C(\tau) \tag{8}
\end{equation*}
$$

where $\tau=t /\left(2 \tau_{B}\right)$ and $C(\tau)=A_{w 1}(\tau) A_{w 2}^{*}(\tau)$ represents the correlation of the two input waveforms. In the hypothesis that $A_{w 1}\left(z_{0}, \tau\right)=A_{w 2}\left(z_{0}, \tau\right)=$ $E\left(t-\beta_{1 s} z_{0}\right), C(\tau)=\left|A_{w 1}\right|^{2}=|E|^{2}$. Eq. 8 has no analytical solution because $C(\tau)$ is a chaotic waveform, however by writing $C(\tau)=C_{0}+\Delta C(\tau)$, where $C_{0}=$ $\sqrt{P_{w 1} P_{w 2}} / \eta A_{e f f}\left(P_{w i}, i=1,2\right.$ are the powers of the pump waves and $A_{e f f}$ the fiber effective area) is the mean of $C(\tau)$ and $\Delta C(\tau)$ the chaotic fluctuations around the mean value, and by averaging eq. 8 , the acoustic wave mean value can be estimated: $\bar{Q}_{0}(\tau)=C_{0}[1-\exp (-\tau)]$. Therefore, after the transient regime, a permanent DBG is sustained at the location $z_{0}$ within the fiber.

In Fig. 2 (left), the analytical solution is compared with the numerical solutions of Eqs. 3-7: the agreement is excellent. In the AO scheme fluctuations around the mean value can be observed; they stem from the small differences in the writing waves, due to the modulation and filtering processes realized to generate the waveform $A_{w 2}$. At the time $t=60 n s$ (indicated by arrows in Fig. 2 left) a short, high power reading pulse is backscattered from the DBGs (see below); the DBG is very weakly depleted and the asymptotic value $C_{0}$ is rapidly recovered.

The DBG is very well localized in space, as shown in Fig. 2 (right), where a time-space contour diagram is shown. Sidelobes appear for the AO scheme at a time interval of about $350 \mathrm{ps} \simeq \Delta t$. Their presence is due to the fact that the chaotic spectrum presents a peak near the relaxation frequency of the external cavity $(\simeq 1 / \Delta t)$ [13]. In the EO scheme this detrimental effect is
highly reduced, because CL amplitude fluctuations are transformed into phase modulations and the term $A_{w 1}(z, \tau) A_{w 2}^{*}\left(z_{0}, \tau\right)$, very sensitive to the waveform phases, can also lead to DBG distruction for $z \neq z_{0}$.

Eventually, if a short, reading pulse $A_{r}$ is launched on the fast axis, the backscattered waveform $A_{\text {out }}$ retains the features of the DBG, thus enabling sensing. In Fig. $3,\left|A_{r}(z=0, t)\right|^{2}$ (red curve) is a Gaussian pulse (FWHM 120 ps , peak power 100 W$) ;\left|A_{\text {out }}(z=0, t)\right|^{2}$ for the EO (AO) scheme is represented by the black (blue) curve (all curves are normalized to the peak values). In the AO scheme the DBGs sidelobes appear also in the backscattered pulse; the peak to sidelobe ratio is about $7.5 d B$. For the EO scheme the ratio increases to $15 d B$. Better optical signal to noise ratios (OSNRs) could be obtained by modifying the chaos spectrum so to reach performance similar to pseudo-random orthogonal sequences [22].


Fig. 3. Pulse backscattered by the DBG for the AO setup (blue curve) and the EO setup (black curve). The red curve refers to the input pulse.

The reflectivity of the DBG is estimated to be $\rho=\left|A_{\text {out }} / A_{r}\right|^{2}=\left(g_{B} C_{0} L_{\text {eff }}\right)^{2}$, [9] where $L_{\text {eff }}$ is the DGB effective length (approximated by half the real length). In this case we had $-57 d B$ for the EO scheme and $-60 d B$ for the AO . The large decrease in $\rho$, with respect to what is achieved with continuous waves, has to be traded for the strong localization of the DBG. However, $\rho$ increases with $C_{0}$, i.e., with the product of input powers; so the reflectivity gains 6 dB per each doubling of the injected powers. Finally, the DBG reflectivity can be increased by about 40 dB in nonsilica fibers [23] and waveguides [24], because $g_{B}$ can be up to 2 orders of magnitude larger than in silica. The tuning of the DBG position can be achieved by inserting a delay line [15].

## 4 Sensing through DBGs

Optical fibers present several interesting properties with regard to sensing: they are small, chemically inert and immune to electromagnetic interference; moreover they present a low-cost and low losses. Because of these features optical fiber can be deployed over large areas, and in harsh environments. Sensing is often based on DBGs [2-5] since SBS is sensitive to temperature and stress.

The main properties of a temperature or stress sensor based on a chaotic waveform generated DBG are addressed below: resolution, OSNR, dynamic range, temperature and stress sensitivity.

The measurement resolution is probably the most valuable property of the chaotically generated DBGs. It physically corresponds to the effective length of the DBG that is fixed by the chaos bandwidth $B_{\text {chaos }}$ which determines the width of the correlation peak $L_{e f f}=\left(2 \beta_{1 s} B_{\text {chaos }}\right)^{-1}$. For the EO setup of Fig. 3, in which sidelobes do not appear, the resolution is $L_{\text {eff }} \simeq 0.8 \mathrm{~cm}$ since $B_{\text {chaos }} \simeq 12 G H z$, comparable to that achieved with pseudo-random signals [11]. Using very broadband optical chaos 32.5 GHz [25], the resolution could be further reduced to 3 mm , a value comparable to frequency domain backscattering techniques.

The main source of noise governing the OSNR in this technique stems from the fact that though the DBG has zero mean intensity, outside the correlation peak, nonetheless the variance of DBG intensity is not zero [11,22]. So for each realization of the reflection the achieved OSNR is given by the ratio between the signal backscattered by the DBG at the correlation peak and that backscattered by the rest of the fiber. From the results of Fig. 3, in the EO setup, the OSNR is $15 d B$, however this value can be made larger in specific applications. For example, if a localized temperature increase must be detected, like in [11], one has to take into account that if the DBG is placed at the sensing position (hot spot) and so the remaining section of the fiber gives a much weaker contribution, due to the SBS gain coefficient frequency dependence. In fact Eqs. 3 to 7 and the results of Fig. 3 refer to the case in which the pump powers frequencies yield the maximum gain over the entire fiber. Therefore, the OSNR can be increased in detecting localized temperature gradients. By assuming the typical Lorentzian SBS gain shape, with a shift of $1.36 \mathrm{MHz} / \mathrm{C}^{\circ}$, a gain FWHM bandwidth of $35 M H z$ at $20 C^{\circ}$ and a bandwidth reduction of $0.1 \mathrm{MHz} / \mathrm{C}^{\circ}$ [26], and since the power reflectivity is directly proportional to $g_{B}^{2}$, the OSNR increase (with respect to the reference temperature of $20 C^{\circ}$ ) is calculated as $g_{B}(T)^{2} / g_{B}\left(20 C^{o}\right)^{2} /$ and it is reported in Fig. 4 (left). The same OSNR increase is calculated for stress detection, using a shift of $594 \mathrm{MHz} / \%$ (and no bandwidth change), in Fig. 4 (right). The dynamic range is dictated


Fig. 4. Enhancement of the OSNR as a function of the hot spot temperature (left) of of the local strain applied (right)
by the DBG reflectivity and by the OSNR. With $\rho \simeq-57 d B$ the fiber length should be limited to a few meters, in order to detect a reflected pulse. However, by increasing the pump powers to 500 mW , the peak reflectivity is about $-45 d B$ which would enable to increase the fiber lenght to several km. However, the OSNR decrase with the fiber length; for example if the acceptable OSNR
is about $10 d B$ for measuring temperatures larger than $80 C^{\circ}$ the OSNR margin of 30 dB is reacghed at a length of about 1 km similarly to [11].

As for the temperature and stress sensitivity these are determined by the change of the reflectivity due to the variations of $g_{B}$. Using the results of Fig. 4 the temperature sensitivity, almost constant, is about $0.4 d B / C^{o}$ while for the strain it ranges from $200 d B / \%$, for small strain values, down to $25 d B / \%$ for large elongation values.

## 5 Conclusions

In conclusion, a method to induce one stable and localized dynamic Brilluoin grating in a polarization maintaining fiber, exploiting optical chaos, is introduced. Two experimental setups were compared: all-optical and electro-optical. The analytical predictions well compare with all the simulations of the full interaction model, based on Brillouin equations. The grating is permanently sustained, stable and well localized in space. The EO scheme provides superior performance with respect to the AO setup. In particular, correlation sidelobes and DBG fluctuations are greatly reduced. The grating reflectivity depend on the applied temperature and stress then enabling sensing applications.

The research was funded by: the University of Padova (Signal processing and sensing based on dynamic Brillouin gratings in optical fibers - CPDA119030), MAE (Directorate General for the Promotion of the National Economy), Fondazione Cassa di Risparmio (SMILAND project) and in the framework of the agreement with ISCTI, Rome.

## References

1.K. Y. Song, W. Zou, Z. He, and K. Hotate. All-optical dynamic grating generation based on Brillouin scattering in polarization-maintaining fiber. Optics Letters, 33: 926-928, 2008.
2.Y. Dong, X. Bao, and L. Chen. Distributed temperature sensing based on birefringence effect on transient Brillouin grating in a polarization-maintaining photonic crystal fiber. Optics Letters, 34: 2590-2592, 2009.
3.W. Zou, Z. He, K.-Y. Song, and K. Hotate. Correlation-based distributed measurement of a dynamic grating spectrum generated in stimulated Brillouin scattering in a polarization-maintaining optical fiber, Optics Letters, 34: 1126-1128, 2009.
4.K. Y. Song, W. Zou, Z. He, and K. Hotate. Optical time-domain measurement of Brillouin dynamic grating spectrum in a polarization-maintaining fiber. Optics Letters, 34: 1381-1383, 2009.
5.K. Y. Song, S. Chin, N. Primerov, and L. Thevenaz. Time-domain distributed fiber sensor with 1 cm spatial resolution based on Brillouin dynamic grating. Journal of Lightwave Technology, 28: 2062-2067, 2010.
6.M. Santagiustina, S. Chin, N. Primerov, L. Ursini, L. Thévenaz. All-optical signal processing using dynamic Brillouin gratings. Submitted to Nature Reports, 2012.
7.L. Thévenaz, S. Chin. Photonic tunable delay lines in optical fibers. Laser \& Photonics Reviews, 6: 728-734, 2012.
8.W. Zou, Z. He, and K. Hotate. Brillouin-based random fiber optic delay line.Proc. Conf. Optical Fiber Communication (OFC), (IEEE/OSA, 2009), JWA6.
9.M. Santagiustina, L. Ursini. Dynamic Brillouin gratings permanently sustained by chaotic lasers,. Optics Letters, 37: 893-895, 2012.
10.Y. Antman, N. Primerov, J. Sancho, L. Thévenaz, A. Zadok. Localized and stationary dynamic gratings via stimulated Brillouin scattering with phase modulated pumps. Optics Express, 20: 7807-7821, 2012.
11.A. Zadok, Y. Antman, N. Primerov, A. Denisov, J. Sancho and L. Thévenaz. Random-access distributed fiber sensing. Laser $\mathcal{G}$ Photonics Reviews 6: L1L5, 2012.
12.P. Colet and R. Roy. Digital communication with synchronized chaotic lasers, Optics Letters, 19: 2056-2058, 1994.
13.C. R. Mirasso, P. Colet, and P. Garcia-Fernandez. Synchronization of chaotic semiconductor lasers: application to encoded communications. IEEE Photonics Technology Letters, 8: 299-301, 1996.
14.A. Argyris, D. Syvridis, L. Larger, V. Annovazzi-Lodi, P. Colet, I. Fischer, J. García-Ojalvo, C.R. Mirasso, L. Pesquera, and K.A. Shore. Chaos-based communications at high bit rates using commercial fibre-optic links. Nature, 438: 343-346, 2005.
15.V. Annovazzi-Lodi, C. Antonelli, G. Aromataris, M. Benedetti, M. Guglielmucci, A. Mecozzi, S. Merlo, M. Santagiustina, and L. Ursini. Chaos encrypted optical communication system, Fiber and Integrated Optics, 27: 308-316, 2008.
16.L. Ursini, M. Santagiustina, and V. Annovazzi-Lodi. Enhancing chaotic communication performances by manchester coding. IEEE Photonics Technology Letters, 20: 401-403, 2008.
17.F. Chiarello, L. Ursini, M. Santagiustina. Securing Wireless Infrared Communications Through Optical Chaos. IEEE Photonics Technology Letters, 23: 564-566, 2011.
18.F.-Y. Lin and J.-M. Liu. Ambiguity functions of laser-based chaotic radar. IEEE Journal of Quantum Electronics, 40: 1732-1738, 2004.
19.F.-Y. Lin and J.-M. Liu, Chaotic lidar. IEEE Jornal of Selected Topics in Quantum Electronics, 10: 991-997, 2004.
20.Y. Wang, B. Wang, and A. Wang. Chaotic correlation optical time domain reflectometer utilizing laser diode. IEEE Photonics Technology Letters, 20: 1636-1638, 2008.
21.A. Mecozzi, C. Antonelli, V. Annovazzi-Lodi, and M. Benedetti. Chaos selfsynchronization in a semiconductor laser. Optics Letters, 34: 1387-1389, 2009.
22.Y. Antman, N. Levanon, A. Zadok. Low-noise delays from dynamic Brillouin gratings based on perfect Golomb coding of pump waves. Optics Letters, 37 : 52595261, 2012.
23.K. Y. Song, K. S. Abedin, K. Hotate, M. González Herráez, and L. Thévenaz. Highly efficient Brillouin slow and fast light using As2Se3 chalcogenide fiber. Optics Express, 14: 5862, 2006.
24.R. Pant, E. Li, C.G. Poulton, D.-Y. Choi, S. Madden, B. Luther-Davies, and B.J. Eggleton. Observation of Brillouin dynamic grating in a photonic chip. Optics Letters, 38: 305-307, 2013.
25.M. Zhang, T. Liu, P. Li, A. Wang, J. Zhang, and Y. Wang. Generation of broadband chaotic laser using dual-wavelength optically injected FabryPérot laser diode with optical feedback. IEEE Photonics Technology Letters, 23: 1872-1874, 2011.
26.M. Niklés, L. Thévenaz, and P. A. Robert. Brillouin Gain Spectrum Characterization in Single-Mode Optical Fibers. Journal of Lightwave Technology, 15: 18421851, 1997.

# Central Configurations in a symmetric five-body problem 

Muhammad Shoaib ${ }^{1}$, Anoop Sivasankaran ${ }^{2}$, and Yehia A. Abdel-Aziz ${ }^{1}$, ${ }^{3}$<br>${ }^{1}$ University of Hail, Department of Mathematical Sciences PO BOX 2440, Saudi Arabia<br>(E-mail: safridi@gmail.com)<br>${ }^{2}$ Khalifa University, Department of Applied Mathematics and Sciences, PO Box:573, Sharjah, UAE<br>(E-mail: anoop.sivasankaran@kustar.ac.ae)<br>${ }^{3}$ National Research Institute of Astronomy and Geophysics (NRIAG), Cairo, Egypt (E-mail: yehia@nriag.sci.eg)


#### Abstract

A central configuration $q=\left(q_{1}, q_{2}, \ldots, q_{n}\right)$ is a particular configuration of the $n$-bodies where the acceleration vector of each body is proportional to its position vector and the constant of proportionality is the same for $n$-bodies. In the three-body problem, it is always possible to find three positive masses for any given three collinear positions given that they are central. This is not possible for more than four-body problems in general. In this paper we model a symmetric five-body problem with with position coordinates for the five bodies as $(-x, 0),(0, y),(x, 0),(0,-y)$ and $\left(c_{1}, c_{2}\right)$. $\left(c_{1}, c_{2}\right)$ is the centre of mass of the system. Regions of central configurations, where it is possible to choose positive masses, are derived using both analytical and numerical tools. We also identify regions in the phase space where no central configurations are possible. A certain relationship exists between the mass placed at the center of mass of the systems i.e $\left(c_{1}, c_{2}\right)$ and the remaining four masses. This relationship is investigated both numerically and analytically. Similarly restrictions on the geometry and restrictions on the inter-body distances are investigated.


Keywords: Central Configurations, n-body problem, five-body problem, inverse problem of central configurations.

## 1 Introduction

The classical equation of motion for the n-body problem has the form

$$
\begin{equation*}
m_{i} \frac{d^{2} \mathbf{q}_{i}}{d t^{2}}=\frac{\partial U}{\partial q_{i}}=\sum_{j \neq i} \frac{m_{i} m_{j}\left(\mathbf{q}_{j}-\mathbf{q}_{i}\right)}{\left|\mathbf{q}_{i}-\mathbf{q}_{j}\right|^{3}} \quad i=1,2, \ldots, n \tag{1}
\end{equation*}
$$

where the units are chosen so that the gravitational constant is equal to one, $\mathbf{q}_{i}$ is a vector in three space,

$$
\begin{equation*}
U=\sum_{1 \preceq i<j \preceq n} \frac{m_{i} m_{j}}{\left|\mathbf{q}_{i}-\mathbf{q}_{j}\right|} \tag{2}
\end{equation*}
$$

is the self-potential, $\boldsymbol{q}_{i}$ is the location vector of the $i$ th body and $m_{i}$ is the mass of the $i$ th body.

A central configuration $q=\left(q_{1}, q_{2}, \cdots, q_{n}\right)$ is a particular configuration of the n -bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for the n-bodies, therefore

$$
\begin{equation*}
\sum_{j=1, j \neq i}^{n} \frac{m_{j}\left(\mathbf{q}_{j}-\mathbf{q}_{k}\right)}{\left|\mathbf{q}_{j}-\mathbf{q}_{k}\right|^{3}}=-\lambda\left(\mathbf{q}_{k}-\mathbf{c}\right) \quad k=1,2, \ldots, n, \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda=\frac{U}{2 I}, \quad I=\sum_{i=1}^{n} m_{i}\left\|\mathbf{q}_{i}\right\|^{2}, \text { and } \mathbf{c}=\frac{\sum_{i=1}^{n} m_{i} \mathbf{q}_{i}}{\sum_{i=1}^{n} m_{i}} \tag{4}
\end{equation*}
$$

So far, in the non-collinear general four and five-body problems the main focus has been on the common question: For a given set of masses and a fixed arrangement of bodies does there exist a unique central configuration ([7], [6]). In this paper, we ask the inverse of the question i.e. given a four or five-body configuration, if possible, find positive masses for which it is a central configuration. Similar question has been answered by Ouyang and Xie (2005) for a collinear four body problem and by Mello and Fernades (2011) for a rhomboidal four and five-body problem. For other recent studies on the rhomboidal problem see [1],[2],[4], and [5]. In this paper we state and prove the following theorems.

Theorem 1. Consider five bodies of masses $\left(m_{1}, m_{2}, m_{3}, m_{4}, m_{0}\right)$ located at $(-x, 0),(y, 0),(x, 0),(0,-y)$ and $(0,0)$ respectively. The mass $m_{0}$ is taken to be stationary at the centre of mass of the system. Let $m_{1}=m_{3}=1, m_{2}=m_{4}=$ $m$.

1. In this particular set up, using polar coordinates, of the rhomboidal five body problem where $m(\theta)>0, m_{0}(\theta)>0$ and $r=1$ will form central configuration when $\theta \in(-1.94,-1.04) \cup(0.74,1.04)$. For all other values of $\theta$ at least one of the masses will become negative.
2. For $r \neq 1$, the central configuration region is given in figure (1).

Theorem 2. Let five bodies of masses $m_{1}=m_{3}=M, m_{2}=m_{4}=m$ be placed at the vertices $m_{1}(-1,0), m_{2}(y, 0), m_{3}(1,0), m_{4}(0,-y)$ and $m_{0}(0,0)$ of a rhombus. The mass $m_{0}$ is taken to be stationary at the centre of mass of the system. There exist a region

$$
\begin{equation*}
R_{1}=\left(R_{1 m} \cup R_{1 m}^{*}\right) \cap\left(R_{1 M} \cup R_{1 M}^{*}\right) \tag{5}
\end{equation*}
$$

in the $y m_{0}-$ plane where it is possible to choose positive masses which will make the configuration central, where

$$
\begin{align*}
R_{1 m} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}>\frac{y^{3}\left(8-\left(1+y^{2}\right)^{3 / 2}\right)}{-8 y^{3}+\left(1+y^{2}\right)^{3 / 2}}\right.\right. \\
& \text { and } y \in(0,2-\sqrt{3}) \cup(2+\sqrt{3}, \infty)\} \tag{6}
\end{align*}
$$

$$
\begin{align*}
R_{1 m}^{*} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}<\frac{y^{3}\left(8-\left(1+y^{2}\right)^{3 / 2}\right)}{-8 y^{3}+\left(1+y^{2}\right)^{3 / 2}}\right. \text { and } y \in(2-\sqrt{3}, 2+\sqrt{3})\right\}(, 7) \\
R_{1 M} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}>\frac{8 y^{3}-\left(1+y^{2}\right)^{3 / 2}}{-8+\left(1+y^{2}\right)^{3 / 2}}\right.\right.  \tag{8}\\
& \text { and } y \in(0,2-\sqrt{3}) \cup(2+\sqrt{3}, \infty)\}, \\
R_{1 M}^{*} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}<\frac{8 y^{3}-\left(1+y^{2}\right)^{3 / 2}}{-8+\left(1+y^{2}\right)^{3 / 2}}\right. \text { and } y \in(2-\sqrt{3}, 2+\sqrt{3})\right\} . \tag{9}
\end{align*}
$$

In the complement of this region no central configurations exist for $m, m_{0}>$ 0.

Theorem 3. Consider five bodies of masses $\left(m_{1}, m_{2}, m_{3}, m_{4}, m_{0}\right)$ located at $(-x, 0),(y, 0),(x, 0),(0,-y)$ and $(0,0)$ respectively. The mass $m_{0}$ is taken to be stationary at the centre of mass of the system. Let $m_{1}=m_{3}=M, m_{2}=m_{4}=$ $m$. There exist a region

$$
\begin{equation*}
R_{3}=\left(\left(R_{d} \cap R_{3 m}\right) \cup\left(R_{d}^{c} \cap R_{3 m}^{c}\right)\right) \cap\left(R_{d} \cap R_{3 M}\right) \cup\left(R_{d}^{c} \cap R_{3 M}^{c}\right), \tag{10}
\end{equation*}
$$

in the xy-plane where it is possible to choose positive masses which will make the configuration central. Here

$$
\begin{align*}
R_{3 m} & =\left\{(x, y) \left\lvert\, r(x, y)>2 y \sqrt[3]{\frac{m_{0}+x^{3}}{m_{0}+y^{3}}}\right., x>0, y>0, m_{0}>0\right\}  \tag{11}\\
R_{3 M} & =\left\{(x, y) \left\lvert\, r(x, y)>2 x \sqrt[3]{\frac{m_{0}+y^{3}}{m_{0}+x^{3}}}\right., x>0, y>0, m_{0}>0\right\} \tag{12}
\end{align*}
$$

In the complement of this region no central configurations exist for $M, m, m_{0}>$ 0 .

Let's consider five bodies of masses $m_{i}, i=0,1,2,3,4$. Four of the masses are placed at the vertices of a rhombus and the fifth mass $m_{0}$ is stationary at the centre of mass of the system. The coordinates for the five bodies are chosen as below:

$$
\begin{align*}
& \mathbf{q}_{0}=\left(c_{1}, c_{2}\right), \mathbf{q}_{1}=(-x, 0), \mathbf{q}_{2}=(0, y),  \tag{13}\\
& \mathbf{q}_{3}=(x, 0), \mathbf{q}_{4}=(0,-y), \tag{14}
\end{align*}
$$

Using (3) and (13) we obtain the following equation for central configurations.

$$
\begin{align*}
& \frac{m_{0} \mathbf{q}_{1}}{x^{3}}+\frac{m_{2} \mathbf{q}_{12}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}+\frac{m_{3} \mathbf{q}_{13}}{8 x^{3}}+\frac{m_{4} \mathbf{q}_{14}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}=-\lambda\left(\mathbf{q}_{1}-\mathbf{c}\right)  \tag{15}\\
& \frac{m_{0} \mathbf{q}_{2}}{y^{3}}+\frac{m_{1} \mathbf{q}_{21}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}+\frac{m_{3} \mathbf{q}_{23}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}+\frac{m_{4} \mathbf{q}_{24}}{8 y^{3}}=-\lambda\left(\mathbf{q}_{2}-\mathbf{c}\right) \tag{16}
\end{align*}
$$

$$
\begin{align*}
& \frac{m_{0} \mathbf{q}_{3}}{x^{3}}+\frac{m_{1} \mathbf{q}_{31}}{8 x^{3}}+\frac{m_{2} \mathbf{q}_{32}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}+\frac{m_{4} \mathbf{r}_{34}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}=-\lambda\left(\mathbf{q}_{3}-\mathbf{c}\right)  \tag{17}\\
& \frac{m_{0} \mathbf{q}_{4}}{y^{3}}+\frac{m_{1} \mathbf{q}_{41}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}+\frac{m_{2} \mathbf{q}_{42}}{8 y^{3}}+\frac{m_{3} \mathbf{q}_{43}}{\left(\sqrt{x^{2}+y^{2}}\right)^{3}}=-\lambda\left(\mathbf{q}_{4}-\mathbf{c}\right) \tag{18}
\end{align*}
$$

## 2 Proof of Theorem 1.

Let $m_{1}=m_{3}=1, m_{2}=m_{4}=m$. As CC's are invariant up to translation and re-scaling therefore we assume that the centre of mass is at the origin. This assumption leads to some simplifications in the CC equations. Therefore from the four CC equations ( $(15$ to 18$)$ the following two linearly independent equations are obtained.

$$
\begin{align*}
-\frac{1}{4 x^{2}}+\frac{m_{0}}{x^{2}}-\frac{2 m x}{\left(x^{2}+y^{2}\right)^{3 / 2}} & =-x \lambda  \tag{19}\\
\frac{m}{4 y^{2}}-\frac{m_{0}}{y^{2}}+\frac{2 y}{\left(x^{2}+y^{2}\right)^{3 / 2}} & =y \lambda \tag{20}
\end{align*}
$$

Let $\lambda=1$. Equations (19 and 20) are solved to obtain $m$ and $m_{0}$ as functions of $x>0$ and $y>0$.

$$
\begin{align*}
m(x, y) & =\frac{8 y^{3}-\left(x^{2}+y^{2}\right)^{3 / 2}\left(1-4 x^{3}+4 y^{3}\right)}{8 x^{3}-\left(x^{2}+y^{2}\right)^{3 / 2}}  \tag{21}\\
m_{0}(x, y) & =\frac{32 x^{3} y^{3}\left(2-\left(x^{2}+y^{2}\right)^{3 / 2}\right)-\left(x^{2}+y^{2}\right)^{3}\left(1-4 x^{3}\right)}{4\left(x^{2}+y^{2}\right)^{3 / 2}\left(8 x^{3}-\left(x^{2}+y^{2}\right)^{3 / 2}\right)} \tag{22}
\end{align*}
$$

It is not possible to explicitly solve for $x$ and $y$ therefore we use polar coordinates to re-write $m(x, y)$ and $m_{0}(x, y)$ as $m(r, \theta)$ and $m_{0}(r, \theta)$, where $x=r \cos \theta$ and $y=r \sin \theta$.

$$
\begin{gather*}
m(r, \theta)=\frac{1+4 r^{3} \cos ^{3} \theta-4\left(2+r^{3}\right) \sin ^{3} \theta}{1-6 \cos \theta-2 \cos 3 \theta} .  \tag{23}\\
m_{0}(r, \theta)=\frac{\left(1-6 \sin 2 \theta+2 \sin 6 \theta-r^{3}(3 \cos \theta-3 \sin 2 \theta+\cos 3 \theta+\sin 6 \theta)\right)}{4(1-6 \cos \theta-2 \cos 3 \theta)} . \tag{24}
\end{gather*}
$$

Let $r=1$. The denominator of both $m(\theta)$ and $m_{0}(\theta)$ becomes zero at $\theta=$ $-\frac{\pi}{3}, \frac{\pi}{3}$. The denominator is negative when $\theta \in\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and is positive elsewhere. The numerator of $m(\theta)$ when $r=1$ is given by $1+\cos ^{3} \theta-12 \sin ^{3} \theta$. This has real zeros at $\theta=-2.61$ and $\theta=0.673$. The numerator is positive


Fig. 1. left: $m_{0}(r, \theta)>0$. Centre: Region, when $m(r, \theta)>0$ Right: Region, when $m_{0}(r, \theta)>0$ and $m(r, \theta)>0$
when $\theta \in(-2.61,0.673)$. Therefore $m(\theta)$ is positive when $\theta \in(-2.61,-1.04) \cup$ (0.673, 1.04).

The numerator of $m_{0}(\theta)$ when $r=1$ is given by $-1+3 \cos \theta+\cos 3 \theta+$ $3 \sin 2 \theta-\sin 6 \theta$. This has real zeros at $\theta=-2.541, \theta=-1.935, \theta=-0.449$, and $\theta=1.248$. The numerator of $m_{0}(\theta)$ is positive when $\theta \in(-\pi,-2.54) \cup$ $(-1.935,-0.449) \cup(1.248, \pi)$. Therefore $m_{0}(\theta)$ is positive when $\theta \in(-\pi,-2.54) \cup$ $(-1.935,-1.04) \cup(-0.449,1.04) \cup(1.248, \pi)$.

Hence, this particular set up of the rhomboidal five body problem where $m(\theta)>0, m_{0}(\theta)>0$ and $r=1$ will form central configuration when $\theta \in$ $(-1.94,-1.04) \cup(0.74,1.04)$. For all other values of $\theta$ at least one of the masses will become negative.

In the case when $r \neq 1$, The central configuration region is given in figure (1)

## 3 Proof of Theorem 2.

Let $\lambda=x=1$. Solve equations (19 and 20) to obtain $m$ and $M$ as functions of $m_{0}$ and $y$.

$$
\begin{align*}
& m\left(y, m_{0}\right)=\frac{4\left(1+y^{2}\right)^{3 / 2} N_{m}\left(y, m_{0}\right)}{\left(1-4 y+y^{2}\right)\left(1+4 y+18 y^{2}+4 y^{3}+y^{4}\right)}  \tag{25}\\
& M\left(y, m_{0}\right)=\frac{4\left(1+y^{2}\right)^{3 / 2} N_{M}\left(y, m_{0}\right)}{\left(1-4 y+y^{2}\right)\left(1+4 y+18 y^{2}+4 y^{3}+y^{4}\right)} \tag{26}
\end{align*}
$$

where

$$
\begin{align*}
N_{m}\left(y, m_{0}\right)= & y^{3}\left(-2+\sqrt{1+y^{2}}\right)\left(5+y^{2}+2 \sqrt{1+y^{2}}\right) \\
& +m_{0}\left(\left(-2 y+\sqrt{1+y^{2}}\right)\left(1+5 y^{2}+2 y \sqrt{1+y^{2}}\right)\right),  \tag{27}\\
N_{M}\left(y, m_{0}\right)= & \left(-2 y+\sqrt{1+y^{2}}\right)\left(1+5 y^{2}+2 y \sqrt{1+y^{2}}\right) \\
& +m_{0}\left(\left(-2+\sqrt{1+y^{2}}\right)\left(5+y^{2}+2 \sqrt{1+y^{2}}\right)\right) . \tag{28}
\end{align*}
$$

The factor $1-4 y+y^{2}$ of the denominator of $m\left(y, m_{0}\right)$ and $M\left(y, m_{0}\right)$ is positive when $y \in(0,2-\sqrt{3}) \cup(2+\sqrt{3}, \infty)$ and is negative when $y \in(2-\sqrt{3}, 2+$ $\sqrt{3})$. Therefore to find the sign of $m\left(y, m_{0}\right)$ and $m\left(y, m_{0}\right)$ we need to analyze $N_{m}\left(y, m_{0}\right)$ and $N_{M}\left(y, m_{0}\right)$. The component of the numerator of $m\left(y, m_{0}\right)$, $N_{m}\left(y, m_{0}\right)$, has two factors i.e. $-2+\sqrt{1+y^{2}}$ and $-2 y+\sqrt{1+y^{2}}$ which can become negative and hence can make $N_{m}\left(y, m_{0}\right)$ negative. The factor $-2+$ $\sqrt{1+y^{2}}>0$ when $y \in(\sqrt{3}, \infty)$ and $-2 y+\sqrt{1+y^{2}}>0$ when $y \in\left(0, \frac{1}{\sqrt{3}}\right)$. As both the intervals have empty intersection therefore we must have the following bound on $m_{0}$ for $N_{m}\left(y, m_{0}\right)$ to be positive.

$$
\begin{equation*}
m_{0}>\frac{y^{3}\left(8-\left(1+y^{2}\right)^{3 / 2}\right)}{-8 y^{3}+\left(1+y^{2}\right)^{3 / 2}} \tag{29}
\end{equation*}
$$

Hence $m\left(y, m_{0}\right)$ will be positive in the following two regions.

$$
\begin{align*}
R_{1 m} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}>\frac{y^{3}\left(8-\left(1+y^{2}\right)^{3 / 2}\right)}{-8 y^{3}+\left(1+y^{2}\right)^{3 / 2}}\right.\right. \\
& \text { and } y \in(0,2-\sqrt{3}) \cup(2+\sqrt{3}, \infty)\},  \tag{30}\\
R_{1 m}^{*} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}<\frac{y^{3}\left(8-\left(1+y^{2}\right)^{3 / 2}\right)}{-8 y^{3}+\left(1+y^{2}\right)^{3 / 2}}\right.\right. \\
& \text { and } y \in(2-\sqrt{3}, 2+\sqrt{3})\} . \tag{31}
\end{align*}
$$

Similarly $M\left(y, m_{0}\right)$ is positive in the following two regions

$$
\begin{align*}
R_{1 M} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}>\frac{8 y^{3}-\left(1+y^{2}\right)^{3 / 2}}{-8+\left(1+y^{2}\right)^{3 / 2}}\right.\right. \\
& \text { and } y \in(0,2-\sqrt{3}) \cup(2+\sqrt{3}, \infty)\}  \tag{32}\\
R_{1 M}^{*} & =\left\{\left(y, m_{0}\right) \left\lvert\, m_{0}<\frac{8 y^{3}-\left(1+y^{2}\right)^{3 / 2}}{-8+\left(1+y^{2}\right)^{3 / 2}}\right. \text { and } y \in(2-\sqrt{3}, 2+\sqrt{3})\right\}( \tag{.33}
\end{align*}
$$

Hence, the central configuration region for this particular set up of the rhomboidal five body problem where both $m\left(x, y, m_{0}\right)$ and $M\left(x, y, m_{0}\right)$ are positive is given by

$$
\begin{equation*}
R_{1}=\left(R_{1 m} \cup R_{1 m}^{*}\right) \cap\left(R_{1 M} \cup R_{1 M}^{*}\right) \tag{34}
\end{equation*}
$$

This completes the proof of theorem 2. This central configuration region is given in figure (2)

## 4 Proof of Theorem 3.

Let $\lambda=1$. Solve equations (19 and 20) to obtain $m$ and $M$ as functions of $x, y$ and $m_{0}$.


$$
\begin{align*}
m\left(x, y, m_{0}\right)= & \frac{4\left(x^{2}+y^{2}\right)^{3 / 2}\binom{y^{3}\left(-8 x^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right.}{+m_{0}\left(-8 y^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)}}{\left(x^{2}-4 x y+y^{2}\right)\left(x^{4}+4 x^{3} y+18 x^{2} y^{2}+4 x y^{3}+y^{4}\right)}  \tag{35}\\
M\left(x, y, m_{0}\right)= & \frac{4\left(x^{2}+y^{2}\right)^{3 / 2}\binom{x^{3}\left(-8 y^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)}{+m_{0}\left(-8 x^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)}}{\left(x^{2}-4 x y+y^{2}\right)\left(x^{4}+4 x^{3} y+18 x^{2} y^{2}+4 x y^{3}+y^{4}\right)} \tag{36}
\end{align*}
$$

It can be immediately seen that the denominator of both $m\left(x, y, m_{0}\right)$ and $M\left(x, y, m_{0}\right)$ becomes singular at $y=(2 \pm \sqrt{3}) x$. Therefore $y=(2 \pm \sqrt{3}) x$ will form two singular curves for the two masses $m$ and $M$. Therefore the denominator will be positive in region $R_{d}$ given below and will be negative in its complement.

$$
\begin{equation*}
R_{d}=\{(x, y) \mid 0<y<(2-\sqrt{3}) x \text { or } y>(2+\sqrt{3}) x, x>0\} \tag{37}
\end{equation*}
$$

It is not possible to explicitly solve the numerator of either $m\left(x, y, m_{0}\right)$ or $M\left(x, y, m_{0}\right)$ for $x$ or $y$ therefore we choose the inter body distance $x^{2}+y^{2}$ to find regions of central configuration where both $m$ and $M$ are positive. In the numerator of $m\left(x, y, m_{0}\right)$ the factor

$$
y^{3}\left(-8 x^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)+m_{0}\left(-8 y^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)=N_{3 m}
$$

can be become negative. By taking $r=\sqrt{x^{2}+y^{2}}$, the factor $N_{3 m}$ is simplified as below.

$$
\begin{equation*}
N_{3 m}=y^{3}\left(-8 x^{3}+r^{3}\right)+m_{0}\left(-8 y^{3}+r^{3}\right) \tag{38}
\end{equation*}
$$

After some algebraic manipulation it can be shown that $N_{3 m}$ is positive in the following region.

$$
\begin{equation*}
R_{3 m}=\left\{(x, y) \left\lvert\, r(x, y)>2 y \sqrt[3]{\frac{m_{0}+x^{3}}{m_{0}+y^{3}}}\right., x>0, y>0, m_{0}>0\right\} \tag{39}
\end{equation*}
$$

$N_{3 m}$ is negative in the complement of $R_{3 m}$. Therefore, in this particular set up, the central configuration region where $m$ is positive is given by

$$
\begin{equation*}
\left(R_{d} \cap R_{3 m}\right) \cup\left(R_{d}^{c} \cap R_{3 m}^{c}\right) . \tag{40}
\end{equation*}
$$

Similarly $N_{3 M}=x^{3}\left(-8 y^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)+m_{0}\left(-8 x^{3}+\left(x^{2}+y^{2}\right)^{3 / 2}\right)$ is positive in the following region.

$$
\begin{equation*}
R_{3 M}=\left\{(x, y) \left\lvert\, r(x, y)>2 x \sqrt[3]{\frac{m_{0}+y^{3}}{m_{0}+x^{3}}}\right., x>0, y>0, m_{0}>0\right\} \tag{41}
\end{equation*}
$$

$N_{3 M}$ is negative in the complement of $R_{3 M}$. Therefore, in this particular set up, the central configuration region where $M$ is positive is given by

$$
\begin{equation*}
\left(R_{d} \cap R_{3 M}\right) \cup\left(R_{d}^{c} \cap R_{3 M}^{c}\right) . \tag{42}
\end{equation*}
$$

Hence, the central configuration region for this particular set up of the rhomboidal five body problem where both $m\left(x, y, m_{0}\right)$ and $M\left(x, y, m_{0}\right)$ are positive is given by

$$
\begin{equation*}
R_{3}=\left(\left(R_{d} \cap R_{3 m}\right) \cup\left(R_{d}^{c} \cap R_{3 m}^{c}\right)\right) \cap\left(R_{d} \cap R_{3 M}\right) \cup\left(R_{d}^{c} \cap R_{3 M}^{c}\right) \tag{43}
\end{equation*}
$$

In the complement of this region no central configurations are possible as at least one of the masses will become negative. This completes the proof of theorem 3.

## 5 Acknowledgments

The authors, Muhammad Shoaib and Yehia A Abdel-Aziz thank the Deanship of Scientific research at the University of Hail, Saudi Arabia for funding this project (reference number SM3 ) for the year 1433 Hijri (2012-2013).

## References

1.B. Lennard, and S. Simmons. Stability of the Rhomboidal Symmetric-Mass Orbit.arXiv preprint arXiv:1208.3183 (2012).
2.D. Diarova and N. I. Zemtsova. The instability of the rhombus-like central configurations in newton 9-body problem. Computer Algebra in Scientific Computing. Springer Berlin Heidelberg, 2006.
3.L. F. Mello and A. C. Fernandes. Co-circular and co-spherical kite central configurations. Qualitative theory of dynamical systems. 10 (2011) 29-41.
4.M. Kulesza, M. Marchesin, and C. Vidal. Restricted rhomboidal five-body problem. Journal of Physica A: Mathematical and Theoritical. 44 (2011) 485204.
5.M. Marcelo, and C. Vidal. Spatial restricted rhomboidal five-body problem and horizontal stability of its periodic solutions. Celestial Mechanics and Dynamical Astronomy (2013): 1-19.
6.M. Shoaib, B.A. Steves, and A. Széll. Stability analysis of quintuple stellar and planetary systems using a symmetric five-body model. New Astronomy. 13 (2008) 639-645.

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

Chaotic Modeling and Simulation (CMSIM) : -, 2013
7.M. Shoaib and I. Faye. Collinear equilibrium solutions of four-body problem. Journal of Astrophysics and Astronomy. 32 (2011) 411-423.
8.T Ouyang and Z. Xie. Collinear central configuration in four-body problem, Ce lestial Mechanics and Dynamical Astronomy. 93 (2005) 147-166.

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

# Observation of Chaotic Behaviour in Physical Ageing of Chalcogenide Glasses 

O. Shpotyuk ${ }^{1,2}$, A. S. Hacınlıyan ${ }^{3,4}$, R. Golovchak ${ }^{1,6}$, Y. Skarlatos ${ }^{3,4}$, V. Balitska ${ }^{1,7}$, I. K. Aybar ${ }^{3}$, A. Kozdras ${ }^{8}$, O. O. Aybar ${ }^{3,5}$<br>${ }^{1}$ Lviv Scientific Research Institute of Materials of SRC "Carat", Lviv, Ukraine<br>E-mail: shpotyuk@novas.lviv.ua<br>${ }^{2}$ Institute of Physics of Jan Dlugosz University, Czestochowa, Poland<br>${ }^{3}$ Yeditepe University, Istanbul, Turkey<br>${ }^{4}$ Bogazici University, Istanbul, Turkey<br>${ }^{5}$ Gebze Institute of Technology, Kocaeli, Turkey<br>${ }^{6}$ Lehigh University, Bethlehem, USA<br>${ }^{7}$ Lviv State University of Vital Activity Safety, Lviv, Ukraine<br>${ }^{8}$ Opole Technical University, Opole, Poland


#### Abstract

Theory of strange attractors is shown to be adequately applied to analyze kinetics of physical ageing revealed in structural relaxation of Se-rich As-Se glasses below glass transition. Kinetics of enthalpy losses induced by prolonged storage in natural conditions is used to determine phase space reconstruction parameters. The observed chaotic behaviour (involving chaos and fractal consideration such as detrended fluctuation analysis, attractor identification using phase space representation, delay coordinates, mutual information, false nearest neighbours, etc.) reconstructed via TISEAN is treated within potential energy landscape as diversity of multiple transitions between different basins-metabasins towards more thermodynamically equilibrium state, minimizing the free energy of the system.


Keywords: Chaotic behaviour, Physical ageing, Chalcogenide glass, Relaxation kinetics.

## 1. Introduction

Chalcogenide glasses (ChG) belong to promising kind of low-phonon-energy materials transparent from visible to infrared, which are perspective for advanced IR telecommunication and different fibre-optic applications [1]. This feature of ChG is due to the heavier chalcogen atoms ( $\mathrm{S}, \mathrm{Se}, \mathrm{Te}$ ), which are the main constituents of their covalent networks. However, the chalcogen-rich glasses possess a pronounced natural physical ageing ( NPhA ) at the ambient conditions hindering their practical implementation [2,3]. The related structural relaxation is a reason for uncontrolled drift in the exploitation characteristics of ChG-based devices and can have extended kinetics from few hours up to tens of years depending on ChG composition and ageing temperature. In general, the PhA originates from non-equilibrium metastable nature of glassy state, causing all ChG prepared by conventional melt-quenching route to approach with time the most equilibrium structure of corresponding supercooled liquid [2,3].
One of possible ways to resolve this problem is an analysis of underlying PhA kinetics. Since the ChG belong to a wide class of disordered solids, which are
typical nonlinear systems being far out of thermodynamic equilibrium, the theory of chaos [4-6] can be used to analyse their time evolution. The $N$-particle system such as ChG can be imagined in terms of a hypothetical energy hypersurface of $3 N+1$ dimensions (energy/enthalpy landscape [7,8]), where the system's energy is determined by positions of all constituting particles and its dynamics is viewed as motion of "state point" described by coordinates of these particles along landscape surface [8]. Such landscape of complex glassy-like system consists of many local minima of energy (inherent structures or attractors) and associated basins. If the system with broken ergodicity can explore more than one (but finite) number of inherent structures at a particular temperature, they can be grouped together in larger metabasins [8]. In such consideration, the NPhA can be viewed as a problem of transitions between different basins/metabasins tending ChG towards thermodynamically equilibrium state, minimizing free energy of the system. In other words, all the trajectories of the particles should be finished within basin of attractions accessible under certain conditions after a sufficiently long time period.
In this work, we shall try to use nonlinear time series analysis [4-6] successfully applied previously for different solid systems including some polymers, such as polymethylmethacrylate and polyethylene glycol [9,10], to analyze kinetics of below- $T_{g}$ relaxation in Se-rich As-Se glasses as typical representatives of ChG.

## 2. Real-time NPhA kinetics in g-As-Se

The samples of glassy $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}, \mathrm{~g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ were prepared by conventional melt quenching route in evacuated quartz ampoules from a mixture of high purity precursors as was described in more details elsewhere [3]. The amorphicity and compositional identity of these ChG were tested by character conch-like fracture, data of X-ray diffraction and photoelectron spectroscopy.
In order to determine the kinetics of enthalpy losses $\Delta H$, the differential scanning calorimetry (DSC) patterns were detected using NETZSCH 404/3/F calorimeter calibrated with a set of standard elements. The DSC traces were recorded in the ambient atmosphere with $5 \mathrm{~K} / \mathrm{min}$ heating rate, the same calibration procedure being repeated each time during each routine measurement. Three independent DSC signals with ChG samples of close masses were performed to confirm the reproducibility of the results.
Typical DSC signals of enthalpy losses $\Delta H$ caused by long-term NPhA of g$\mathrm{As}_{10} \mathrm{Se}_{90}$ is shown in Figure 1. The NPhA behaviour is revealed by DSC technique as appearance of strong endothermic peak superimposed on endothermic step of glass transition signal and its displacement towards higher temperatures with PhA duration [2,3]. Difference in the area under DSC signal of aged and rejuvenated ChG is directly proportional to the enthalpy losses $\Delta \mathrm{H}$. It was established that microstructural origin of NPhA in Se-based ChG relies on twisting of bridge chalcogen atoms between specific configuration states possessing a so-called double-well potentials in nearest-atomic interaction [11,12]. From this point, three possible environments for $\mathbf{S e}$ atom can be distinguished in g-As-Se: Se-Se-Se fragments within $\mathrm{Se}_{\mathrm{n}}$ chains (number $n$ of Se atoms in the chain means number of chalcogen atoms inserted between As

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey
atoms), As-Se-Se and As-Se-As. Each double-well potential associated with these fragments has different energetic barrier and configurational parameters of state. Therefore, the activation energies for over-barrier transitions or tunnelling of Se atoms between two neighbouring states within double-well potential are different, giving variety of possibilities to be externally activated during prolonged NPhA. It is shown that twisting of Se atoms within double-well potential associated with floppy $\mathrm{Se}-\mathrm{Se}-\mathrm{Se}$ fragments is responsible for fast component of NPhA [12]. Such twisting leads to alignment of longer $\mathrm{Se}_{\mathrm{n} 23}$ chains, their better space utilization followed by a fast shrinkage of surrounding glassy network. According to "chains crossing" model [3], long $\mathrm{Se}_{\mathrm{n}}$ chains with $n \geq 3$ should fully disappear in $g-\mathrm{As}_{\mathrm{x}} \mathrm{Se}_{100-\mathrm{x}}$ at $x \geq 25$. This leads to vanishing of relatively fast alignment-shrinkage in these ChG. On the other hand, twisting of Se atoms within double-well potential of As-Se-Se fragments and associated shrinkage of under-constrained glassy network (g-As-Se with $Z<2.4$ are considered as under-constrained networks in full respect to Phillips-Thorpe rigidity theory $[13,14]$ ) are shown to have very slow kinetics in a dark (as testified by NPhA at room temperature) [3,15,16]. The kinetics of changes in glass transition temperature $T_{g}$, partial area A under the endothermic peak in DSC experiments, enthalpy losses $\Delta H$ and fictive temperature $T_{F}$ caused by dark storage of Se-rich g-As-Se at room temperature exhibited a well-expressed stepwise character, showing some kinds of plateaus and steep regions [16]. The phenomenological description was sufficiently derived from alignmentshrinkage mechanism of NPhA , showing that relaxation kinetics of experimentally obtained enthalpy losses $\Delta H$ is caused by a superposition of parallel/sequent alignment-shrinkage processes with different relaxation times.


Fig. 1. DSC curves showing kinetics of room-temperature NPhA in g-As ${ }_{10} \mathrm{Se}_{90}$.

## 3. Chaotic behavior observed in NPhA of g-As-Se

We observe irregular transient enthalpy losses $\Delta H$ characteristics for g-As-Se under NPhA as shown in Figure 2. One way to understand this irregularity is to take increasing enthalpy losses $\Delta H$ as a slowly varying parameter with the data split into equal time periods using a so-called nonlinear time series analysis [4$6,17]$. After splitting, the delay times (referred to as delay or lag) are analysed using the delay-coordinate embedding theorem by F. Takens and T. Sauer et al. $[18,19]$. If the embedding is performed correctly, the theorem guarantees that the reconstructed dynamics of the system should be identical to true dynamics and dynamical invariants should be also identical.


Fig. 2. Enthalpy $v s$. time $\Delta H-t$ graph for NPhA
in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ (red curve), $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ (green curve) and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ (blue curve).
To work on time series at first we build up the delay vectors $x(T), x(T+t), \ldots$, $x(T+(m-1) t)$, here $t$ and $m$ represents delay time and embedding dimension, respectively. The reconstructed invariants (basically, its fractal dimension) of the attractor as found by this approach remain unchanged (invariant) with respect to the unknown, original system that generated the series.

### 3.1. Mutual information

In contrast to linear dependence measured by autocorrelation, the mutual information $I(t)$, provides a measure of general dependence [17]. Therefore $I(t)$ is expected to provide a better measure of the transition from small to large times $t$ with nonlinear systems. Mutual information answers the following question: given the observation of $S(T)$, at time $T$, how accurately can one predict $S(T+t)$ after a delay of $t$, so that successive delay coordinates are interpreted as relatively independent when the mutual information is small?

According to mutual information identification as shown in Figure 3, the delay time of NPhA in all As-Se ChG are quite similar being close to 250 time steps despite glass composition. This specificity can be explained by preferential input of the same structural entities responsible for primary changes associated with long-term NPhA in all ChG. In respect to microstructure study on NPhA in Se-rich g-As-Se [11,12], these governing ageing-related relaxation events can be associated with twisting of central Se atoms inserted in heteropolar environment (As and Se ) within character double-well potential (As-Se-Se fragments).


Fig. 3. Average mutual information $v s$. delay time graphs $I-t$ for NPhA in $g-\mathrm{As}_{10} \mathrm{Se}_{90}$ (red curve), $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ (green curve) and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ (blue curve).

### 3.2. Embedding Dimension

Most of the systems in nature wander chaotically on a set of points called strange attractors. A related difficulty with attractor reconstruction involves the choice of the embedding dimension $m$. After choosing an acceptable time delay, we need a sufficiently large embedding dimension for the reconstructed phase space that avoids projecting the system onto a lower dimensional space. As mentioned before, the embedding theorem $[18,19]$ tells us that if the box counting dimension of the attractor defined as $n$, an embedding dimension $m$ that is greater $2 n$, will absolutely allow unfolding the system in the reconstructed phase space. Normally, one has no a priori knowledge regarding the topological dimension, and it is unclear what a proper value of $m$ would be. One needs a criterion for the minimum embedding dimension, sufficient to unfold the attractor. At this point the false nearest neighbors method [17] is a useful tool to give an estimate for the embedding dimension. Suppose that a space reconstruction in dimension $m>m_{0}$ is carried out, where $m_{0}$ is the minimum dimension that unfolds the reconstructed attractor. In $m$-dimensional
space, the reconstructed attractor becomes one to one image of the attractor in the original phase space. Conservation of topological properties in actual phase space satisfies mapping neighboring points of a given point of the original system onto neighbors of the image of that given point in the reconstructed space [17]. This is usually understood as equivalent tangent spaces.
By taking the delay time as given above, we analyzed the minimum embedding dimension to reconstruct the attractor by the false neighbors method (Figure 4). One may choose the smallest embedding dimension $m$ that yields a convergent result. Due to the fact that chaotic systems are stochastic when embedded in a phase space that is very small to accommodate the true dynamics, we assume the embedding dimensions $m$ as 4, 3 and 2 in that given order for NPhA for g$\mathrm{As}_{10} \mathrm{Se}_{90}, \mathrm{~g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$. It is important to note that this sequence in $m$ values corresponds to smoothing tendency in NPhA kinetics, when multiple step-wise trends observed in more Se-rich ChG (especially in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ ) disappear in samples with greater Se content $\left(g-\mathrm{As}_{20} \mathrm{Se}_{80}\right.$ and $\left.g-\mathrm{As}_{30} \mathrm{Se}_{70}\right)$ [15]. It should be mentioned that the general structure of similar results for polymer matewrials studied previously $[9,10]$ is the same as those for ChG samples.


Fig. 4. Fraction of false nearest neighbors $v s$. embedding dimension graphs for NPhA in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ (red curve), $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ (green curve) and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ (blue curve).

### 3.3. Detrended Fluctuation Analysis (DFA)

Confirmation of these characteristics mentioned in previous sections requires a more detailed analysis of short- and long-range structural correlations in ChG. To investigate this, we have applied a scaling analysis used to estimate longrange power-law correlation exponents known as DFA method [17].
It was established that ChG subjected to NPhA have different characteristic properties concerning more than one regime as it follows from Figure 5.


Fig. 5. DFA for for NPhA
in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ (red curve), $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ (green curve) and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ (blue curve).
The starting stages of NPhA in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ and $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ samples has similar characteristics with slope of $\sim 1.57$, changing this slope towards $\sim 1.90$ with further stages of NPhA, which is also the slope for NPhA in g-As ${ }_{30} \mathrm{Se}_{70}$ sample. Figure 6 gives details about Rescaled Range Analysis (R/S-Hurst) of NPhA for studied g-As-Se ChG.


Fig. 6. R/S for NPhA
in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ (red curve), $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ (green curve) and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ (blue curve).

The NPhA of $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ and $g-\mathrm{As}_{30} \mathrm{Se}_{70}$ demonstrate one-regime behaviour with similar characteristics and $\sim 0.42$ slope, but NPhA in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ has smaller $\sim 0.25$ slope at the end of ageing demonstrating a character two-regime behaviour. The fact that R/S-Hurst analysis for $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ gives final point that is too low compared to the remaining ones may also be noticed (see Figure 6).
Finally, in order to ascertain whether the linear interpolation caused this uniformity, we have added $5 \%$ noise and used a stretched exponential extrapolation to construct the data set. The results and slopes are given in Figure 7. Except increase in the embedding dimension and appearance of two distinct regions (attributable to the noise introduced), the similar conclusions follow.


Fig. 9. DFA graph of NPhA (in $\Delta \mathrm{H}$ determination)
for $\mathrm{As}_{10} \mathrm{Se}_{90}, \mathrm{As}_{20} \mathrm{Se}_{80}, \mathrm{As}_{30} \mathrm{Se}_{70}$ glasses and glass data with $5 \%$ noise added.

## 4. Conclusions

Typical DSC traces of enthalpy losses $\Delta H$ caused by long-term NPhA of Se-rich As-Se glasses demonstrate an obvious evidence of chaotic behaviour with (1) character delay time in mutual information presentation close to 250 time steps despite glass composition and (2) embedding dimensions decreasing in 4-3-2 sequence in a row of $\mathrm{As}_{10} \mathrm{Se}_{90}-\mathrm{As}_{20} \mathrm{Se}_{80}-\mathrm{As}_{30} \mathrm{Se}_{70}$ glasses (with increase in average covalent bonding). Within Detrended Fluctuation Analysis, it was established that starting stages of NPhA in $g-\mathrm{As}_{10} \mathrm{Se}_{90}$ and $\mathrm{g}-\mathrm{As}_{20} \mathrm{Se}_{80}$ has similar characteristics with slope of $\sim 1.57$, tending towards $\sim 1.90$ with further stages of ageing, which is also the slope for $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$. In contrast, within Rescaled Range Analysis, it was demonstrated that NPhA of $g-\mathrm{As}_{20} \mathrm{Se}_{80}$ and $\mathrm{g}-\mathrm{As}_{30} \mathrm{Se}_{70}$ show one-regime behaviour with similar characteristics and $\sim 0.42$ slope, while NPhA in $\mathrm{g}-\mathrm{As}_{10} \mathrm{Se}_{90}$ demonstrates typical two-regime behaviour expressed in smaller slope of $\sim 0.25$ at the end of ageing.

Observed chaoticity in NPhA of Se-rich As-Se glasses is attributed to complex nature of underlying structural transformations evolving multiply-repeated cycles of Se atoms twisting within nearest-neighbour chain environments of double-well potentials followed by atomic shrinkage at larger length scales. This chaotic behaviour in NPhA can be treated within potential energy landscape as diversity of transitions between different basins-metabasins towards more thermodynamically equilibrium state, minimizing free energy of the system.

## Acknowledgements

The authors are kindly grateful to TUBITAK 'The Scientific and Technical Research Council of Turkey' and 'The State Agency of Ukraine for Science, Innovation and Information Technology' for providing financial support of this research under common Ukrainian-Turkish collaborative project. This study is also partially supported by the Bogazici University Research Fund.

## References

1. A. Zakery and S.R. Elliott. Optical Non-Linearities in Chalcogenide Glasses and Their Applications, Springer-Verlag, Berlin-Heidelberg, 2007.
2. L. C. E. Struik. Physical Ageing in Amorphous Polymers and Other Materials, Elsevier, New York, 1978.
3. R. Golovchak, Gz. Gorecki, A. Kozdras and O. Shpotyuk. Physical ageing effects in vitreous arsenic selenides, Soid. State Communication, vol. 137, 67-69, 2006.
4. F. H. Stillinger. A Topographic View of Supercooled Liquids and Glass Formation, Science, vol. 267, 1935 (1995).
5. A. Heuer. Exploring the Potential Energy Landscape of Glass-Forming Systems: from Inherent Structures via Metabasins to Macroscopic Transport, Journal of Phys.: Condensed Matter, vol. 20, 373101-1-56, 2008.
6. H. Kantz and T. Schreiber. Nonlinear Time Series Analysis, Cambridge University Press, Cambridge, 2003.
7. H. D. I. Abarbanel, R. Brown, J. J. Sidorowich and L.Sh. Tsimring. The analysis of observed chaotic data in physical systems, Reviews of Modern Physics, vol. 65, no. 4, 1331-1392, 1993.
8. T. Schreiber. Interdisciplinary Application of Nonlinear Time Series Methods, Physics Reports, vol. 308, 1-64, 1999.
9. A. Hacinliyan, Y. Skarlatos, G. Sahin and G. Akin. Signals of chaotic behavior in PMMA, Chaos, Solitons and Fractals, vol. 17, 575-583, 2003.
10. K. Atak, O. O. Aybar, G. Sahin, A. Hacinliyan and Y. Skarlatos. Chaoticity analysis of the current through pure, hydrogenated and hydrophobically modified PEG-Si thin films under varying relative humidity, Central European Journal of Physics, vol. 7 , no 3, 568-574, 2009.
11. R. Golovchak, O. Shpotyuk, A. Kozdras, B. Bureau, M. Vlcek, A. Ganjoo and H. Jain. Atomistic model of physical ageing in Se-rich As-Se glasses, Philosophical Magazine, vol. 87, no. 28, 4323-4334, 2007.
12. R. Golovchak, A. Ingram, A. Kozdras, M. Vlcek, C. Roiland, B. Bureau and O. Shpotyuk. Initial stage of physical ageing in network glasses, Philosophical Magazine, vol. 92, no. 33, 4182-4193, 2012.
13. J. C. Phillips. Topology of covalent non-crystalline solids. I: Short-range order in chalcogenide alloys, Journal of Non-Crystalline Solids, vol. 34, 153-181, 1979.
14. M. F. Thorpe. Continuous deformations in random networks, Journal of NonCrystalline Solids, vol. 57, 355-370, 1983.
15. R. Golovchak, O. Shpotyuk, A. Kozdras, M. Vlcek, B. Bureau, A. Kovalskiy and H. Jain. Long-term physical ageing in As-Se glasses with short chalcogen chains, Journal of Physics: Condensed Matter, vol. 20, 245101-1-7, 2008.
16. R. Golovchak, A. Kozdras, V. Balitska and O. Shpotyuk. Step-wise kinetics of natural physical ageing in arsenic selenide glasses, Journal of Physics: Condensed Matter, vol. 24, 505106-1-10, 2012.
17. R. Hegger, H. Kantz and T. Schreiber. Practical implementation of nonlinear time series methods: The TISEAN package, Chaos, vol. 9, no 2, 413-435 (1999).
18. F. Takens. Detecting Strange Attractors in Turbulence, Lecture Notes in Mathematics, vol. 898, Springer-Verlag, New York, 1981.
19. T. Sauer, J. Yorke and M. Casdagli. Embedology, Journal of Statistical Physics, vol. 65, no. 3-4, 579-616, 1991.

# An efficient computational approach for global regularisation schemes 

Anoop Sivasankaran ${ }^{1}$ and Muhammad Shoaib ${ }^{2}$<br>${ }^{1}$ Khalifa university of Sciences Technology and Research, Department of Applied Mathematics and Sciences, PO Box 573, Sharjah UAE<br>(E-mail: Anoop.Sivasankaran@kustar.ac.ae)<br>${ }^{2}$ University of Hail, Department of Mathematic, PO BoX 2440, Hail Saudi Arabia. (E-mail: safridi@gmail.com)


#### Abstract

Due to collisional singularities appearing in gravitational few-body problems, one needs regularisation techniques for their stable approximate solution. We present an efficient computational approach for numerically integrating a symmetrical five body problem called the Caledonian Symmetric Five Body Problem (CS5BP) which is a five-body system with a symmetrically reduced phase space. The proposed global regularisation scheme consists of adapted versions of several known regularisation transformations such as the Levi-Civita-type coordinate transformations together with a time transformation which enables the numerical exploration of the systems as they pass through two-body close encounters. An algebraic optimisation algorithm is adapted for numerically implementing the regularisation scheme which make use of the reverse mode algorithmic differentiation. We show that the proposed regularisation algorithm is numerically and computationally very efficient in handling various two-body close encounters appearing in the CS5BP.


Keywords: Regularisation, singularity, celestial mechanics, few-body problem, optimisation.

## 1 Introduction

There is a growing interest in studying gravitational few-body problems (with $n>3$ ) which makes use of the symmetric boundary conditions to reduce the mathematical complexity of the problem [13],[14], [8], [7].

Several papers in the last decade have studied the the Caledonian Symmetric Four-Body Problem (CSFBP) which is a restricted coplanar four-body system with a symmetrically reduced phase space [5], [12]. The model involves two pairs of non-equal masses moving in coplanar, initially circular orbits, starting in a collinear arrangement [5]. The authors have shown that the global stability of the CSFBP system depends on a parameter called the Szebehely constant $C_{0}$. The Szebehely constant $C_{0}=-\frac{c^{2} E}{G^{2} M^{5}}$ is a dimensionless function of the total energy $(E)$ and the magnitude of the angular momentum of the system
(c), where $G$ is the gravitational constant, and $M$ is the total mass. A generalization of the CSFBP named the Caledonian Symmetric Five-Body problem (CS5BP) was done by introducing a stationary mass to the centre of mass of the CSFBP with the same analytical stability criteria [8].

Existing numerical integration schemes were inadequate to study orbits with strong close encounters, as the numerical integration fails due to collision singularities [15], [16]. In gravitational few-body problems, singularities normally appear when the distance between objects undergoing orbital motion becomes very small. As a result, the equations describing the dynamics of the system tend towards singular and the numerical integration falls apart [3]. Use of regularisation algorithms to numerically integrate gravitational few-body problems which involve near collisions or close encounters has been widely acknowledged [3], [1]. Recently a global regularisation scheme for the CSFBP is prsented in [11]. In this paper, we extend the regularisation scheme to the Caledonian Symmetric Five-Body problem (CS5BP).

## 2 Definition of the Caledonian Symmetric Five Body Problem(CS5BP)

Let us consider five bodies $P_{0}, P_{1}, P_{2}, P_{3}, P_{4}$ of masses $m_{0}, m_{1}, m_{2}, m_{3}, m_{4}$ respectively existing in three dimensional Euclidean space [6]. The radius and velocity vectors of the bodies with respect to the centre of mass of the five body system are given by $\mathbf{r}_{i}$ and $\dot{\mathbf{r}}_{i}$ respectively, $i=0,1,2,3,4$. Let the centre of mass of the system be denoted by $O$.

The CS5BP involves two types of symmetries; past-future symmetry and dynamical symmetry [8]. Past future symmetry exists in an n-body system when the dynamical evolution of the system after $t=0$ is a mirror image of the dynamical evolution of the system before $t=0$. It occurs whenever the system passes through a mirror configuration, i.e. a configuration in which the velocity vectors of all the bodies are perpendicular to all the position vectors from the centre of mass of the system [5].

Dynamical symmetry exists when the dynamical evolution of two bodies on one side of the centre of mass of the system is paralleled by the dynamical evolution of the two bodies on the other side of the centre of mass of the system. The resulting configuration is always a parallelogram, but of varying length, width and orientation [8]. See Figure 1 for the configuration of the CS5BP for $t>0$.

The CS5BP has the following conditions:

1. All five bodies are finite point masses with:

$$
\begin{equation*}
m_{1}=m_{3}, \quad m_{2}=m_{4} \tag{1}
\end{equation*}
$$

2. $P_{0}$ is stationary at origin $O$, the centre of mass of the system. $P_{1}$ and $P_{3}$ are moving symmetrically to each other with respect to the centre of mass of the system. Likewise $P_{2}$ and $P_{4}$ are moving symmetrically to each other. Thus dynamical symmetry is maintained for all time $t$;


Fig. 1. The configuration of the coplanar CS5BP for $t>0$

$$
\begin{gather*}
\mathbf{r}_{1}=-\mathbf{r}_{3}, \quad \mathbf{r}_{2}=-\mathbf{r}_{4}, \quad \mathbf{r}_{0}=0 \\
\mathbf{V}_{1}=\dot{\mathbf{r}}_{1}=-\dot{\mathbf{r}}_{3}, \quad \mathbf{V}_{2}=\dot{\mathbf{r}}_{2}=-\dot{\mathbf{r}}_{4}, \quad \mathbf{V}_{0}=\dot{\mathbf{r}}_{0}=0 . \tag{2}
\end{gather*}
$$

3. At time $t=0$ the bodies are collinear with their velocity vectors perpendicular to their line of position. This ensures the past-future symmetry and is described by:

$$
\begin{equation*}
\mathbf{r}_{1} \times \mathbf{r}_{2}=0, \quad \mathbf{r}_{1} \cdot \dot{\mathbf{r}}_{1}=0, \quad \mathbf{r}_{2} \cdot \dot{\mathbf{r}}_{2}=0 \tag{3}
\end{equation*}
$$

We define the masses as ratios to the total mass. Let the total mass $M$ of the system be

$$
\begin{equation*}
M=2\left(m_{1}+m_{2}\right)+m_{0} \tag{4}
\end{equation*}
$$

Let $\mu_{i}$ be the mass ratios defined as $\mu_{i}=\frac{m_{i}}{M}$ for $i=0,1,2,3,4$ and $\mu=\frac{\mu_{1}}{\mu_{2}}$. Equation (4) then becomes

$$
\begin{equation*}
2\left(\mu_{1}+\mu_{2}\right)+\mu_{0}=1 \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
0 \leq \mu_{0} \leq 1,0 \leq \mu_{1} \leq 0.5,0 \leq \mu_{2} \leq 0.5 \tag{6}
\end{equation*}
$$

## 3 The regularisation scheme

The proposed regularisation scheme consists of a combination of several known regularisation techniques: a Levi-Civita type coordinate transformation, a time transformation function similar to that of [1] and the global formulation of [3]. In general, the proposed scheme follows the transformations described in [4]. We extend the regularisation procedure of the CSFBP [11] into the case of the CS5BP.

Let the position coordinates of the four bodies in cartesian coordinates be $\mathbf{r}_{1}=\left(x_{1}, x_{2}\right), \mathbf{r}_{2}=\left(x_{3}, x_{4}\right), \mathbf{r}_{3}=\left(-x_{1},-x_{2}\right), \mathbf{r}_{4}=\left(-x_{3},-x_{4}\right)$, with corresponding momenta $\left(\omega_{1}, \omega_{2}\right)=\mu_{1} M\left(\dot{x_{1}}, \dot{x_{2}}\right),\left(\omega_{3}, \omega_{4}\right)=\mu_{2} M\left(\dot{x_{3}}, \dot{x_{4}}\right),\left(-\omega_{1},-\omega_{2}\right)$, $\left(-\omega_{3},-\omega_{4}\right)$.

For simplicity, we set the gravitational constant $G$ and total mass $M$ to be equal to unity. According to the symmetrical restrictions, the Hamiltonian function can be written as

$$
\begin{align*}
H & =\frac{1}{\mu_{1} M}\left(\omega_{1}^{2}+\omega_{2}^{2}\right)+\frac{1}{\mu_{2} M}\left(\omega_{3}^{2}+\omega_{4}^{2}\right)-2 G \mu_{1} \mu_{2} M^{2}\left(\frac{1}{r_{12}}+\frac{1}{r_{14}}\right)  \tag{7}\\
& -\frac{G \mu_{1}^{2} M^{2}}{r_{13}}-\frac{G \mu_{2}^{2} M^{2}}{r_{24}}-4 G \mu_{0} M\left(\frac{\mu_{1} M}{2 r_{13}}+\frac{\mu_{2} M}{2 r_{24}}\right)
\end{align*}
$$

where the corresponding inter-body distances are given by

$$
\begin{align*}
& r_{12}=\left(\left(x_{1}-x_{3}\right)^{2}+\left(x_{2}-x_{4}\right)^{2}\right)^{1 / 2}=r_{34}, \\
& r_{14}=\left(\left(x_{1}+x_{3}\right)^{2}+\left(x_{2}+x_{4}\right)^{2}\right)^{1 / 2}=r_{23} \\
& r_{13}=\left(\left(2 x_{1}\right)^{2}+\left(2 x_{2}\right)^{2}\right)^{1 / 2}, \\
& r_{24}=\left(\left(2 x_{3}\right)^{2}+\left(2 x_{4}\right)^{2}\right)^{1 / 2} \tag{8}
\end{align*}
$$

These four inter-body distances result in collision singularities which is characterised by the following four types of two-body close encounters [10].

1. "12"-type double binary collision: collisions occurring in the binary formed between $P_{1}$ and $P_{2}$ and the symmetrical binary formed between $P_{3}$ and $P_{4}$.
2. "14"-type double binary collision: collisions occurring in the binary formed between $P_{1}$ and $P_{4}$ and the symmetrical binary formed between $P_{2}$ and $P_{3}$.
3. "13"-type single binary collision: collision occurring in the binary formed between $P_{1}$ and $P_{3}$.
4. "24"-type single binary collision: collision occurring in the binary formed between $P_{2}$ and $P_{4}$.

Note that $P_{0}$ is stationary at $O$, the centre of mass of the system and thus $P_{0}$ has no influence in deciding the kinetic energy of the system and the collisions.

In order to regularise these singularities first we will map the $\left(x_{i}, \omega_{i}\right)$ physical plane into the ( $Q_{i}, P_{i}$ ) parametric plane using a series of transformation equations so that the new Hamiltonian function will have no singularities as it passes through a two-body close encounter. There are three important steps in the regularisation scheme [9].

## Step 1: Coordinate transformation

We first transform the coordinate system to inter-body coordinates.

$$
\begin{align*}
& q_{1}=x_{1}-x_{3}, \quad q_{2}=x_{2}-x_{4},  \tag{9}\\
& q_{3}=x_{3}+x_{1}, \quad q_{4}=x_{4}+x_{2},  \tag{10}\\
& q_{5}=2 x_{1}, \quad q_{6}=2 x_{2},  \tag{11}\\
& q_{7}=2 x_{3}, \quad q_{8}=2 x_{4} . \tag{12}
\end{align*}
$$

This will make sure that all the possible two-body close encounters in the CS5BP system are regularised [11].

We introduce a generating function $F_{1}\left(p_{k}, q_{k}\right)$ to obtain conjugate momenta $p_{k}$ of the corresponding $q_{k}$

$$
\begin{align*}
F_{1}\left(p_{k}, q_{k}\right) & =p_{k} q_{k}=\left(x_{1}-x_{3}\right) p_{1}+\left(x_{2}-x_{4}\right) p_{2}+\left(x_{3}+x_{1}\right) p_{3}+\left(x_{4}+x_{2}\right) p_{4} \\
& +2 x_{1} p_{5}+2 x_{2} p_{6}+2 x_{3} p_{7}+2 x_{4} p_{8} \tag{13}
\end{align*}
$$

which will give

$$
\begin{equation*}
\omega_{i}=\frac{\partial F_{1}}{\partial x_{i}} \tag{14}
\end{equation*}
$$

where $i=1$ to 4 and $k=1$ to 8 .
Next we find an expression for new momenta, $p_{k}$ 's, in terms of old momenta, $\omega_{i}$, using an arbitrary relation which is similar to that for the $q^{\prime}$ s (i.e. $q_{5}$ -$q_{7}-2 q_{1}=0, q_{5}+q_{7}-2 q_{3}=0, q_{6}+q_{8}-2 q_{4}=0, q_{6}-q_{8}-2 q_{2}=0$ ), we set

$$
\begin{gather*}
p_{5}-p_{7}-2 p_{1}=0 \\
p_{5}+p_{7}-2 p_{3}=0 \\
p_{6}+p_{8}-2 p_{4}=0  \tag{15}\\
p_{6}-p_{8}-2 p_{2}=0
\end{gather*}
$$

Using equation (14) and (15), we can deduce a set of new conjugate momenta $p$ 's as

$$
\begin{align*}
& p_{1}=\frac{1}{6}\left(\omega_{1}-\omega_{3}\right), \quad p_{2}=\frac{1}{6}\left(\omega_{2}-\omega_{4}\right), \\
& p_{3}=\frac{1}{6}\left(\omega_{1}+\omega_{3}\right), \quad p_{4}=\frac{1}{6}\left(\omega_{2}+\omega_{4}\right), \\
& p_{5}=\frac{1}{3} \omega_{1}, \quad p_{6}=\frac{1}{3} \omega_{2},  \tag{16}\\
& p_{7}=\frac{1}{3} \omega_{3}, \quad p_{8}=\frac{1}{3} \omega_{4} .
\end{align*}
$$

Now we perform the Levi-Civita type coordinate transformation on each inter-body coordinate. We introduce the regularising function using the LeviCivita transformation, in a complex form

$$
\begin{equation*}
q_{j}+i q_{j+1}=\left(Q_{j}+i Q_{j+1}\right)^{2} \tag{17}
\end{equation*}
$$

where $j=1,3,5,7$. Here note that $\left(q_{j}, q_{j+1}\right)$ refers to a physical plane and $\left(Q_{j}, Q_{j+1}\right)$ refers to a parametric plane. Their corresponding conjugate momenta $P_{k}$ 's are given by

$$
\begin{equation*}
P_{k}=\frac{\partial F_{2}\left(p_{k}, Q_{k}\right)}{\partial Q_{k}} \tag{18}
\end{equation*}
$$

where $k=1$ to 8 and $F_{2}\left(p_{k}, Q_{k}\right)$ is the generating function of the form

$$
F_{2}\left(p_{k}, Q_{k}\right)=p_{j} f\left(Q_{j}, Q_{j+1}\right)+p_{j+1} g\left(Q_{j}, Q_{j+1}\right)
$$

Using these relations, we can write

$$
\begin{align*}
P_{j} & =2 p_{j} Q_{j}+2 p_{j+1} Q_{j+1} \\
P_{j+1} & =-2 p_{j} Q_{j+1}+2 p_{j+1} Q_{j} \tag{19}
\end{align*}
$$

Step 2: Time transformation
In the next step, we introduce a fictitious time $\tau$, which is a key factor for the regularising effect. The basic principle of regularisation theory is to transform physical coordinates to a parametric plane and physical time to an artificial time by a differential time transformation, which consequently smooths collision effects in the Hamiltonian system. In the literature, we can find a variety of choices for the time transformation function which has a general form

$$
d t=g d \tau=R^{n} d \tau
$$

where $R$ is the separation between the colliding binaries, $g$ is the time re-scaling factor and $n$ has various choices according to the application. We had tried a few arbitrary values for $g$ and we found that, to preserve conservation of energy, it is advantageous to choose a time re-scaling factor of the form

$$
\begin{align*}
\frac{d t}{d \tau} & =g=\frac{r_{12} r_{13} r_{14} r_{24}}{\left(r_{12}+r_{13}+r_{14}+r_{24}\right)^{5 / 2}}  \tag{20}\\
& =\frac{\left(Q_{1}^{2}+Q_{2}^{2}\right)\left(Q_{3}^{2}+Q_{4}^{2}\right)\left(Q_{5}^{2}+Q_{6}^{2}\right)\left(Q_{7}^{2}+Q_{8}^{2}\right)}{\left(Q_{1}^{2}+Q_{2}^{2}+Q_{3}^{2}+Q_{4}^{2}+Q_{5}^{2}+Q_{6}^{2}+Q_{7}^{2}+Q_{8}^{2}\right)^{5 / 2}}
\end{align*}
$$

Step 3: Fixing the energy
With the introduction of the time rescaling factor, the new Hamiltonian $\tilde{H}\left(Q_{i}, P_{i}\right)$ takes the following form in the extended phase space

$$
\begin{equation*}
\Gamma\left(Q_{i}, P_{i}\right)=g\left(\tilde{H}-h_{0}\right) \tag{21}
\end{equation*}
$$

where $\Gamma$ is the transformed Hamiltonian $\tilde{H}\left(Q_{i}, P_{i}\right)$ in the extended phase space and $h_{0}$ is the total energy or the initial value of $\tilde{H}$. For any particular orbit, $\tilde{H}(\tau)=h_{0}$, a constant and $\Gamma(\tau)=0$. We have not shown the transformed Hamiltonian $\Gamma\left(Q_{i}, P_{i}\right)$ in this paper, as the right hand side of the expression is very lengthy due to a large number of multiplicative terms. The numerator terms in the time rescaling factor $g$ cancel out the singular terms in the denominator of the Hamiltonian function and prevent the increase of the velocity to infinity at the collision stages.

We can derive the Hamilton equations of motion with respect to the fictitious time, using this transformed Hamiltonian in the new set of parametric coordinates:

$$
\begin{align*}
\frac{d Q_{i}}{d \tau} & =\frac{\partial \Gamma}{\partial P_{i}}  \tag{22}\\
\frac{d P_{i}}{d \tau} & =-\frac{\partial \Gamma}{\partial Q_{i}}
\end{align*}
$$

Equation (22) is the final regularised equation of motion, which is a set of ordinary differential equations whose solution is a function of the fictitious time $\tau$ and these equations are regular, for any $q_{i} \rightarrow 0$.

There can be singularities when all $q_{i} \rightarrow 0$, where $i=1$ to 8 . This situation is only possible for a CS5BP system with $C_{0}=0$. This corresponds to a singularity at the origin in the physical plane. For $C_{0} \neq 0$; regions of forbidden motion appear very close to the origin and a total central collision is not theoretically possible.

## 4 Optimisation of the regularised Hamiltonian

An optimisation strategy is not generally required for restricted few-body problems for $n<4$, since the equations of motion derived using standard regularisation schemes usually contain algebraic terms which can be easily handled by most of the standard numerical integrators. However, the transformed Hamiltonian $\Gamma\left(Q_{i}, P_{i}\right)$ in Equation (21) is determined using a large number of algebraic multiplications. It is evident that the symbolic differentiation to derive the gradient of $\Gamma\left(Q_{i}, P_{i}\right)$ will produce a large number of additive and multiplicative terms, leading to an inefficient evaluation of the right hand side of the Equation (22). The direct numerical integration of the regularised Equation (22) (i.e. without using any optimisation techniques) required an excessive amount of computational time even for a very small time period of 10 due to the large number function evaluations involved.

We adapt an algebraic optimisation algorithm of [2] to simplify the Equation 22. The first step in the optimisation process is to rewrite the regularised Hamiltonian $\Gamma\left(Q_{i}, P_{i}\right)$ in terms of the most frequently appearing terms as a MAPLE procedure [9]. Then we split up the product terms in the MAPLE procedure in calculating the regularised Hamiltonian to avoid the generation of common subexpressions while computing its partial derivatives [2].

We also make use of the reverse-mode algorithmic differentiation to reduce the total number of multiplicative operations (multiplication and addition) to derive the partial derivatives of the regularised Hamiltonian $\Gamma\left(Q_{i}, P_{i}\right)$. The reverse-mode of automatic differentiation allows computation of gradients at a small cost of computing functions by decomposing the function into a sequence of elementary assignments. The forward-mode differentiation of $\Gamma\left(Q_{i}, P_{i}\right)$ will generate more than 2100 multiplicative terms, whereas the reverse mode algorithmic differentiation leads to a procedure with only about 320 multiplications. Then we convert repeating symbolic expressions into computation sequences needed for the algorithmic differentiation using the built-in MAPLE functions. In general, this algebraic optimisation procedure can be extended to majority of the global regularisation schemes used in gravitational few-body problems (with $n \geq 3$ ) and fast numerical realization can be achieved.

## 5 Numerical experiments

We show some preliminary numerical results using the non-regularised and regularised integration schemes for a regular quasi-periodic orbit. The initial conditions for integrating equation (20) and (22) were fixed using the energy and angular momentum equations of the CS5BP. Numerical experiments were


Fig. 2. A quasi-periodic orbit over the time $[0,20]\left(\mu=1, \mu_{0}=0, E=-7, C_{0}=60\right.$ initial $\mathbf{r}_{1}=0.80$ and $\mathbf{r}_{2}=0.06$ ); with a) non-regularised; b) regularised equations. I. Trajectories of $P_{1}$ (green) and $P_{2}$ (blue) in the $x y$-plane of motion; II. Energy error over the time period $[0,20]$


Fig. 3. Time step variations over the time $[0,20]$ with a) non-regularised; b) regularised equations.
conducted using the standard MATLAB multi-step integrator ode113 which is a variable order Adams-Bashforth-Moulton PECE solver. The orbital trajectories in the xy-plane of motion are shown in Figure 2.1.A central binary is formed (with $P_{2}$ and $P_{4}$ ) and the other symmetrical pair $P_{1}$ and $P_{3}$ orbit around the binary's centre of mass. Only the positions of masses $m_{1}\left(x_{1}, x_{2}\right)$ (green) and $m_{2}\left(x_{3}, x_{4}\right)$ (blue) are shown. The orbits are well separated and remained bounded for some reasonable amount of integration time.

Figure 2.II shows the numerical energy error versus time over a 20 time unit period. Although the orbital trajectories appear to be identical, the regularised integration scheme exhibits a better energy error profile by a factor of 100. Figure 3 shows the corresponding time step variations for the above integrations. The regularised integration scheme has improved the CPU workload

[h]
Fig. 4. Comparisons of the errors with variable absolute tolerance error and relative tolerance error; green (non-regularised) and blue (regularised).
by a factor of 1.4 by allowing the integrator to choose bigger step-sizes resulting in decreased number of time steps. Figure 4 shows a comparison between the CPU time and the maximum observed energy error for the given simulation time. It is clear that the regularised scheme allows better accuracy with improved CPU run time. Despite the regularity of the orbit and the absence of extreme close encounters, our numerical tests indicate that the overall CPU workload has been improved. The computational cost involved in each time step differs for both the non-regularised and regularised integrations, since the regularised scheme has twice as many equations in the non-regularised scheme and it involved a large number of algebraic multiplications and additions due to several coordinate transformations forward and backwards. The regularised treatment combined with the algebraic optimisation scheme outperforms the non-regularised approach in terms of computational efficiency and numerical accuracy.

## 6 Conclusions

We developed a global regularisation scheme that consists of adapted versions of several known regularisation transformations such as the Levi-Civita-type coordinate transformations; that together with a time transformation, removes all the singularities due to colliding pairs of masses in the CS5BP. An algebraic optimisation algorithm is proposed for numerically implementing the regularisation scheme. Regardless of the nature of the orbits, it was found that the regularised integration scheme outperformed the standard non-regularised integration schemes in terms of computational performance and improved numerical accuracy characterized by stable energy profiles.

## 7 Acknowledgement

The authors are grateful to Prof. Bonnie Steves (Glasgow Caledonian University, Scotland) and Dr Winston Sweatman (Massey University, New Zealand)
for many enlightening discussions that motivated our work. The basic outline of the regularisation algorithm was generously provided by Dr Winston Sweatman.

## References

1.S. J. Aarseth and K. Zare. A regularisation of the three body problem. Celestial Mechanics 10:185-205, 1974.
2.D. Gruntz and J. Waldvogel. Orbits in the Planar Three-Body Problem, In W. Gander and J. Hrebicek (eds) Solving Problems in Scientific Computing Using Maple and Matlab, pp 51-72, 1997.
3.D. C. Heggie. A Global Regularisation of the Gravitational N-Body Problem. Celestial Mechanics 10:217-241, 1974.
4.D. C. Heggie and W. L. Sweatman. Three-body scattering near triple collision or expansion. Monthly Notices of Royal Astronomical Society 250:555-575, 1991.
5.A. E. Roy and B. A. Steves. The Caledonian Symmetrical Double Binary Four-Body Problem: Surfaces of zero velocity using the energy integral. Celestial Mechanics and Dynamical Astronomy 78:299-318, 2000.
6.M. Shoaib. Many Body Symmetrical Dynamical Systems. Glasgow Caledonian University, 2004.
7.M. Shoaib and I. Faye. Collinear equilibrium solutions of four-body problem. Journal of Astrophysics and Astronomy. 32:411-423, 2011.
8.M. Shoaib, B. A. Steves and A. Széll. Stability analysis of quintuple stellar and planetary systems using a symmetric five-body model. New Astronomy 13:639645, 2008.
9.A. Sivasankaran. Stability of the Caledonian Symmetric Four-Body Problem with close encounters. Glasgow Caledonian University, 2010.
10.A. Sivasankaran and B. A. Steves and W. L. Sweatman. Close encounters in the the Caledonian Symmetric Four-Body Problem. In H. Varvoglis and S. Ferraz-Mello (eds). Proceedings of the International Conference on the Dynamics of Celestial Bodies, Publication of Astronomical Observatory of Belgrade pp 57-61, 2009.
11.A. Sivasankaran and B. A. Steves and W. L. Sweatman. A global regularisation for integrating the Caledonian Symmetric Four-Body Problem. Celestial Mechanics and Dynamical Astronomy 107:157-168, 2010.
12.B. A. Steves and A. E. Roy. Surfaces of separation in the Caledonian Symmetrical Double Binary Four-Body Problem, In B. A. Steves and A. J. Maciejewski (eds) The Restless Universe: Application of Gravitational N-Body Dynamics to Planetary, Stellar and Galactic systems, IOP Publishing, pp 301-325, 2001.
13.W. L. Sweatman. The symmetrical one-dimensional Newtonian four-body problem: A numerical investigation. Celestial Mechanics and Dynamical Astronomy 82:179-201, 2002.
14.W. L. Sweatman. A family of Schubart-like interplay orbits and their stability in the one-dimensional four-body problem. Celestial Mechanics and Dynamical Astronomy 94:37-65, 2006.
15.A. Széll and B. A. Steves and B. Érdi. The hierarchical stability of quadruple stellar and planetary systems using the Caledonian Symmetric Four-Body Problem. Astronomy and Astrophysics, 427:1145-1154, 2004.
16.A. Széll and B. Érdi and Z. Sándor and B. Steves. Chaotic and stable behavior in the Caledonian Symmetric Four-Body Problem. Monthly Notices of Royal Astronomical Society, 347:380-388, 2004.

# Pattern Formation Dynamics in Diverse PhysicoChemical Systems 

Tony Karam, Houssam El-Rassy, Victor Nasreddine, Farah Zaknoun, Samia El-Joubeily, Amal Zein Eddin, Hiba Farah, Jad Husami, Samih Isber $\dagger$ and Rabih Sultan*<br>Departments of Chemistry and $\dagger$ Physics, American University of Beirut, Beirut, Lebanon<br>E-mail: rsultan@aub.edu.lb


#### Abstract

Complex reaction-transport dynamics can lead to the formation of ordered structures. A constant dissipation of free energy is a requirement for sustaining macroscopic order, especially in solution. In the solid phase, the evolved pattern can be locked for days, months or even years. Liesegang bands are stratified stripes of precipitate that appear and persist, when co-precipitate ions interdiffuse in a gel medium. A host of interesting properties characterize such rich dynamical systems: band spacing laws (direct and revert), band splitting, rhythmic multiplicity, multiple precipitate formation and band redissolution are but a few manifested characteristics, emerging from a complex dynamics with a great diversity of scenarios. The familiar and well-known band formation in rocks could be the result of a complex coupled diffusion-percolation-chemical reaction mechanism. Similarities between geochemical self-organization and the Liesegang phenomenon are surveyed and analyzed. The simulation of band generation in a rock bed is realized and carried out insitu, by injection and infusion of the reagent components into the rock medium. Ramified, tree-like structures (dendrites) are obtained during the electrodeposition or simple electroless redox deposition of metal systems. A great variety of morphologies just resembling tree branches are observed and characterized as fractal structures. Keywords: Liesegang, dendrites, reaction-diffusion, rock banding.


## 1. Liesegang Banding

In 1896, Raphael Eduard Liesegang discovered an intriguing phenomenon [1] whereby precipitation in a gel medium takes place in banded form, just like the superb display of bands that we commonly observe in rocks [2-4]. Various specimens of Liesegang patterns, prepared for different precipitates, are shown in Fig. 1.


Figure 1: A panorama of colorful Liesegang patterns in gel.
In the laboratory, the Liesegang experiment [5-7] is quite simple: a concentrated electrolyte containing a certain co-precipitate ion ( $s a y \mathrm{~Pb}^{2+}$ ) is allowed to diffuse into a gel containing its insoluble salt counterpart (such as $\mathrm{I}^{-}$to form $\mathrm{PbI}_{2}$ ); normally one order of magnitude more dilute. Due to the coupling of diffusion to a cycle of supersaturation, nucleation and depletion, known as the Ostwald cycle [8], the precipitation takes place in the form of beautifully stratified bands, as displayed in Fig. 1.
We highlight the main features of such a rich dynamical phenomenon, but also shed light on abnormalities, curiosities and strange behavior exhibited by such systems under certain conditions. The observations common to most Liesegang systems are summarized by the four well-known empirical laws [9,10]:

| Time law: | $x_{n}=\sqrt{a t_{n}}$ | Spacing law: $\rho_{n}=\frac{x_{n+1}}{x_{n}} \rightarrow 1+p$ |
| :--- | :--- | :--- |
|  | as as $n$ is large. |  |

where $n$ denotes band number, $x$ is location and $w$ is band width. The spacing law formula suggests that the spacing between consecutive bands increases as we move away from the electrolytes junction. Although $90 \%$ of the Liesegang patterns follow this so-called Jablczynski spacing law [11], some systems exhibit an opposite trend, known as revert spacing [12,13]. The distinction between direct and revert spacing Liesegang patterns is depicted in Fig. 2.


Figure 2: a. Liesegang pattern of $\mathrm{CuCrO}_{4}$ showing direct (normal) spacing. b. Plot of fraction of adsorbed $\mathrm{CrO}_{4}{ }^{2-}$ on the copper chromate precipitate $(h)$ with band number. c. Liesegang patterns of $\mathrm{PbCrO}_{4}$ showing revert spacing. d. Plot of fraction of adsorbed $\mathrm{CrO}_{4}{ }^{2-}$ on the lead chromate precipitate ( $f$ ) with band number. We see that $h$ decreases while $f$ increases.

In a recent study [13], we showed that the fraction of $\mathrm{CrO}_{4}{ }^{2-}$ adsorbed ( $f$ ) on the lead chromate precipitate increases with band number $n$ (see Fig. 2d); whereas the opposite trend was observed for the adsorption on copper chromate (the fraction $h$ decreases with band number $n$; as seen in Fig. 2b). Hence the increased extent of adsorption causes the bands to form closer and closer as $n$ increases. It seems that more $\mathrm{CrO}_{4}{ }^{2-}$ adsorbed attract the $\mathrm{Pb}^{2+}$ in the gel closer than in the preceding band, thus causing the precipitate band to from closer, and the spacing to become narrower. The opposite behavior (decreasing extent of adsorption with band number as in Fig. 2b) results in a normal Liesegang pattern with direct spacing (Figure 1a).
Liesegang systems exhibit a great diversity of special features. A pattern of bands seemingly 'migrates' if redissolution of the bands at the top is synchronized with the band formation. Such scenario occurs in systems where the precipitate redissolves to form a complex ion. Typical studied examples include the $\mathrm{Co}(\mathrm{OH})_{2}$ [14,15], $\mathrm{Cr}(\mathrm{OH})_{3}$ [16] and $\mathrm{HgI}_{2}$ [17] systems. When $\mathrm{Co}(\mathrm{OH})_{2}$ is precipitated from $\mathrm{Co}^{2+}$ and $\mathrm{NH}_{4} \mathrm{OH}$, the precipitate redissolves in excess $\mathrm{NH}_{4} \mathrm{OH}$ to form the hexaammine cobalt (II) complex ion, $\mathrm{Co}\left(\mathrm{NH}_{3}\right)_{6}{ }^{2+}$, according to the reaction:

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

$$
\mathrm{Co}(\mathrm{OH})_{2}(s)+6 \mathrm{NH}_{4}^{+}(a q) \longrightarrow \mathrm{Co}\left(\mathrm{NH}_{3}\right)_{6}{ }^{2+}(a q)+4 \mathrm{H}^{+}(a q)+2 \mathrm{H}_{2} \mathrm{O}
$$


a.
c.

b.



Figure 3: a. Propagating $\mathrm{Co}(\mathrm{OH})_{2}$ Liesegang pattern via a concerted band formation and band redissolution scenario. b. Correlation plot showing the linear correlation between the distance of last band $(d l b)$ and distance of first band $(d f b)$. c. Plot of $d f b$ versus time. d. Plot of $d l b$ versus time. The two parameters are controlled by diffusion.

The precipitation-redissolution-propagation of the $\mathrm{Co}(\mathrm{OH})_{2}$ pattern of bands is illustrated in Fig. 3a. The distance of the top edge of the propagation zone (dfb) and the distance of the last band $(d l b)$ are plotted versus time in days. The plots are shown in Figs. 3c and 3d. We see that the propagation at the top and the bottom is dominated by diffusion. The correlation between $d f b$ and $d l b$ is almost perfectly linear [14], as revealed by the correlation plot in Fig. 3b.
A host of other diverse features are observed in Liesegang systems. To name but a few, we report secondary banding [18], spiral and helicoidal patterns [19] and two-precipitate dynamics [20].

## 2. Geochemical Banding

Perhaps the most common and most spread resemblance between Liesegang patterns and natural phenomena is the landscape of bands that we observe in rocks $[21,2,3]$. Many studies have emphasized such similarity, presented
coherent explanations and proposed mechanisms. Theoretical modeling studies are extensive in the literature [21]. Possible scenarios range from cyclicity in large mafic-ultramafic layered intrusions, to fractional crystallization in magmatic processes, to temperature-pressure changes in both first and second-order phase transitions, to nonlinear reaction-diffusion dynamics.
In a recent work, we attempted to simulate geochemical banding (or selforganization) in-situ, i.e. inside the rock bed [22,23].


Figure 4: Liesegang bands in a rock bed behind a reaction front. The infiltrating water carries a co-precipitate ion that meets its counter ion in the rock medium and thus precipitation takes place; but it does so but in banded form, just resembling a Liesegang pattern.

Consider a porous rock infiltrated from one side by an inlet flow of reactive water, that causes the dissolution of certain constituent rock minerals. The water flow, acting as a sink of co-precipitate ions for the altered rock, can provoke the precipitation and deposition of other insoluble minerals. In many such situations, the minerals deposition occurs in banded form, in a way that just resembles the Liesegang bands obtained in a lab experiment. Such a plausible scenario is illustrated in Fig. 4.
In the lab, a ferruginous limestone rock with a planar surface (Figure 5) was infiltrated through a thin tube inserted at its center by a $4.30 \mathrm{M} \mathrm{H}_{2} \mathrm{SO}_{4}$ solution by means of a multi-rate infusion pump. The acid causes the dissolution of calcite $\left(\mathrm{CaCO}_{3}\right)$ and the precipitation of the acid-insoluble gypsum $\left(\mathrm{CaSO}_{4}\right)$ and anhydrite $\left(\mathrm{CaSO}_{4} .2 \mathrm{H}_{2} \mathrm{O}\right)$ according to the reaction:

$$
\mathrm{CaCO}_{3}+\mathrm{H}_{2} \mathrm{SO}_{4}(a q) \longrightarrow \mathrm{CaSO}_{4}+\mathrm{CO}_{2}+\mathrm{H}_{2} \mathrm{O}
$$

Due to the spatio-temporal flow, the deposition of $\mathrm{CaSO}_{4}$ is anticipated to occur in banded form in accordance with the above described Liesegang dynamics (Sect. 1).
The experiment was kept running for about two years (692 days). The appearance of the various banded zones at $t=202$ days is depicted in Fig. 5a. The latter were delineated and labeled by tracing contours defining the inner and outer edges of each zone (Figure 5 b at 692 days). The gypsum/anhydrite content of regions 1 through 7 of Fig. 5 b was determined by powder X-ray diffraction. The results are shown in Table 1.

Table 1: $\mathrm{CaSO}_{4}$ composition over the various zones of Fig. 5

| Region | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\%$ CaSO $_{4}$ | 100 | 97.5 | 98.9 | 28.4 | 85.8 | 17.6 | 5.4 |



Figure 5: Acidization of a ferruginous limestone rock, by slow injection of $\mathrm{H}_{2} \mathrm{SO}_{4}$ at the center causing the dissolution of calcite $\left(\mathrm{CaCO}_{3}\right)$. The front is accompanied by the deposition of gypsum $\left(\mathrm{CaSO}_{4}\right)$ and anhydrite $\left(\mathrm{CaSO}_{4} \cdot 2 \mathrm{H}_{2} \mathrm{O}\right)$. a. At $t=202$ days. b. 'Concentric' deposition zones exhibiting oscillation in the $\mathrm{CaSO}_{4}$ content at $t=692$ days.

We clearly see that beyond the central region where the deposition of $\mathrm{CaSO}_{4}$ is maximal (bands 1-3), the $\mathrm{CaSO}_{4}$ content starts oscillating.
Very few other simulations of rock banding in-situ were attempted by a number of investigators. Rodriguez-Navarro et al. [24] observed Liesegang rings by monitoring the slow carbonation of traditional, aged lime mortars. A portlandite $\left[\mathrm{Ca}(\mathrm{OH})_{2}\right] /$ quartz mortar kept for a long time under excess, $\mathrm{CO}_{2}$-rich water gives rise to a calcite $\left(\mathrm{CaCO}_{3}\right)$ deposit, via the reaction:

$$
\mathrm{Ca}(\mathrm{OH})_{2}+\mathrm{CO}_{2}(a q) \longrightarrow \mathrm{CaCO}_{3}+\mathrm{H}_{2} \mathrm{O} .
$$

The carbonation process yields 3D Liesgeang patterns consisting of concentric ellipsoids of alternating calcite and calcite-free zones. The rings exhibit revert spacing instead of direct spacing and obey Jablczynski's spacing law. The revert
nature of the pattern was attributed to the decrease in $\mathrm{CO}_{2}$ uptake and diffusion as the process progresses toward the core.

## 3. Dendritic Metal Deposits

Another intriguing class of pattern formation in solid structures is the ramified, tree-like structures we observe in metal deposits $[25,26]$. Two routes are known for obtaining metal deposits: electrolytic and electroless. In the former, metal ions are reduced by standard electrolysis at the cathode. In the latter, a spontaneous redox reaction is carried out in the supporting medium. We perform such a study on Ag metal deposits, by growing the latter via both methods.

## Electroless

Silver metal was deposited by reduction of $\mathrm{Ag}^{+}$with metallic copper according to the following scheme:

$$
\begin{array}{lll}
\text { Oxidation: } & \mathrm{Cu} \rightarrow \mathrm{Cu}^{2+} 2 e^{-} & E_{\mathrm{Cu}^{2+} / \mathrm{Cu}}^{0}=+0.34 \mathrm{~V} \\
\text { Reduction: } & \mathrm{Ag}^{+}+e^{-} \rightarrow \mathrm{Ag} & E_{\mathrm{Ag}^{+} / \mathrm{Ag}_{g}}=+0.80 \mathrm{~V}
\end{array}
$$

The overall reaction is:

$$
\mathrm{Cu}+2 \mathrm{Ag}^{+} \rightarrow 2 \mathrm{Ag}+\mathrm{Cu}^{2+}
$$

To that end, a shallow methacrylate glass (plexiglass) dish of 10.5 cm diameter was manufactured, mounted with a peripheral ring of 0.3 mm height acting as a spacer, on top of which a plexiglass cover can rest. The solution layer thickness will thus be 0.3 mm . The cover has a 1.50 cm hole, wherein a well-fitted metallic disc (here Cu ) can be inserted.
With the perforated cover on, a solution of silver nitrate of known concentration (say 0.10 M ), was carefully poured through the cover hole, until it spread evenly and without air bubbles throughout the dish area. Once such a thin solution film is achieved, the copper disk is placed at the center, marking the start of the spontaneous reaction (1). One important variant from other electroless growth experiments is the bare solution medium, without soaking in a filter paper to lock the pattern. After big experimental challenges, the preliminary appearance of the fractal growth (seemingly promising) is displayed in Fig. 6.


Figure 6: Silver deposits showing dendritic structure growth. a. Circular disc of reductant $(\mathrm{Cu})$. b. Square Cu disc.

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

An interesting observation is that the ramifications display straight, stringy branches in the circular core, whereas they exhibit curved branching with the square core. Different regions of the Ag deposits were cut, and the images transformed into black and white, for good contrast. Samples are depicted in Fig. 7.


Figure 7: Selected regions from the deposits in Fig. 6a after transformation of the image to black and white. The three regions (a-c) essentially exhibit the same value of the fractal dimension.

The dendrites exhibited a fractal dimension of $1.58 \pm 0.04$.

a.
b.

Figure 8: a. Ag deposits by electrolysis in a circular field with potential difference of 3.09 V . b. Ag deposits via reduction of $\mathrm{Ag}^{+}$by Cu in a horizontal magnetic field of 0.50 T.

## Electrolysis

Figure 8a shows a 'rosette' obtained by electrodeposition at a graphite electrode immersed in the solution at the dish center. The anode is a circular tungsten wire electrode of 0.5 mm diameter thickness.

Figure 8 b displays electroless Ag deposits from the reduction of $\mathrm{Ag}^{+}$by metallic Cu , in the presence of a horizontal magnetic field of 0.50 T applied across the dish. The striking differences in the morphology reveal the importance,
complexity and rich dynamics of metal deposition and growth. These observations are under continuing investigation at the present time.
Other dynamical studies of complex fractal structure in metal deposition systems include the simultaneous growth of two metals [27,28] and the effect of electric [29] and magnetic fields [30,31] in electroless and electrolytic systems.

Acknowledgements: This work was supported by the University Research Board (URB) of the American University of Beirut (AUB); and the Lebanese National Council for Scientific Research (LNCSR).

## References

1. R. E. Liesegang. Chemische Fernwirkung. Lieseg. Photograph. Arch. 1896, 37:305; continued in 37:331, 1896.
2. R. Sultan and A. Abdel-Rahman. On Dynamic Self-Organization: Examples from Magmatic and Other Geochemical Systems. Latin American Journal of Solids and Structures (LAJSS) 10:59-73, 2013.
3. Samar Sadek and Rabih Sultan. Liesegang Patterns in Nature: A Diverse Scenery Across the Sciences, a review paper. In: Lagzi I. (ed) Precipitation Patterns in Reaction-Diffusion Systems. Research Signpost publications, Trivandrum, Chapter 1, pp. 1-43. 2011.
4. J. H. Kruhl (ed). Fractals and Dynamic Systems in Geoscience, Springer-Verlag, Berlin, 1994.
5. R. A. Schibeci and C. Carlsen. An Interseting Student Chemistry Project: Investigating Liesegang Rings. J. Chem. Educ. 65:365-366, 1988.
6. H. K. Henisch. Crystals in Gels and Liesegang Rings, Cambridge University Press, Cambridge, 1988.
7. R. Sultan and S. Sadek. Patterning Trends and Chaotic Behavior in $\mathrm{Co}^{2+} / \mathrm{NH}_{4} \mathrm{OH}$ Liesegang Systems. J. Phys. Chem.100:16912-16920,1996.
8. Wi. Ostwald, Lehrbuch der Allgemeinen Chemie, 2. Aufl., Band II, 2. Teil: Verwandtschaftslehre, Engelmann, Leipzig, 1899, p. 779.
9. T. Antal, M. Droz, J. Magnin, Z. Rácz and M. Zrinyi. Derivation of the MatalonPackter law for Liesegang patterns. J. Chem. Phys. 109:9479-9486, 1998.
10. Ferenc Izsák and István Lagzi, A New Universal Law for the Liesegang Pattern Formation. J. Chem. Phys. 122:184707-184713, 2005.
11. C. K. Jablczynski. La formation Rythmique des Précipités: Les Anneaux de Liesegang. Bull. Soc. Chim. Fr. 33:1592, 1923.
12. N. Kanniah, F. D. Gnanam and P. Ramasamy. Proc. Indian Acad. Sci., Chem. Sci. 93:801-811, 1984.
13. T. Karam, H. El-Rassy and R. Sultan. Mechanism of Revert Spacing in a $\mathrm{PbCrO}_{4}$ Liesegang System. J. Phys. Chem. A, 115:2994-2998, 2011.
14. V. Nasreddine and R. Sultan. Propagating Fronts and Chaotic Dynamics in $\mathrm{Co}(\mathrm{OH})_{2}$ Liesegang Systems. J. Phys. Chem. A 103:2934-2940, 1999.
15. R. Sultan. Propagating Fronts in Periodic Precipitation Systems with Redissolution. Phys. Chem. Chem. Phys. (PCCP) 4:1253-1261, 2002.
16. M. Zrínyi, L. Gálfi, É. Smidróczki, Z. Rácz and F. Horkay. Direct Observation of a Crossover from Heterogeneous Traveling Wave to Liesegang Pattern Formation. J. Phys. Chem. 95:1618-1620, 1991.
17. Das, A. Pushkarna and N. R. Argawal. Chemical Waves and Light-Induced Spatial Bifurcation in the Mercuric Chloride-Potassium Iodide System in Gel Media. J. Phys. Chem. 93:7269-7275, 1989.
18. R. Feeney, P. Ortoleva, P. Strickholm, S. Schmidt and J. Chadam. Periodic Precipitation and Coarsening Waves: Applications of the Competitive Particle Growth Modela. J. Chem. Phys. 78:1293-1311, 1983.
19. D. S. Chernavskii, A. A. Polezhaev and S. C. Müller. A model of Pattern Formation by Precipitation. Physica D 54, 160-170, 1991.
20. R. Sultan, N. Al-Kassem, A. Sultan and N. Salem. Periodic Trends in Precipitate Patterning Schemes Involving Two Salts. Phys. Chem. Chem. Phys. (PCCP) 2, 3155-3162, 2000.
21. P. Ortoleva. Geochemical Self-Organization, Oxford University Press, New York, 1994.
22. M. Msharrafieh, M. Al-Ghoul and R. Sultan. Simulation of Geochemical Banding in Acidization-Precipitation Experiments In-Situ. In: M. M. Novak (ed) Complexus Mundi: Emergent Patterns in Nature. World Scientific, Singapore, pp. 225-236, 2006.
23. F. Zaknoun, H. El-Rassi, M. Al-Ghoul, S. Al-Joubeily, Th. Mokalled and R. Sultan. Simulation of Geochemical Self-Organization: Acid Infiltration and Mineral Deposition in a Porous Ferruginous Limestone rock. In: C. H. Skiadas and I. Dimotikalis (eds) Chaotic Systems: Theory and Applications. World Scientific, pp. 385-394, 2010.
24. C. Rodriguez-Navarro, O. Cazalia, K. Elert and E. Sebastian. Liesegang Pattern Development in Carbonating Traditional Lime Mortars. Proc. R. Soc. Lond. A 458:2261-2273, 2002.
25. M. Matsushita, M. Sano, Y. Hayakawa, H. Honjo and Y. Sawada. Fractal Structures of Zinc Metal Leaves Grown by Electrodeposition. Phys. Rev. Lett. 53:286-289, 1984.
26. R. Saab and R. Sultan. Density, Fractal Angle and Fractal Dimension in Linear Zn Electrodeposition Morphology. J. Non-Equilib. Thermodyn. 30:321-336, 2005.
27. E. Nakouzi and R. Sultan. Fractal Structures in Two-Metal Electrodeposition Systems I: Pb and Zn. Chaos 21:043133, 2011.
28. E. Nakouzi and R. Sultan. Fractal Structures in Two-Metal Electrodeposition Systems II: Cu and Zn . Chaos 22:023122, 2012.
29. C. M. Cronemberger and L. C. Sampaio. Growth of Fractal Electrochemical Aggregates Under electric and Magnetic Fields Using a Modified Diffusion-Limited Aggregation Algorithm Phys. Rev. E 73:041403, 2006.
30. I. Mogi, M. Kamiko and S. Okubo. Magnetic Field Effects on Fractal Morphology in Electrochemical Deposition. Physica B 211:319-322, 1995.
31. Y. Tanimoto, A. Katsuki, H. Yano and Sh. Watanabe. Effect of High Magnetic Field on the Silver Deposition from its Aqueous Solution. J. Phys. Chem. A 101:7359-7363, 1997.

# Stochastic synchronization of chaotic recurrent neural networks with time varying delays using adaptive feedback control 

R.Suresh ${ }^{1}$<br>Department of Mathematics<br>Vel-Tech DR. RR DR.SR technical University<br>Avadi - 600062, Tamilnadu, India


#### Abstract

The asymptotic mean square stability of stochastic synchronization of chaotic recurrent neural networks with time varying delays using adaptive feedback control is analyzed. In this paper, by utilizing tracking controller,the adaptive synchronization control is designed. Which ensures that the synchronization error of chaotic recurrent neural network system is asymptotically mean square stable.


Key Words: Asymptotic mean square stable,Lyapunov functional, adaptive feed back control.

## 1. Introduction

Chaos occurs in dynamics nonlinear deterministic systems. The causes that determine the occurrence of chaos are not fully known. There are three possible causes identified for determine the occurrence of chaos; first one is the growth of control factor to a high value, second one is the nonlinear interaction of two or more physical operations and another one is the effect of noise presence.A chaotic process is highly sensitive to the variations of the parameters describing the initial state [13, 14]. For example any change of the initial conditions of today's atmosphere conditions will causes of major change of the tomorrow's atmosphere conditions. A chaotic process is initially generated in deterministic system yet its evolution in time is apparently random [11]. A chaotic system dimension in the phase state is characterized by the value called Lyapunov exponent. A chaotic process will have at least one positive Lyapunov exponent. Its magnitude versus time indicates the starting movement from which the process become chaotic. A negative Lyapunov exponent indicates how rapidly the system restores its initial state after a perturbation.

[^1]A neural network is a mathematical model relaying on the model of biological neurons. A neural networks can acquire knowledge from the environment through a learning process and the inter neuron connection strength is used to store the knowledge. Artificial neural networks offer qualitative methods for business and economic systems. Chaotic dynamics have been observed in many artificial neural systems either in a continuous systems $[1,12,15]$ or discrete systems $[2,7,8,16]$.The idea of chaotic synchronizing of two independent copies of identical chaotic dynamical systems have been increasing recent interest. Chaotic synchronization plays a crucial role in information processing in living organisms and could lead to important applications in speech and image processing. Moreover due to the important role that secure communications plays in industrial and banking communications, the potential application of neuro chaotic synchronization to secure communications is receiving increased attention. Recently the synchronization of chaotic recurrent neural networks with time varying delays using adaptive feedback control was proposed [3, 10] .In this paper, synchronization of noise-perturbed synchronization of chaotic recurrent neural networks with time varying delays using adaptive feedback control is analytically investigated.In this paper we organize a synchronization strategy by noise-perturbed synchronization of chaotic recurrent neural networks with time varying delays using adaptive feedback control in section 2 . In section 3 we present control design and outline of stochastic synchronization in recurrent neural network with time varying delays. Finally concluding remarks and references close the paper.

## 2. The Synchronization Description and preliminaries

Consider the chaotic neural network

$$
\begin{equation*}
d x(t)=[-c x(t)+A f(x(t))+B f(x(t))+B f(x(t-\tau(t))+J] d t \tag{1}
\end{equation*}
$$

where $x(t)=\left(x_{1}(t), x_{2}(t), x_{3}(t), \ldots, x_{n}(t)\right)^{T} \in \mathbb{R}^{n}$ is the state vector of the neural network; C is a diagonal matrix with $c_{i}>0, i=1,2,3, . . n ., A=\left(a_{i j}\right)_{n \times n}$ is a weight matrix; $B=\left(b_{i j}\right)_{n \times n}$ is the delayed weight matrix; $J=\left(J_{1}, J_{2}, \ldots, J_{n}\right)^{T} \in \mathbb{R}^{n}$ is the input vector function; $\tau(t)$ is the transmission delay; $f(x(t))$ is the activation function.

In order to obtain our main results,assume the following condition hold.
$(A 1)$ The activation function $f$ is bounded and satisfy the Lipschitz condition

$$
\left|f\left(x_{1}\right)-f\left(x_{2}\right)\right| \leq k_{j}\left|x_{1}-x_{2}\right|, \quad \forall x_{1}, x_{2} \in \mathbb{R}
$$

$\left(A_{2}\right) \cdot \tau(t) \geq 0$ is a differential function with $\tau^{*}=\max (\tau(t))$ and $0 \leq \dot{\tau}(t) \leq \sigma<1$, for all t.
According to the drive-response concept, the controlled chaotic neural network system can be described by the following equation
$d y(t)=\left[-c y(t)+A f(y(t))+B f(y(t))+B f(y(t-\tau(t))+J+u(t)] d t+\sum_{i=1}^{n}\left[\sigma_{i j}(t, e(t), e(t-\tau(t)))\right] d w_{j}(t)\right.$
where $y(t)=\left(y_{1}(t), y_{2}(t), y_{3}(t), \ldots, y_{n}(t)\right)^{T} \in \mathbb{R}^{n}$ and $\mathrm{u}(\mathrm{t})$ is driving signal, then the initial condition of the controlled network system can be described by

$$
\begin{equation*}
y_{i}(t)=\chi_{i}(t) \tag{3}
\end{equation*}
$$

Let us define the synchronization error $e(t)=x(t)-y(t)$.Therefore, the dynamics of synchronization error between the master and slave systems given in equations (1) and (2) can be described by

$$
\begin{align*}
& \dot{e}=\dot{X}-\dot{Y} \\
& d e(t)=d x(t)-d y(t) \\
& d e(t)=[-c x(t)+A f(x(t))+B f(x(t))+B f(x(t-\tau(t))+J] d t- \\
& {[[-c y(t)+A f(y(t))+B f(y(t))+B f(y(t-\tau(t))+J+u(t)] d t} \\
& \left.+\sum_{i=1}^{n}\left[\sigma_{i j}(t, e(t), e(t-\tau(t)))\right] d w_{j}(t)\right] \\
& d e(t)=[-c(x(t)-y(t))+A(f(x(t))-f(y(t)))+B[f(x(t-\tau(t)))-f(x(t-\tau(t)))-u(t)] d t \\
& \left.-\sum_{i=1}^{n}\left[\sigma_{i j}(t, e(t), e(t-\tau(t)))\right] d w_{j}(t)\right]  \tag{6}\\
& d e(t)=\left[-c(e(t))+A(g(e(t)))+B[g(e(t-\tau(t)))-u(t)] d t-\sum_{i=1}^{n}\left[\sigma_{i j}(t, e(t), e(t-\tau(t)))\right] d w_{j}(t)\right] \tag{7}
\end{align*}
$$

Lemma 2.1. (Schur Complement) Given constant matrices $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ with appropriate dimensions, where $\Omega_{1}^{T}=\Omega_{1}$ and $\Omega_{2}^{T}=\Omega_{2}>0$, then

$$
\Omega_{1}+\Omega_{3}^{T} \Omega_{2}^{-1} \Omega_{3}<0
$$

if and only if

$$
\left[\begin{array}{cc}
\Omega_{1} & \Omega_{3}^{T} \\
* & -\Omega_{2}
\end{array}\right]<0, \quad \text { or, } \quad\left[\begin{array}{cc}
-\Omega_{2} & \Omega_{3} \\
* & \Omega_{1}
\end{array}\right]<0 .
$$

Lemma 2.2. [6] Let $\sum_{1}, \sum_{2}, \sum_{3}$ be the real matrices of appropriate dimensions with $\sum_{3}>0$, then for any vectors $x$ and $y$ with appropriate dimensions

$$
2 x^{T} \Sigma_{1}^{T} \Sigma_{2} y \leq x^{T} \Sigma_{1}^{T} \Sigma_{3} \Sigma_{1} y+x^{T} \Sigma_{2}^{T} \Sigma_{3}^{-1} \Sigma_{2} y
$$

Lemma 2.3. [5] Given a continuous non linear system $\dot{x}=g(x(t), t)$ where $x(t)$ is an $n \times 1$ vector; let $v(x, t)$ be the associated Lyapunov function with the following properties

$$
\begin{align*}
\left(\lambda_{1}\|x\|\right)^{2} & \leq v(x, t) \leq\left(\lambda_{2}\|x\|\right)^{2}, \forall x, t \in \mathbb{R}^{n} \times \mathbb{R}  \tag{8}\\
\dot{v}(x, t) & \leq-\lambda_{3} v(x, t)+\lambda_{4} e^{-\alpha t}, \forall x, t \in \mathbb{R}^{n} \times \mathbb{R}, \tag{9}
\end{align*}
$$

where $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ and $\alpha$ are positive constants. If the Lyapunov function satisfies (8) and (9), the state $x(t)$ is exponentially stable in the sense that

$$
\begin{align*}
\|x\| & \leq\left[\left(\frac{\lambda_{2}}{\lambda_{1}}\|x(0)\|\right)^{2} e^{-\lambda_{3} t}+\frac{\lambda_{4}}{\lambda_{1}^{2}} e^{-\alpha t}\right]^{\frac{1}{2}}, \lambda_{3}=\alpha \\
& \leq\left[\left(\frac{\lambda_{2}}{\lambda_{1}}\|x(0)\|\right)^{2} e^{-\lambda_{3} t}+\frac{\lambda_{4}}{\lambda_{1}^{2}\left(\lambda_{1}-\alpha\right)}\left(e^{-\alpha t}-e^{-\lambda_{3} t}\right)\right]^{\frac{1}{2}}, \lambda_{3} \neq \alpha \tag{10}
\end{align*}
$$

## 3. MAIN RESULT

Controller design and adaptive synchronization scheme:
The controller should have perfect tracking capacity in order to allow the possibility of tracking convergence. This means that $e_{q} \rightarrow 0$ as $t \rightarrow \infty$ and for $\dot{e}_{q} \rightarrow 0$ as $t \rightarrow \infty$.An adaptive controller differs from an ordinary controller in that the controller parameters are time-varying and there is a mechanism for adjusting these parameters. The basic idea in adaptive control is to estimate uncertain signals and use the estimated parameters in the controller computation. For adaptive control the unknown parameters have to be constants or must vary considerably slower than the controller is to achieve proper tracking control behavior in systems.For the possibility of tracking convergence the controller should define the adaptive control $u(t)$

$$
u(t)=\gamma P e(t)+\frac{1}{2} \rho \frac{\| e\left(t-\tau(t)\left\|^{2}\right\| M \|\right.}{P\|e(t)\|}-\frac{1}{2} \frac{k^{2} e^{T}(t) e(t) P^{-1} Q_{1}^{T} Q_{1} e(t)}{k\left\|Q_{1}\right\|\|e(t)\|^{2}+\epsilon e^{-\alpha t}}
$$

Theorem 3.1. Suppose (A1) and (A2) hold.Consider system (7) with the control law(12). Then the controlled slave system(2) will be globally synchronized with the master system(1) in the sense of equation(10) with

$$
\lambda_{1}=\sqrt{\lambda_{\min }(P)}, \lambda_{2}=\sqrt{\lambda_{\max }(P)}, \lambda_{3}=\frac{\lambda_{\min }(\Omega)}{\lambda_{\max }(P)}, \lambda_{4}=\epsilon
$$

and if $\lambda_{3}<1$ and here exist matrices $P, Q_{1}, Q_{2}$ a diagonal matrix $K>0$ and a positive scalar $\gamma>0, \tau>0$ and $\rho>0$ such that the following LMI holds

$$
\left[\begin{array}{cccc}
-P C-C^{T} P+\frac{1+\tau^{*}}{1-\sigma} K^{T} Q_{2} K+K^{T} Q_{1} K+M^{T} \rho M & P & P A & P B \\
* & \gamma^{-1} & 0 & 0 \\
* & 0 & -Q_{1} & 0 \\
* & 0 & 0 & -Q_{2}
\end{array}\right]<0 .
$$

proof: Consider the following Lyapunov functional

$$
V(t)=e^{T}(t) P e(t)+\frac{1}{1-\sigma} \int_{t-\tau(t)}^{t} g^{T}(e(s)) Q_{2} g(e(s)) d s
$$

then its derivative can be obtained by ito formula, that

$$
d V(t)=\mathbb{L} V(t) d t+2 e^{T}(t) P \sigma(e(t), e(t-\tau(t)) d w(t)
$$

where

$$
\mathbb{L} V(t)=V_{t}(e(t), t)+V_{e}(e(t), t) f(t)+\frac{1}{2} \operatorname{trace}\left[\sigma(e(t), e(t-\tau(t))) V_{e e} \sigma^{T}(e(t), e(t-\tau(t)))\right],
$$

here, $V_{t}(e(t), t)=\frac{g^{T}(e(t)) Q_{2} g(e(t))}{1-\sigma}-g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t))), V_{e}(e(t), t)=2 e^{T}(t) P$ and $V_{e e}(e(t), t)=2 P$, then the equation (15) become

$$
\begin{aligned}
\mathbb{L} V(t)= & \frac{1}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))-g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t)))+ \\
& 2 e^{T}(t) P[-c(e(t))+A(g(e(t)))+B[g(e(t-\tau(t)))-u(t)]+ \\
& \operatorname{trace}\left[\sigma(t, e(t), e(t-\tau(t))) P \sigma^{T}(t, e(t), e(t-\tau(t)))\right] \\
= & 2 e^{T}(t) P[-c(e(t))+A(g(e(t)))+B[g(e(t-\tau(t)))-u(t)]+ \\
& \frac{1}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))-g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t)))+
\end{aligned}
$$

$$
\begin{equation*}
\operatorname{trace}\left[\sigma(t, e(t), e(t-\tau(t))) P \sigma^{T}(t, e(t), e(t-\tau(t)))\right] \tag{16}
\end{equation*}
$$

consider the assumption
$\operatorname{trace}\left[\sigma(t, e(t), e(t-\tau(t))) P \sigma^{T}(t, e(t), e(t-\tau(t)))\right] \leq \rho\left[e^{T}(t) M^{T} M e(t)+e^{T}(t-\tau(t)) M^{T} M e(t-\tau(t))\right.$,
substitute the consideration in equation(16), we get

$$
\begin{array}{r}
\leq e^{T}\left[-P C-C^{T} P\right] e(t)+2 e^{T}(t) P A(g(e(t)))+2 e^{T}(t) P B g(e(t-\tau(t)))-2 e^{T}(t) P u(t)+ \\
\frac{1}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))-g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t)))+ \\
\rho\left[e^{T}(t) M^{T} M e(t)+e^{T}(t-\tau(t)) M^{T} M e(t-\tau(t)),\right.
\end{array}
$$

from lemma (2.3) and tacking $\Sigma_{3}$ as the identity matrix

$$
\begin{aligned}
2 e^{T}(t) P A(g(e(t))) & \leq e^{T} P A Q_{1}^{-1} A^{T} P e(t)+g^{T}(e(t)) Q_{1} g(e(t)) \\
2 e^{T}(t) P B g(e(t-\tau(t))) & \leq e^{T} P B Q_{2}^{-1} B^{T} P e(t)+g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t))),
\end{aligned}
$$

therefore the equation become

$$
\begin{align*}
\mathbb{L} V(t) \leq & e^{T}(t)\left[-P C-C^{T} P+P A Q_{1}^{-1} A^{T} P+P B Q_{2}^{-1} B^{T} P\right] e(t)+g^{T}(e(t)) Q_{1} g(e(t))+ \\
& g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t)))-2 e^{T}(t) P u(t)+\frac{1}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))- \\
& g^{T}(e(t-\tau(t))) Q_{2} g(e(t-\tau(t)))+\rho\left[e^{T}(t) M^{T} M e(t)+e^{T}(t-\tau(t)) M^{T} M e(t-\tau(t))\right. \\
= & e^{T}(t)\left[-P C-C^{T} P+P A Q_{1}^{-1} A^{T} P+P B Q_{2}^{-1} B^{T} P+M^{T} \rho M\right] e(t)+ \\
& \frac{1}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))+e^{T}(t-\tau(t)) M^{T} \rho M e(t-\tau(t))-2 e^{T}(t) P u(t) . \tag{17}
\end{align*}
$$

consider the assumption

$$
\begin{aligned}
g^{T}(e(t)) Q_{1} g(e(t)) & \leq e^{T}(t) K^{T} Q_{1} K e(t) \\
g^{T}(e(t)) Q_{2} g(e(t)) & \leq e^{T}(t) K^{T} Q_{2} K e(t)
\end{aligned}
$$

where $K$ is a positive constant matrix, then the equation (17) become

$$
\begin{aligned}
\mathbb{L} V(t) \leq & e^{T}(t)\left[-P C-C^{T} P+P A Q_{1}^{-1} A^{T} P+P B Q_{2}^{-1} B^{T} P+M^{T} \rho M\right] e(t)+e^{T}(t) K^{T} Q_{1} K e(t)+ \\
& \frac{1}{1-\sigma} e^{T}(t) K^{T} Q_{2} K e(t)+e^{T}(t-\tau(t)) M^{T} \rho M e(t-\tau(t))-2 e^{T}(t) P u(t) \\
= & e^{T}(t)\left[-P C-C^{T} P+P A Q_{1}^{-1} A^{T} P+P B Q_{2}^{-1} B^{T} P+M^{T} \rho M+K^{T} Q_{1} K+\right. \\
& \left.\frac{1}{1-\sigma} K^{T} Q_{2} K\right] e(t)++e^{T}(t-\tau(t)) M^{T} \rho M e(t-\tau(t))-2 e^{T}(t) P u(t) \\
\leq & e^{T}(t)\left[-P C-C^{T} P+P A Q_{1}^{-1} A^{T} P+P B Q_{2}^{-1} B^{T} P+M^{T} \rho M+K^{T} Q_{1} K+\right. \\
& \left.\frac{1}{1-\sigma} K^{T} Q_{2} K+\frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K\right] e(t)++e^{T}(t-\tau(t)) M^{T} \rho M e(t-\tau(t))-\frac{1}{1-\sigma} K^{T} Q_{2} K+ \\
& \frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K-2 e^{T}(t) P u(t) \\
\leq & -e^{T}(t)\left[P C+C^{T} P-P A Q_{1}^{-1} A^{T} P-P B Q_{2}^{-1} B^{T} P-M^{T} \rho M-K^{T} Q_{1} K-\right. \\
& \left.\frac{1}{1-\sigma} K^{T} Q_{2} K-\frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K\right] e(t)+\|e T(t-\tau(t))\|^{2} \rho\|M\|-\frac{1}{1-\sigma} K^{T} Q_{2} K+ \\
& \frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K-2 e^{T}(t) P u(t) .
\end{aligned}
$$

now substitute nonlinear adaptive feed back control

$$
u(t)=\gamma P e(t)+\frac{1}{2} \rho \frac{\| e\left(t-\tau(t)\left\|^{2}\right\| M \|\right.}{P\|e(t)\|}-\frac{1}{2} \frac{k^{2} e^{T}(t) e(t) P^{-1} Q_{1}^{T} Q_{1} e(t)}{k\left\|Q_{1}\right\|\|e(t)\|^{2}+\epsilon e^{-\alpha t}}
$$

then,

$$
\begin{aligned}
\mathbb{L} V(t) \leq & -e^{T}(t)\left[P C+C^{T} P-P A Q_{1}^{-1} A^{T} P-P B Q_{2}^{-1} B^{T} P-M^{T} \rho M-K^{T} Q_{1} K-\right. \\
& \left.\frac{1}{1-\sigma} K^{T} Q_{2} K-\frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K\right] e(t)-\frac{\tau^{*}}{1-\sigma} g^{T}(e(t)) Q_{2} g(e(t))+ \\
& \frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}} \\
\leq & -e^{T}(t) \Omega e(t)+\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{\tau^{*}}{1-\sigma} K^{T} e^{T}(t) Q_{2} K e(t) \\
\leq & -e^{T}(t) \Omega e(t)+\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{\tau^{*}}{1-\sigma} K^{T} e^{T}(t) Q_{2} K e(t),
\end{aligned}
$$

where $\Omega=P C+C^{T} P-P A Q_{1}^{-1} A^{T} P-P B Q_{2}^{-1} B^{T} P-M^{T} \rho M-K^{T} Q_{1} K-\frac{1}{1-\sigma} K^{T} Q_{2} K-\frac{\tau^{*}}{1-\sigma} K^{T} Q_{2} K$,

$$
\begin{align*}
\mathbb{L} V(t) & =\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{\tau^{*}}{1-\sigma} K^{T} e^{T}(t) Q_{2} K e(t) \\
d V(t) & \leq\left[\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{\tau^{*}}{1-\sigma} K^{T} e^{T}(t) Q_{2} K e(t)\right] d t+2 e^{T}(t) P \sigma(e(t), e(t-\tau(t)) d w(t) . \tag{18}
\end{align*}
$$

Taking expectation to equation(18), then

$$
\begin{aligned}
\dot{V}(t) & =\left[\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{\tau^{*}}{1-\sigma} K^{T} e^{T}(t) Q_{2} K e(t)\right] \\
& =\frac{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\| \epsilon e^{-\alpha t}}{k\left|\|e(t)\|^{2}\right|\left\|Q_{1}\right\|+\epsilon e^{-\alpha t}}-\frac{1}{1-\sigma} \int_{t-\tau(t)}^{t} g^{T}(e(t)) Q_{2} g e(t) d t,
\end{aligned}
$$

using the relation $0 \leq \frac{a b}{a+b} \leq a$ and by an lemma (2.3), we get

$$
\dot{V}(t) \leq-\frac{\lambda_{\min }(\Omega)}{\lambda_{\min }(P)} v(t)+\epsilon e^{-\alpha t}
$$

by using Lyapunov stability theorem, the error dynamic system is exponentially mean square stable and the controlled slave system is globally synchronized with the master system.

## 4. Numerical Simulation

Consider the master chaotic neural network system [9] is $\dot{x}(t)=[-c x(t)+A f(x(t))+B f(x(t))+$ $B f(x(t-\tau(t))+J]$, where

$$
\begin{array}{r}
C=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \quad A=\left[\begin{array}{cc}
2.1 & -0.12 \\
-5.1 & 3.2
\end{array}\right], \quad B=\left[\begin{array}{cc}
-1.6 & -0.1 \\
-0.2 & -2.4
\end{array}\right] \quad J=-\left[\begin{array}{c}
0.01 \\
0.1
\end{array}\right], \\
x(t)=\left[\begin{array}{c}
\sinh x_{1}(t) \\
\sinh x_{2}(t)
\end{array}\right], \quad f(x(t))=\left[\begin{array}{c}
\tanh x_{1}(t) \\
\tanh x_{2}(t)
\end{array}\right]
\end{array}
$$

therefore the master chaotic system is described by

$$
\begin{aligned}
& \dot{x}_{1}(t)=-\sinh x_{1}(t)+2.1 \tanh x_{1}(t)-0.12 \tanh x_{2}(t)-1.6 x_{1}(t-\tau(t))-0.1 x_{2}(t-\tau(t))-0.01 \\
& \dot{x}_{2}(t)=-\sinh x_{2}(t)+5.1 \tanh x_{1}(t)+3.2 \tanh x_{2}(t)-0.2 x_{1}(t-\tau(t))-2.4 x_{2}(t-\tau(t))-0.1
\end{aligned}
$$

and their corresponding slave chaotic network system is
$\dot{y}(t)=\left[-c y(t)+A f(y(t))+B f(y(t))+B f(y(t-\tau(t))+J+u(t)] d t+Z d w_{( } t\right)$, where

$$
Z=\left[\begin{array}{c}
-0.001 \\
0.001
\end{array}\right]
$$

here the adaptive feed back control $u(t)=0.00136$ then the slave chaotic system is described by
$\dot{y}_{1}(t)=-\sinh y_{1}(t)+2.1 \tanh _{1}(t)-0.12 \tanh _{2}(t)-1.6 y_{1}(t-\tau(t))-0.1 y_{2}(t-\tau(t))-0.01-0.001 d w_{( }()$
$\dot{y}_{2}(t)=-\sinh _{2}(t)+5.1 \tanh _{1}(t)+3.2 \tanh _{2}(t)-0.2 y_{1}(t-\tau(t))-2.4 y_{2}(t-\tau(t))-0.1+0.001 d w(t)$.

Now we assume the values $\tau^{*}=0.1, \sigma=0.5, \gamma=\frac{3}{4}$

$$
K=\left[\begin{array}{cc}
0.2 & 0 \\
0 & 0.2
\end{array}\right] \text { and } \quad M=\left[\begin{array}{cc}
0.1 & 0 \\
0 & 0.2
\end{array}\right]
$$

Solving the LMI(13) in theorem(3.1) by invoking the MATLAB LMI control Toolbox[4], we obtain the solution as $\rho=0.00136$

$$
\begin{gathered}
P=1.0 e-0004 \times\left[\begin{array}{ll}
0.1380 & 0.0120 \\
0.0120 & 0.0218
\end{array}\right], \quad Q_{1}=1.0 e-0003 \times\left[\begin{array}{cc}
0.1844 & -0.0235 \\
-0.0235 & 0.1435
\end{array}\right] \text { and } \\
Q_{2}=1.0 e-0004 \times\left[\begin{array}{cc}
-0.1312 & 0.0280 \\
0.0280 & -0.1440
\end{array}\right] .
\end{gathered}
$$

When $\rho$ is 0.00136 , the stochastic recurrent neural system is marginally synchronized and the $\rho$ value is less than zero the system is exponentially mean square stable. The above results show that all conditions stated in theorem (3.1) have been satisfied and hence the synchronization of chaotic recurrent neural networks with stochastic time varying is exponentially mean square stable. The synchronization portrait between the master and corresponding slave chaotic system with initial condition $[0.1,0.1]^{T}$, noise $[0.01,0.01]^{T}$, and their corresponding time delay $\tau=1$ is as follows

A new sufficient condition is derived to guarantee the exponential mean square stability of the equilibrium point for Stochastic synchronization of chaotic recurrent neural networks with time varying delays. To the best of our knowledge, the results presented here have been not appeared in the related literature. The synchronization stability criteria is expressed in terms of LMIs, which is less conservative and can be easily verified by using MATLAB LMI control Toolbox.

## References

[1] K.E.Kurten, J.W.Clark, Chaos in neural system.Physical Letter A, 114 (1986) 413-418.
[2] X.Wang, Period-doubling to chaos in a simple neural network. Proceeding of IIJCNN-91 seattle, 2 (1991) 333-339.
[3] Baotong Cui, Xuyang Lou, Synchronization of chaotic recurrent neural networks with time-varying delays using non-linear feedback control .Chaos Solitons and Fractals, Article in press.
[4] Boyed S et al. Linear matrix inequalities in system and theory, Philadelphea;.SIAM; (1994).
[5] Dawson D, Qu Z, Carroll JC.On the state observation and output feedback problems for non-linear uncertain dynamics systems.Sys Cont Lett, 18 (1992) 217-222.
[6] XuS,Chen T, Lam .Robust $H_{\infty}$ filtering for uncertain Markovian jump systems with mode-dependent time delays.IEEE Trans Automat Contr, 48 (2003) 900-7.
[7] W.Chen Guan, X.Liu, Delay-dependent exponential stability of uncertain stochastic systems with multiple delays; an LMI approachSystem and Control Letter, 54 (2005) 547-555.
[8] X.Liao,X.Mao, Exponential stability of stochastic delay interval, System and Control Letter, 50 (2000) 171-181.
[9] Wenwn Yu, Jinde Cao, Synchronization control of stochastic delayed neural networks, Physica A, 373 (2007) 252260.
[10] J.Cao,J.Lu, Adaptive synchronization of neural networks with or without time varying delay, Chaos, 16 (2006)013133.
[11] W.Lin, W.He, Complete Synchronization of the noise-perturbed Chua's circuits, Chaos, 15 (2006) 023705.
[12] W.Wang, J. Cao, Synchroniztion in an array of linearly coupled networks with time-varying delay, Physica $A$, 366 (2006) 197-211.
[13] L.M.Pecora, T.L.Carroll, Synchronization in chaotic systemPhysics Review Letter, 64 (1990) 821-824.
[14] E.Ott,C.Gerbogi,J.A.Yorke,Controlling Chaos,Physics Review Letter, 64 (1990) 1196-99.
[15] Cao J, Huang DS, Qu Y. Global robust stability of delayed recurrent neural networksChaos solitons and Fractal, 23 (2005), 221-229.
[16] Lu HT. Chaotic attractors in delayed neural networksPhysics Letter A, 298 (2002),109-16.

# Dynamics of multiple pendula without gravity 

Wojciech Szumiński<br>Institute of Physics, University of Zielona Góra<br>(E-mail: uz88szuminski@gmail.com)


#### Abstract

We present a class of planar multiple pendula consisting of mathematical pendula and spring pendula in the absence of gravity. Among them there are systems with one fixed suspension point as well as freely floating joined masses. All these systems depend on parameters (masses, arms lengths), and possess circular symmetry $\mathbb{S}^{1}$. We illustrate the complicated behaviour of their trajectories using Poincaré sections. For some of them we prove their non-integrability analysing properties of the differential Galois group of variational equations along certain particular solutions of the systems.


Keywords: Hamiltonian systems, Multiple pendula, Integrability, Non-integrability, Poincaré sections, Morales-Ramis theory, Differential Galois theory.

## 1 Introduction

The complicated behaviour of various pendula is well known but still fascinating, see e.g. books $[2,3]$ and references therein as well as also many movies on youtube portal. However, it seems that the problem of the integrability of these systems did not attract sufficient attention. According to our knowledge, the last found integrable case is the swinging Atwood's machine without massive pulleys [1] for appropriate values of parameters. Integrability analysis for such systems is difficult because they depend on many parameters: masses $m_{i}$, lengths of arms $a_{i}$, Young modulus of the springs $k_{i}$ and unstretched lengths of the springs. In a case when the considered system has two degrees of freedom one can obtain many interesting information about their behaviour making Poincaré cross-sections for fixed values of the parameters

However, for finding new integrable cases one needs a strong tool to distinguish values of parameters for which the system is suspected to be integrable. Recently such effective and strong tool, the so-called Morales-Ramis theory [5] has appeared. It is based on analysis of differential Galois group of variational equations obtained by linearisation of equations of motion along a non-equilibrium particular solution. The main theorem of this theory states that if the considered system is integrable in the


Fig. 1. Simple double pendulum. Liouville sense, then the identity component of the differential Galois group of the variational equations is Abelian. For a precise definition of the differential Galois group and differential Galois theory, see, e.g. [6].

The idea of this work arose from an analysis of double pendulum, see Fig. 1. Its configuration space is $\mathbb{T}^{2}=\mathbb{S}^{1} \times \mathbb{S}^{1}$, and local coordinates are ( $\phi_{1}, \phi_{2}$ ) $\bmod 2 \pi$. A double pendulum in a constant gravity field has regular as well as chaotic trajectories. However, a proof of its non-integrability for all values of parameters is still missing. Only partial results are known, e.g., for small ratio of pendulums masses one can prove the non-integrability by means of Melnikov method [4]. On the other hand, a double pendulum without gravity is integrable. It has $\mathbb{S}^{1}$ symmetry, and the Lagrange function depends on difference of angles only. Introducing new variables $\theta_{1}=\phi_{1}$ and $\theta_{2}=\phi_{2}-\phi_{1}$, we note that $\theta_{1}$ is cyclic variable, and the corresponding momentum is a missing first integral.

The above example suggests that it is reasonable to look for new integrable systems among planar multiple-pendula in the absence of gravity when the $\mathbb{S}^{1}$ symmetry is present. Solutions of such systems give geodesic flows on product of $\mathbb{S}^{1}$, or products of $\mathbb{S}^{1}$ with $\mathbb{R}^{1}$. For an analysis of such systems we propose to use a combination of numerical and analytical methods. From the one side, Poincaré section give quickly insight into the dynamics. On the other hand, analytical methods allow to prove strictly the non-integrability.

In this paper we consider: two joined pendula from which one is a spring pendulum, two spring pendula on a massless rod, triple flail pendulum and triple bar pendulum. All these systems possess suspension points. One can also detach from the suspension point each of these systems. In particular, one can consider freely moving chain of masses (detached multiple simple pendula), and free flail pendulum. We illustrate the behaviour of these systems on Poicaré sections, and, for some of them, we prove their non-integrability. For the double spring pendulum the proof will be described in details. For others the main steps of the proofs are similar.

In order to apply the Morales-Ramis method we need an effective tool which allows to determine the differential Galois group of linear equations. For considered systems variational equations have two-dimensional subsystems of normal variational equations. They can be transformed into equivalent second order equations with rational coefficients. For such equations there exists an algorithm, the so-called the Kovacic algorithm [7], determining its differential Galois groups effectively.

## 2 Double spring pendulum

The geometry of this system is shown in Fig. 2. The mass $m_{2}$ is attached to $m_{1}$ on a spring with Young modulus $k$. System has $\mathbb{S}^{1}$ symmetry, and $\theta_{1}$ is a cyclic coordinate. The corresponding momentum $p_{1}$ is a first integral. The reduced system has two degrees of freedom with coordinates $\left(\theta_{2}, x\right)$, and momenta ( $p_{2}, p_{3}$ ). It depends on parameter $c=p_{1}$.

The Poincaré cross sections of the reduced


Fig. 2. Double spring pendulum. system shown in Fig. 3 suggest that the system is not integrable. The main


Fig. 3. The Poincaré sections for double spring pendulum. Parameters: $m_{1}=m_{2}=$ $a_{1}=a_{2}=1, k=0.1, p_{1}=c=0$ cross-plain $x=1$.
problem is to prove that in fact the system is not integrable for a wide range of the parameters. In Appendix we prove the following theorem.

Theorem 1. Assume that $a_{1} m_{1} m_{2} \neq 0$, and $c=0$. Then the reduced system descended from double spring pendulum is non-integrable in the class of meromorphic functions of coordinates and momenta.

## 3 Two rigid spring pendula

The geometry of the system is shown in Fig. 4. On a massless rod fixed at one end we have two masses joined by a spring; the first mass is joined to fixed point by another spring. As generalised coordinates angle $\theta$ and distances $x_{1}$ and $x_{2}$ are used. Coordinate $\theta$ is a cyclic variable and one can consider the reduced


Fig. 4. Two rigid spring pendula. system depending on parameter $c$ - value of momentum $p_{3}$ corresponding to $\theta$. The Poincaré cross sections in Fig. 5 and in Fig. 6 show the complexity of the system. We are able to prove non-integrability only under assumption $k_{2}=0$.

Theorem 2. If $m_{1} m_{2} k_{1} c \neq 0$, and $k_{2}=0$, then the reduced two rigid spring pendula system is non-integrable in the class of meromorphic functions of coordinates and momenta.

Moreover, we can identify two integrable cases. For $c=0$ the reduced Hamilton equations become linear equations with constant coefficients and they are solvable. For $k_{1}=k_{2}=0$ original Hamiltonian simplifies to

$$
H=\frac{1}{2}\left(\frac{p_{1}^{2}}{m_{1}}+\frac{p_{2}^{2}}{m_{2}}+\frac{p_{3}^{2}}{m_{1} x_{1}^{2}+m_{2} x_{2}^{2}}\right)
$$

and is integrable with two additional first integrals $F_{1}=p_{3}, F_{2}=m_{2} p_{2} x_{1}-$ $m_{2} p_{1} x_{2}$.


Fig. 5. The Poincaré sections for two rigid spring pendula. Parameters: $m_{1}=m_{2}=$ $a_{1}=a_{2}=1, k_{1}=k_{2}=1 / 10, p_{3}=c=1 / 10$, cross-plain $x_{1}=0, p_{1}>0$.


Fig. 6. The Poincaré sections two rigid spring pendula. Parameters: $m_{1}=1, m_{2}=3$, $k_{1}=0.1, k_{2}=1.5, a_{1}=a_{2}=0, p_{3}=c=0.1$, cross-plain $x_{1}=0, p_{1}>0$

## 4 Triple flail pendulum

In Fig. 7 the geometry of the system is shown. Here angle $\theta_{1}$ is a cyclic coordinate. Fixing value of the corresponding momentum $p_{1}=c \in \mathbb{R}$, we consider the reduced system with two degrees of freedom. Examples of Poincaré sections for this system are shown in Fig. 8 and 9. For more plots and its interpretations see [11]. One can also prove that this system is not integrable, see [9].


Fig. 7. Triple flail pendulum.

Theorem 3. Assume that $l_{1} l_{2} l_{3} m_{2} m_{3} \neq 0$, and $m_{2} l_{2}=m_{3} l_{3}$. If either (i) $m_{1} \neq 0, c \neq 0, l_{2} \neq l_{3}$, or (ii) $l_{2}=l_{3}$, and $c=0$, then the reduced flail system is not integrable in the class of meromorphic functions of coordinates and momenta.


Fig. 8. The Poincaré sections for flail pendulum. Parameters: $m_{1}=1, m_{2}=3, m_{3}=$ $2, a_{1}=1, a_{2}=2, a_{3}=3, p_{1}=c=1$, cross-plain $\theta_{2}=0, p_{2}>0$.


Fig. 9. The Poincaré sections for flail pendulum. Parameters: $m_{1}=1$, $m_{2}=m_{3}=2, a_{1}=2, a_{2}=a_{3}=1, p_{1}=c=\frac{1}{2}$, cross-plain $\theta_{2}=0, p_{2}>0$.

## 5 Triple bar pendulum

Triple bar pendulum consists of simple pendulum of mas $m_{1}$ and length $a_{1}$ to which is attached a rigid weightless rod of length $d=d_{1}+d_{2}$. At the ends of the rod there are attached two simple pendula with masses $m_{2}, m_{3}$, respectively, see Fig.10. Like in previous cases fixing value for the first integral $p_{1}=c$ corresponding to cyclic variable $\theta_{1}$, we


Fig. 10. Triple bar pendulum. obtain the reduced Hamiltonian depending only on four variables $\left(\theta_{2}, \theta_{3}, p_{2}, p_{3}\right)$. Therefore we are able to make Poincaré cross sections, see Fig. 11, and also to prove the following theorem [10].:


Fig. 11. The Poincaré sections for bar pendulum. Parameters: $m_{1}=m_{2}=1$, $m_{3}=2, a_{1}=1, a_{2}=2, a_{3}=1, d_{1}=d_{2}=1, p_{1}=c=\frac{1}{2}$, cross-plain $\theta_{2}=0, p_{2}>0$.

Theorem 4. Assume that $l_{2} l_{3} m_{1} m_{2} m_{3} \neq 0$, and $m_{2} l_{2}=m_{3} l_{3}, d_{1}=d_{2}$. If either (i) $c \neq 0, l_{2} \neq l_{3}$ or (ii) $l_{2}=l_{3}$, and $c=0$, then the reduced triple bar system governed by Hamiltonian is not integrable in the class of meromorphic functions of coordinates and momenta.

## 6 Simple triple pendulum

Problem of dynamics of a simple triple pendulum in the absence of gravity field was numerically analysed in [8]. Despite the fact that $\theta_{1}$ is again cyclic variable, and the corresponding momentum $p_{1}$ is constant, the Poincaré sections suggest that this system is also nonintegrable, see Fig.13. One can think, that the approach applied to the previous pendula can be used for this system. However, for this pendulum we only found particular solutions that after reductions become equilibria and then the Morales-Ramis theory does not give any obstructions to the integrability.

## 7 Chain of mass points

We consider a chain of $n$ mass points in a plane. The system has $n+1$ degrees of freedom. Let $\boldsymbol{r}_{i}$ denote radius vectors of points in the center of mass frame. Coordinates of these vectors $\left(x_{i}, y_{i}\right)$ can be expressed in terms of $\left(x_{1}, y_{1}\right)$ and relative angles $\theta_{i}, i=2, \ldots, n$. In the


Fig. 12. Simple triple pendulum.


Fig. 14. Chain of mass points


Fig. 13. The Poincaré sections for simple triple pendulum: $m_{1}=2, m_{2}=1, m_{3}=$ $1, a_{1}=2, a_{2}=a_{3}=1, p_{1}=c=1$, cross-plain $\theta_{2}=0, p_{2}>0$.
centre of mass frame we have $\sum m_{i} \boldsymbol{r}_{i}=\mathbf{0}$, thus we can expressed $\left(x_{1}, y_{1}\right)$ as a function of angles $\theta_{i}$. Lagrange and Hamilton functions do not depend on $\left(x_{1}, y_{1}\right),\left(\dot{x}_{1}, \dot{y}_{1}\right)$, and $\theta_{2}$ is a cyclic variable thus the corresponding momentum $p_{2}$ is a first integral. The reduced system has $n-2$ degrees of freedom. Thus the chain of $n=3$ masses is integrable. Examples of Poincaré sections for reduced system of $n=4$ masses are given in Fig. 15. In the case when $m_{3} a_{4}=m_{2} a_{2}$ a non-trivial particular solution is known and non-integrability analysis is in progress.


Fig. 15. The Poincaré sections for chain of 4 masses. Parameters: $m_{1}=m_{3}=1$, $m_{2}=2, m_{4}=3, a_{2}=1, a_{3}=1, a_{4}=3, p_{2}=c=\frac{3}{2}$, cross-plain $\theta_{3}=0, p_{3}>0$.

## 8 Unfixed triple flail pendulum

One can also unfix triple flail pendulum described in Sec.4, and allow to move it freely. As the generalised coordinates we choose coordinates $\left(x_{1}, y_{1}\right)$ of the first mass, and relative angles, see Fig. 16. In the center of masses frame coordinates $\left(x_{1}, y_{1}\right)$, and their derivatives


Fig. 16. Chain of mass points ( $\dot{x}_{1}, \dot{y}_{1}$ ) disappear in Lagrange function, and $\theta_{2}$ is a cyclic variable. Thus we can also consider reduced system depending on the value of momentum $p_{2}=c$ corresponding to $\theta_{2}$. Its Poincaré sections are presented in Fig. 17. One can also find a non-trivial particular solution when $a_{3}=a_{4}$. The non-integrability analysis is in progress.


Fig. 17. The Poincaré sections for unfixed flail pendulum. Parameters: $m_{1}=2$, $m_{2}=1, m_{3}=2, m_{4}=1, a_{2}=a_{3}=a_{4}=1, p_{2}=c=\frac{3}{2}$, cross-plain $\theta_{3}=0, p_{3}>0$.

## 9 Open problems

We proved non-integrability for some systems but usually only for parameters that belong to a certain hypersurface in the space of parameters. It is an open question about their integrability when parameters do not belong to these hypersurfaces. Another problem is that for some systems we know only very simple particular solutions that after reduction by one degree of freedom transform into equilibrium. There is a question how to find another particular solution for them.

## 10 Appendix: Proof of non-integrability of the double spring pendulum, Theorem 1

Proof. The Hamiltonian of the reduced system for $p_{1}=c=0$ is equal to

$$
\begin{align*}
H= & {\left[m_{2} p_{2}^{2} x^{2}+2 a_{1} m_{2} p_{2} x\left(p_{2} \cos \theta_{2}+p_{3} x \sin \theta_{2}\right)+a_{1}^{2}\left(m _ { 1 } \left(p_{2}^{2}+x^{2}\left(p_{3}^{2}\right.\right.\right.\right.} \\
& \left.\left.\left.+k m_{2}\left(x-a_{2}\right)^{2}\right)\right)+m_{2}\left(p_{2} \cos \theta_{2}+p_{3} x \sin \theta_{2}\right)^{2}\right] /\left(2 a_{1}^{2} m_{1} m_{2} x^{2}\right) \tag{1}
\end{align*}
$$

and its Hamilton equations have particular solutions given by

$$
\begin{equation*}
\theta_{2}=p_{2}=0, \quad \dot{x}=\frac{p_{3}}{m_{2}}, \quad \dot{p}_{3}=k\left(a_{2}-x\right) \tag{2}
\end{equation*}
$$

We chose a solution on the level $H\left(0, x, 0, p_{3}\right)=E$. Let $\left[\Theta_{2}, X, P_{2}, P_{3}\right]^{T}$ be variations of $\left[\theta_{2}, x, p_{2}, p_{3}\right]^{T}$. Then the variational equations along this particular solution are following

$$
\left[\begin{array}{c}
\dot{\Theta}_{2}  \tag{3}\\
\dot{X} \\
\dot{P}_{2} \\
\dot{P}_{3}
\end{array}\right]=\left[\begin{array}{cccc}
\frac{p_{3}\left(a_{1}+x\right)}{a_{1} m_{1} x} & 0 & \frac{a_{1}^{2} m_{1}+m_{2}\left(a_{1}+x\right)^{2}}{a_{1}^{2} m_{1} m_{2} x^{2}} & 0 \\
0 & 0 & 0 & \frac{1}{m_{2}} \\
-\frac{p_{3}^{2}}{m_{1}} & 0 & -\frac{p_{3}\left(a_{1}+x\right)}{a_{1} m_{1} x} & 0 \\
0 & -k & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\Theta_{2} \\
X \\
P_{2} \\
P_{3}
\end{array}\right]
$$

where $x$ and $p_{3}$ satisfy (2). Equations for $\Theta_{2}$ and $P_{2}$ form a subsystem of normal variational equations and can be rewritten as one second-order differential equation

$$
\begin{align*}
& \ddot{\Theta}+P \dot{\Theta}+Q \Theta=0, \quad \Theta \equiv \Theta_{2}, \quad P=\frac{2 a_{1} p_{3}\left(a_{1}\left(m_{1}+m_{2}\right)+m_{2} x\right)}{m_{2} x\left(a_{1}^{2} m_{1}+m_{2}\left(a_{1}+x\right)^{2}\right)} \\
& Q=\frac{k\left(a_{1}+x\right)\left(x-a_{2}\right)}{m_{1} a_{1} x}-\frac{2 a_{1}^{2} p_{3}^{2}}{m_{2} a_{1} x\left(a_{1}^{2} m_{1}+m_{2}\left(a_{1}+x\right)^{2}\right)} \tag{4}
\end{align*}
$$

The following change of independent variable $t \longrightarrow z=x(t)+a_{1}$, and then a change of dependent variable

$$
\begin{equation*}
\Theta=w \exp \left[-\frac{1}{2} \int_{z_{0}}^{z} p(\zeta) \mathrm{d} \zeta\right] \tag{5}
\end{equation*}
$$

transforms this equation into an equation with rational coefficients

$$
\begin{equation*}
w^{\prime \prime}=r(z) w, \quad r(z)=-q(z)+\frac{1}{2} p^{\prime}(z)+\frac{1}{4} p(z)^{2} \tag{6}
\end{equation*}
$$

where

$$
\begin{aligned}
& p=\left[a_{1}^{2} m_{1}\left(-4 E+k\left(2 a_{2}^{2}+3 a_{1}\left(a_{1}-2 z\right)+5 a_{2}\left(a_{1}-z\right)\right)+3 k z^{2}\right)+m_{2} z\left(2 a_{1} a_{2}^{2} k\right.\right. \\
& \left.\left.+a_{1}\left(-4 E+a_{1} k\left(2 a_{1}-3 z\right)\right)+k z^{3}+a_{2} k\left(a_{1}-z\right)\left(4 a_{1}+z\right)\right)\right] /\left[\left(a_{1}^{2} m_{1}+m_{2} z^{2}\right)\right. \\
& \left.\times\left(z-a_{1}\right)\left(-2 E+k z^{2}-\left(a_{1}+a_{2}\right) k\left(2 z-a_{1}-a_{2}\right)\right)\right] \\
& q=\frac{m_{2}\left(a_{1}^{2} m_{1}\left(4 E-k\left(2\left(a_{1}+a_{2}\right)-3 z\right)\left(a_{1}+a_{2}-z\right)\right)+k m_{2}\left(a_{1}+a_{2}-z\right) z^{3}\right)}{a_{1} m_{1}\left(-2 e+k\left(a_{1}+a_{2}-z\right)^{2}\right)\left(z-a_{1}\right)\left(a_{1}^{2} m_{1}+m_{2} z^{2}\right)}
\end{aligned}
$$

We underline that both transformations do not change identity component of the differential Galois group, i.e. the identity components of differential Galois groups of equation (4) and (6) are the same.

Differential Galois group of (6) can be obtained by the Kovacic algorithm [7]. It determines the possible closed forms of solutions of (6) and simultanously its differential Galois group $\mathcal{G}$. It is organized in four cases: (I) Eq. (6) has an exponential solution $w=P \exp \left[\int \omega\right], P \in \mathbb{C}[z], \omega \in \mathbb{C}(z)$ and $\mathcal{G}$ is a triangular group, (II) (6) has solution $w=\exp \left[\int \omega\right]$, where $\omega$ is algebraic function of degree 2 and $\mathcal{G}$ is the dihedral group, (III) all solutions of (6) are algebraic and $\mathcal{G}$ is a finite group and (IV) (6) has no closed-form solution and $\mathcal{G}=\mathrm{SL}(2, \mathbb{C})$. In cases (II) and (III) $\mathcal{G}$ has always Abelian identity component, in case (I) this component can be Abelian and in case (IV) it is not Abelian.

Equation (6) related with our system can only fall into cases (I) or (IV) because its degree of inifinity is 1 , for definition of degree of infinity, see [7]. Moreover, one can show that there is no algebraic function $\omega$ of degree 2 such that $w=\exp \left[\int \omega\right]$ satisfies (6) thus $\mathcal{G}=\operatorname{SL}(2, \mathbb{C})$ with non-Abelian identity component and the necessary integrability condition is not satisfied.

## References

1.O. Pujol, J. P. Pérez, J. P. Ramis, C. Simó, S. Simon, and J. A. Weil. Swinging Atwood machine: experimental and numerical results, and a theoretical study. Phys. D, 239(12):1067-1081, 2010.
2.G. L. Baker and J. A. Blackburn. The Pendulum. A Case Study in Physics. Oxford University Press, Oxford, 2005.
3.M. Gitterman.The Chaotic Pendulum. World Scientific, 2010.
4.A. V. Ivanov. Problem of non-integrability for the double mathematical pendulum. In Progress in nonlinear science, Vol. 1 (Nizhny Novgorod), 2001, pages 251-256. RAS, Inst. Appl. Phys., Nizhni Novgorod, 2002.
5.J. J. Morales Ruiz. Differential Galois theory and non-integrability of Hamiltonian systems. Progress in Mathematics, volume 179, Birkhäuser Verlag, Basel, 1999.
6.M. van der Put, M. F. Singer. Galois theory of linear differential equations. SpringerVerlag, Berlin, 2003.
7.J. J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. J. Symbolic Comput., 2(1):3-43, 1986.
8.V.N. Salnikov. On numerical approaches to the analysis of topology of the phase space for dynamical integrability. arXiv:1206.3801 [math.DS].
9.M. Przybylska, W. Szumiński. Non-integrability of flail triple pendulum. Chaos, Solitons \& Fractals. After revision. arXiv:1211.6045 [nlin.CD].
10.M. Przybylska, W. Szumiński. Non-integrability of triple bar pendulum. In preparation.
11.W. Szumiński. Dynamics of multiple pendula. Bachelor thesis, University of Zielona Góra, 2012. accessible at http://midori-yama.pl/?q=publications.

# Experimental demonstration of time-irreversible, self-ordering evolution processes in macroscopic quantum systems. 

A. Titov, I. Malinovsky ${ }^{+}$.<br>Physics Department, Yeditepe University, Istanbul, Turkey; atitov@yeditepe.edu.tr. "National Metrology Institute, INMETRO, Rio de Janeiro, Brazil, laint@inmetro.gov.br.


#### Abstract

Using the recently developed method, which presents the combination of the modulation technique with synchronous differential thermal measurements, we have demonstrated experimentally the existence of thermal surface energy (TSE) in metallic blocks with signal-to-noise ratio of several thousands. The TSE arises when there are changes of energy and momentum of the coupled field-particle system inside the material artifact, produced by the irradiation of the artifact surface by an external EM field. It is shown that the magnitude of TSE and the direction of its increase are defined by the Poynting vector of the external field. The fundamental features of the TSE - the lack of symmetry in space and the irreversible character of the process of its creation in time are sufficient for the observation of the thermal hysteresis effect, whose hysteresis loop is reported. As the principle of superposition is demonstrated to be invalid in case of TSE, the thermal hysteresis curve converts in case of a continuous sweep in time into helicaltype curve, for which the form and the magnitude of each cycle are slightly different as a result of the non-linear interaction of heat sources of the Universe through TSE. As a result of non-linear character of interaction of quantum objects with EM field (established theoretically by N. F. Ramsey and experimentally by P. Kusch), the selfordering evolution process, observed for the thermal EM field, inevitably results in the same type of the evolution process in the whole energy spectrum of the EM field. The number of influence parameters in case of TSE is absolutely enormous, in confirmation of the previous theoretical studies of F. W. Cummings and Ali Dorri.


Keywords: surface energy, thermodynamic temperature, hysteresis, synthesis.

## 1. Introduction

This communication we want to start with reminding of the theoretical prediction by Albert Einstein made in [1] that "classical thermodynamics can no longer be looked upon as applicable with precision...For the calculation of the free energy, the energy and the entropy of the boundary surface should also be considered". The advancement of these ideas we find in [2a], where the thermal surface energy (TSE) is defined as the energy of boundary zones, located between the macroscopic parts of the system (sub-systems), in which the quasiequilibrium thermal conditions are realized. It is stated in [2a] that the TSE is proportional to the area of contact between the two sub-systems, and that the internal energy of the system can be considered as additive, only when the value of the TSE can be regarded as negligible. It is clear that in case of experimental demonstration of the TSE, the concept of thermodynamic temperature [2b] should be somehow modified and it should be, at least, in agreement with the notion of "temperature", which is traditionally used in the J. Fourier thermal conduction theory [2c] and which definitely refers to thermal non-equilibrium
conditions. Meanwhile, in accordance with A. Einstein requirements formulated in [3], thermodynamics can be applied only to isolated systems, and additionally, when all the transients are finished [3].

## 2. 2. Experiment.

The presented studies are based on the variation principle - one of the most general and powerful principles in experimental Physics. We have used a recently developed multi-channel synchronous detection technique (MSDT) [4a], which presents some modification of the famous R. Dicke's method of synchronous detection. The specific feature of MSDT is that the modulation of the heat input to the system is realized through thermometer in one of the channels, and the detection is realized by several temperature sensors of the other channels [4a], which are located at different positions relative to the modulation source (Fig.1). In this case, the temperature information from the modulation channel can be used to find the synchronous temperature differences between the different points of the system, and, consequently, the propagation of the thermal signals can be precisely characterized both in time and in space.


Fig.1. Simultaneous records of the resistance variations of the platinum resistance thermometer (PRT) and of two thermistors R6 and R3, located symmetrically relative to the PRT on the surface of the gauge block (as shown in the insert). During the current modulation cycle in the PRT, its current for $1 / 4$ of the modulation period is kept at the level of 5 mA and $3 / 4$ of the period is kept at 1 mA . The sensitivities of the thermistors are equal. The location of one of the gauging surfaces is shown by the arrow.

A schematic outline of our experimental set-up and an example of unprocessed results of the measurements, performed on a homogeneous steel artifact, are presented in Fig.1. A steel (or tungsten carbide) gauge block (GB), with dimensions $9 \times 35 \times 100 \mathrm{~mm}$, is located horizontally on three small-radius, polished spheres inside a closed Dewar. The Dewar is kept in a temperature controlled room, where typical temperature variations can be characterized by a

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey
standard deviation $\sigma$ of $\sim 50 \mathrm{mK}$. Two thermistors R6 and R3, belonging to channels 1 and 2 respectively, are installed on the surface of the GB in copper adapters, whose axes are parallel to the gauging surfaces. A 100-Ohm platinum resistance thermometer (PRT), also in a copper adapter, is located parallel to thermistors and at equal distances from their adapters. The PRT is connected to MI-bridge T615 (Canada), in which the current I is changed by step from 1 to 5 mA (Fig.1). The period of the modulation cycle is $\sim 148$ minutes, and for 37 minutes the current I is 5 mA , and for the rest of the modulation cycle it is held at 1 mA level. In Fig.1, the PRT measurements correspond to the record with faster transients. Two other records show the variations of resistances of the two temperature calibrated thermistors R6 and R3, which have negative temperature coefficients. The thermistors are connected to high-precision multi-meters HP35 a , and are calibrated together with the multi-meters, using the procedure described in [4a]. Both thermistors have, practically, equal sensitivities. From Fig. 1 it follows that the temperature difference between the channels T[1,2] for the last 25 minutes of the first cycle (shown in Fig.1) was $465.6 \mu \mathrm{~K}$ with the standard deviation $\sigma$ for a single measurement point $3.3 \mu \mathrm{~K}$. For the last 25 minutes of the next cycle, the value of $\mathrm{T}[1,2]$ was $469.5 \mu \mathrm{~K}$, with a $\sigma$-value of $2.3 \mu \mathrm{~K}$. Using a linear fit to the indicated reference points, the induced temperature variations $\Delta \mathbf{T}[\mathbf{1 , 2}]$ (at $\mathrm{I}=5 \mathrm{~mA}$ ) can be determined very precisely. When our detection system was moved as a whole, a fast decrease of the TSE with increase of the distance from the nearest gauging surface was found.


Fig.2. Variations in time of the vector quantity $\mathbf{\Delta V}[\mathbf{1 , 2}]$, observed during the heating period ( $\mathrm{I}=5 \mathrm{ma}$ ) of the modulation cycle (dots) and during the cooling period ( $\mathrm{I}=1 \mathrm{~mA}$ ) of the modulation cycle (rhombi). Reference points are shown as squares.

In Fig. 2 we present the dependence of the vector quantity $\boldsymbol{\Delta V} \mathbf{V}, \mathbf{2}]$, which corresponds to the difference in temperature velocities, observed in the channels 1 and 2 . This quantity describes the difference in the energy current densities, entering the elementary volumes inside the artifact in the vicinity of the thermistors R6 and R3. This follows from the Poynting's theorem (see eq.(6.109) in [5]) and the continuity equation, written for the total energy density of a field-particle system [6]. The theorem says that the rate of change of the EM energy plus the total rate of work of the fields over the charged particles within the volume of a material artifact is equal to the flux of the Poynting vector, $\mathbf{S}$, entering the volume of the artifact through its boundary surface (see eq.(6.111) in [5]). The vector $\mathbf{S}$ describes the energy current density of a dielectric material with arbitrary level of losses [6], and the continuity equation for the total energy density for the coupled field-particle system can be presented in the form (see eq.(2.17) in [6]):

$$
\begin{equation*}
\frac{\partial}{\partial t} W+m \mathbb{T}\left(\frac{\partial}{\partial t} s\right)^{2}=-\boldsymbol{V} \boldsymbol{s} \tag{1}
\end{equation*}
$$

Here, W presents the total energy density, which contains the kinetic and potential energy densities of the of the optic vibrational mode and the energy density of the EM field; $s$ is the relative spatial displacement field of two ions in the primitive unit cell; $m$ is the reduced mass of two ions in the primitive unit cell, and $\Gamma$ is the damping rate of the optical mode.

The rate of energy variations, described by the first two terms, is detected by resistance thermometers and corresponds to the experimentally measured thermal velocity at the specified point of the material artifact, as the reading of a thermometer depends on the power, absorbed from the external EM field.


Fig.3. The dependences of the quantity $\mathbf{\Delta V}[\mathbf{1 , 2}]$ on the PRT increment in power, obtained for the separations of the R6 axis from the nearest gauging surface of $\mathrm{L}=4.5 \mathrm{~mm}$ (squares) and $\mathrm{L}=13.5 \mathrm{~mm}$ (dots).

The dependences of Fig. 3 demonstrate the established experimentally linear relation between the Poynting vector $\mathbf{S}$ of the external EM field and the vector quantity $\mathbf{\Delta V}[\mathbf{1 , 2}]$. It is important that the excessive anti-symmetric energy flux does exist only at the beginning of the heating and cooling periods of the cycle.


Fig.4. Experimental demonstration of the anti-symmetric property of the quantity $\mathbf{\Delta V}[\mathbf{1 , 2}]$ on the heat source position: squares are used to show the dependence, corresponding to the separation of the R6 axis from the nearest gauging surface of $\mathrm{L}=4.5 \mathrm{~mm}$, while dots correspond to the same separation of the R 3 axis from the other gauging surface.

Two dependencies in Fig. 4 show that the vector quantity $\mathbf{\Delta V}[\mathbf{1 , 2}]$ presents an anti-symmetric function of the position of the heat source relative to the centre of the corresponding surface of the block. It means that the total thermal energy has no symmetry in space, as the major part of it is a symmetric function.
Under the approximations of [6], for one dimensional case the cycled-averaged value of the total-energy current density in the z -direction $<\mathbf{S}_{\mathbf{z}}>$ (the only nonzero component of the Poynting vector) is related to the cycle-averaged energy density $\langle\mathrm{W}\rangle$ (see Eq. (2.19) in [6]) by a simple relation (4.16):

$$
<\mathbf{S}_{\mathbf{z}}>=\mathbf{v}_{\mathbf{e}}<\mathrm{W}>\quad \ldots(3)
$$

where $\mathbf{v}_{\mathbf{e}}$ is the velocity vector of the energy propagation in the material. Both parameters, velocity $\mathbf{v}_{\mathbf{e}}$ and the energy density W , can be precisely determined from our experimental data. So, the energy current density of a guided EM wave, which cannot be calculated theoretically (as constitutive relations for the medium are not known [5]), can be measured experimentally. This also refers to the cycle averaged value of the corresponding component of wave momentum density $<\mathrm{G}_{z}>$, which is equal to the ratio of total-energy density $<\mathrm{W}>$ and the value of the phase velocity $\mathrm{v}_{\mathrm{p}}$. As the cycle-averaged rate of the energy conversion into heat $\left.<\mathrm{R}_{\mathrm{H}}\right\rangle$ [6] is given by the ratio $<\mathrm{S}_{\mathrm{z}}>/ \mathrm{L}$, (where L is the characteristic length of the decay of the field intensity), and $<\mathrm{R}_{\mathrm{H}}>$ defines the force density in the medium, the thermal hysteresis loop can be presented.


Fig.5. The thermal hysteresis loop for the quantity $\boldsymbol{\Delta T}[\mathbf{1 , 2}$, corresponding to the temperature records of Figs. 1 and 2. The heating period of the cycle is shown by dots, while the cooling part is presented by rhombi. The time interval for the data points between arrows 1 and 3 is increased, as the temperature variations are negligible.

The result of primary importance is illustrated by the dependences of Fig.6, from which it follows that the principle of superposition of EM fields is not valid for thermal energy.


Fig.6. . The records of the quantity $\Delta \mathbf{T}[1,2]$, that were obtained for the tungsten carbide block for the temperature differences between the channels $\mathrm{T}[1,2]$, which were produced by an external heat source and which were equal to -1.72 mK (dots); -7.2 mK (squares) and -12 mK (rhombi).
Here, we present the variations of the quantity $\boldsymbol{\Delta T}[\mathbf{1 , 2}]$ as a function of time in the presence of an additional heat source, when the measurements were

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey
performed on a $100-\mathrm{mm}$ tungsten carbide (TC) block, in which the process of the build-up of the TSE is found to be about 3 times faster than in the steel GB. From the plots of Fig. 6 we infer that the induced temperature variations (produced by the PRT current modulation) are also affected by the presence of the auxiliary source of energy. As it follows from Figs.1-7, the evolution process is specific for any point inside the artefact, is irreversible in time and is described by enormous (practically, infinite) number of external parameters. (The distances and the orientations of all the interacting bodies are the necessary influence parameters for the description of TSE.)

## 3. Conclusions

Thermal evolution process, with the spiral-type curve and with the lack of symmetries in time and space, results from the existence of the surface energy and irreversible character of the Earth's rotation.
Experimental confirmation of the whole series of theoretical papers [7-9], dealing with the interaction of the EM field with an ensemble of atoms has been obtained. In agreement with [7,8], the evolution process is shown to depend on the number of particles, it is irreversible in time, and is characterized by the infinite number of influence parameters, as predicted by [9].

## References

1. A. Einstein, Investigations on the Theory of the Brownian Motion, Annalen der Physik, v. 17, pp.549-566, (1905).
2. D. V. Sivukhin, General Course of Physics, Physmatlit, Mscow, (2008) V2, Thermodynamics, pp.44-71 [2a]; pp.85-110 [2b]; pp.162-182 [2c].
3. A. Einstein, A Theory of the Foundations of Thermodynamics, Annalen der Physik, 11, pp.170-187 (1903).
4. A. Titov, I. Malinovsky, Nanometrology and high-precision temperature measurements under varying in time temperature conditions, Proc. SPIE, 5879, ed by J. E. Decker and Gwo-Sheng Peng, 587902-01 - 587902-11 (2005) [4a];
A. Titov, I. Malinovsky', New techniques and advances in high-precision temperature measurements of material artefacts, Can. J. of Scientific and Industrial Research, V.2, No.2, pp. 59-81 (2011) [4b];
Titov, et al., Precise certification of the temperature measuring system of the original Kösters interferometer and ways of its improvement, Proc. SPIE 5879, ed by J. E. Decker and Gwo-Sheng Peng, 587904-01 - 587904-11 (2005) [4c].
5. J. D. Jackson, Classical Electrodynamics, J. Willey and Sons, 3-ed, pp.259-262 (1999).
6. R. Loudon, L. Allen and D. F. Nelson, Propagation of electromagnetic energy and momentum through an absorbing dielectric, Phys. Rev., E, 55, 1071-1085 (1997).
7. R. H. Dicke, Coherence in Spontaneous Radiation Processes, Phys. Rev. 93, pp. 99-110 (1954).
8. C. R. Stroud, et. al, Superradiant Effects in Systems of Two-Level Atoms, Phys. Rev. A, 5, pp. 1094-1104 (1972).
9. F. W. Cummings and Ali Dorri, Exact Solution for spontaneous emission in the presence of $N$ atoms, Phys. Rev. A, 28, pp.2282-2285 (1982).

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

# Positive Solutions for Second Order Multi-Point Boundary Value Problems 

Fatma Tokmak ${ }^{1}$ and Ilkay Yaslan Karaca ${ }^{2}$<br>${ }^{1}$ Ege University, Department of Mathematics, 35100 Bornova, Izmir, Turkey (E-mail: fatma.tokmakk@gmail.com)<br>${ }^{2}$ Ege University, Department of Mathematics, 35100 Bornova, Izmir, Turkey (E-mail: ilkay.karaca@ege.edu.tr)


#### Abstract

By using double fixed point theorem, we study the existence of at least two positive solutions of a second order multi-point boundary value problem. Keywords: Positive solutions, Fixed point theorem, Boundary value problems.


## 1 Introduction

In this paper we consider the second order multi-point boundary value problem (BVP)

$$
\left\{\begin{array}{l}
\left(\phi\left(u^{\prime}(t)\right)\right)^{\prime}+q(t) f(t, u(t))=0, \quad t \in(0,1),  \tag{1}\\
\phi\left(u^{\prime}(0)\right)=\sum_{i=1}^{m-2} a_{i} \phi\left(u^{\prime}\left(\xi_{i}\right)\right), \quad u(1)=\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right),
\end{array}\right.
$$

where $\xi_{i}, \eta_{i} \in(0,1)(i=1,2, \ldots, m-2)$ with $0<\xi_{1}<\xi_{2}<\ldots<\xi_{m-2}<1, \quad 0<$ $\eta_{1}<\eta_{2}<\ldots<\eta_{m-2}<1, \phi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0)=0$. A projection $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:
(i) If $x \leq y$, then $\phi(x) \leq \phi(y)$, for all $x, y \in \mathbb{R}$;
(ii) $\phi$ is continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(x y)=\phi(x) \phi(y)$, for all $x, y \in \mathbb{R}$, where $\mathbb{R}=(-\infty, \infty)$.

We assume that the following conditions are satisfied:
(A1) $f \in \mathcal{C}\left([0,1] \times \mathbb{R}^{+}, \mathbb{R}^{+}\right), q \in \mathcal{C}[0,1]$ is nonnegative,
(A2) $a_{i} \in[0, \infty), b_{i} \in[0, \infty), i=1,2, \ldots, m-2$ with $0<\sum_{i=1}^{m-2} a_{i}<1$ and $0<\sum_{i=1}^{m-2} b_{i}<1$.

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseer [1]. Since then, there has been a lot of recent attention focused on the study of nonlinear multi-point boundary value problems, see $[2-5]$. We cite some appropriate references here [6-9].

In [8], Ji et al. studied the existence of multiple positive solutions for onedimensional p-Laplacian boundary value problem

$$
\left\{\begin{array}{l}
\left(\phi\left(u^{\prime}(t)\right)\right)^{\prime}+q(t) f(t, u(t))=0, \quad t \in(0,1)  \tag{2}\\
u(0)=\sum_{i=1}^{n} \alpha_{i} u\left(\xi_{i}\right), \quad u(1)=\sum_{i=1}^{n} \beta_{i} u\left(\xi_{i}\right)
\end{array}\right.
$$

The authors established the existence of multiple positive solutions (2) by using fixed point theorem in a cone.

In [9], Ma et al. studied the existence of positive solutions for multi-point boundary value problem with p-Laplacian operator

$$
\left\{\begin{array}{l}
\left(\phi\left(u^{\prime}(t)\right)\right)^{\prime}+q(t) f(t, u(t))=0, \quad t \in(0,1),  \tag{3}\\
u^{\prime}(0)=\sum_{i=1}^{n} \alpha_{i} u^{\prime}\left(\xi_{i}\right), \quad u(1)=\sum_{i=1}^{n} \beta_{i} u\left(\xi_{i}\right) .
\end{array}\right.
$$

In this paper, motivated by the above research efforts on multi-point boundary value problems, criteria for the existence of at least two positive solutions of the BVP (1) are established by using the double fixed point theorem. Thus, our results are new for differential equations.

This paper is organized as follows. In Section 2, we give some preliminary lemmas which are key tools for our proof. The main result is given in Section 3.

## 2 Preliminaries

In this section, we give some lemmas which are useful for our main result.
We consider the Banach space $\mathbb{B}=\mathcal{C}^{1}[0,1]$ endowed with the norm

$$
\|u\|=\max _{0 \leq t \leq 1}|u(t)| .
$$

Define the cone $\mathcal{P} \subset \mathbb{B}$ by
$\mathcal{P}=\{u \in \mathbb{B}: u$ is a concave, nonnegative and nonincreasing function,

$$
\left.u(1)=\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right)\right\} .
$$

Lemma 1. If $u \in \mathcal{P}$, then $\min _{0 \leq t \leq 1} u(t) \geq M\|u\|$, where $M=\frac{\sum_{i=1}^{m-2} b_{i}\left(1-\eta_{i}\right)}{1-\sum_{i=1}^{m-2} b_{i} \eta_{i}}$.
Proof. Since $u \in \mathcal{P}$, nonnegative and nonincreasing $\|u\|=u(0), \quad \min _{0 \leq t \leq 1} u(t)=$ $u(1)$. On the other hand, $u(t)$ is concave on $[0,1]$. So, for every $t \in[0,1]$, we have

$$
u(t) \geq(1-t) u(0)+t u(1)
$$

Therefore,

$$
\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right) \geq \sum_{i=1}^{m-2} b_{i}\left(1-\eta_{i}\right) u(0)+\sum_{i=1}^{m-2} b_{i} \eta_{i} u(1)
$$

This together with $u(1)=\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right)$, implies that

$$
u(1) \geq \frac{\sum_{i=1}^{m-2} b_{i}\left(1-\eta_{i}\right)}{1-\sum_{i=1}^{m-2} b_{i} \eta_{i}} u(0)
$$

So, the proof of Lemma is completed.
Lemma 2. Assume that $(A 1),(A 2)$ hold. Then $u \in \mathcal{C}^{1}[0,1]$ is a solution to problem (1) if and only if $u$ is a solution to the integral equation:

$$
\begin{align*}
u(t)= & \int_{t}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s \\
& +\frac{1}{1-\sum_{i=1}^{m-2} b_{i}} \sum_{i=1}^{m-2} b_{i} \int_{\eta_{i}}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s \tag{4}
\end{align*}
$$

where

$$
\begin{equation*}
A=\frac{1}{1-\sum_{i=1}^{m-2} a_{i}} \sum_{i=1}^{m-2} a_{i} \int_{0}^{\xi_{i}} q(s) f(s, u(s)) d s \tag{5}
\end{equation*}
$$

Proof. First, suppose that $u \in \mathcal{C}^{1}[0,1]$ is a solution of problem (1). Integrating the equation (1) from 0 to $t$, one has

$$
\begin{equation*}
-\phi\left(u^{\prime}(t)\right)+\phi\left(u^{\prime}(0)\right)=\int_{0}^{t} f(s, u(s)) d s \tag{6}
\end{equation*}
$$

and taking $t=\xi_{i}$ in (6), we get

$$
\sum_{i=1}^{m-2} a_{i} \phi\left(u^{\prime}\left(\xi_{i}\right)\right)=\sum_{i=1}^{m-2} a_{i} \phi\left(u^{\prime}(0)\right)-\sum_{i=1}^{m-2} a_{i} \int_{0}^{\xi_{i}} q(s) f(s, u(s)) d s
$$

Since $\phi\left(u^{\prime}(0)\right)=\sum_{i=1}^{m-2} \alpha_{i} \phi\left(u^{\prime}\left(\xi_{i}\right)\right)$, we have

$$
\begin{equation*}
\phi\left(u^{\prime}(0)\right)=-\frac{1}{1-\sum_{i=1}^{m-2} \alpha_{i}} \sum_{i=1}^{m-2} \alpha_{i} \int_{0}^{\xi_{i}} q(s) f(s, u(s)) d s=-A \tag{7}
\end{equation*}
$$

Substituting (7) into (6), we get

$$
\begin{equation*}
u^{\prime}(t)=-\phi^{-1}\left(\int_{0}^{t} q(s) f(s, u(s)) d s+A\right) \tag{8}
\end{equation*}
$$

Integrating the equation (8) from $t$ to 1 , one has

$$
\begin{equation*}
u(t)=u(1)+\int_{t}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) \tag{9}
\end{equation*}
$$

and taking $t=\eta_{j}$ in (9), we get

$$
\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right)=u(1) \sum_{i=1}^{m-2} b_{i}+\sum_{i=1}^{m-2} b_{i} \int_{\eta_{i}}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s
$$

Since $u(1)=\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right)$,

$$
\begin{equation*}
u(1)=\frac{1}{1-\sum_{i=1}^{m-2} b_{i}}\left[\sum_{i=1}^{m-2} b_{i} \int_{\eta_{i}}^{1} \phi_{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s\right. \tag{10}
\end{equation*}
$$

Substituting (10) into (9), we get (4), which completes the proof of sufficiency.
Conversely, if $u \in C^{1}[0,1]$ is a solution to (4), apparently

$$
\begin{aligned}
\left(\phi\left(u^{\prime}(t)\right)\right)^{\prime} & =-q(t) f(t, u(t)), \\
\phi\left(u^{\prime}(0)\right) & =\sum_{i=1}^{m-2} a_{i} \phi_{p}\left(u^{\prime}\left(\xi_{i}\right)\right), \quad u(1)=\sum_{i=1}^{m-2} b_{i} u\left(\eta_{i}\right) .
\end{aligned}
$$

The proof is complete.
Now define an operator $T: \mathcal{P} \longrightarrow \mathbb{B}$ by

$$
\begin{align*}
T u(t)= & \int_{t}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s \\
& +\frac{1}{1-\sum_{i=1}^{m-2} b_{i}} \sum_{i=1}^{m-2} b_{i} \int_{\eta_{i}}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau+A\right) d s \tag{11}
\end{align*}
$$

Lemma 3. Assume that $(A 1)-(A 2)$ hold. Then $T: \mathcal{P} \rightarrow \mathcal{P}$ is a completely continuous operator.

Proof. It is clear that $T \mathcal{P} \subset \mathcal{P}$ and $T: \mathcal{P} \rightarrow \mathcal{P}$ is a completely continuous operator by a standard application of the Arzela-Ascoli theorem.

## 3 Main Results

In this section we state and prove our main result. The following fixed point theorem is fundamental and important to the proof of main result.

For a nonnegative continuous functional $\gamma$ on a cone $\mathcal{P}$ in a real Banach space $\mathbb{B}$, and each $d>0$, we set

$$
\mathcal{P}(\gamma, d)=\{x \in \mathcal{P} \mid \gamma(x)<d\} .
$$

Lemma 4. (Double Fixed Point Theorem) [10] Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let $\alpha$ and $\gamma$ be increasing, nonnegative, continuous functionals on $\mathcal{P}$, and let $\theta$ be a nonnegative, continuous functional on $\mathcal{P}$ with $\theta(0)=0$ such that, for some $c>0$ and $M>0$,

$$
\gamma(u) \leq \theta(u) \leq \alpha(u) \text { and }\|u\| \leq M \gamma(u)
$$

for all $u \in \overline{\mathcal{P}(\gamma, c)}$. Suppose that there exist positive numbers $a$ and $b$ with $a<b<c$ such that

$$
\theta(\lambda u) \leq \lambda \theta(u), \quad \text { for } \quad 0 \leq \lambda \leq 1 \quad \text { and } \quad u \in \partial \mathcal{P}(\theta, b)
$$

and

$$
T: \overline{\mathcal{P}(\gamma, c)} \rightarrow \mathcal{P}
$$

is a completely continuous operator such that:
(i) $\gamma(T u)>c$, for all $u \in \partial \mathcal{P}(\gamma, c)$;
(ii) $\theta(T u)<b$, for all $x \in \partial \mathcal{P}(\theta, b)$;
(iii) $K(\alpha, a) \neq \emptyset$, and $\alpha(T u)>a$, for all $u \in \partial K(\alpha, a)$.

Then $T$ has at least two fixed points, $u_{1}$ and $u_{2}$ belonging to $\overline{\mathcal{P}(\gamma, c)}$ such that

$$
a<\alpha\left(u_{1}\right), \quad \text { with } \quad \theta\left(u_{1}\right)<b,
$$

and

$$
b<\theta\left(u_{2}\right), \quad \text { with } \quad \gamma\left(u_{2}\right)<c .
$$

Let us define the increasing, nonnegative, continuous functionals $\gamma, \beta$, and $\alpha$ on $\mathcal{P}$ by

$$
\begin{aligned}
& \gamma(u)=\min _{0 \leq t \leq \xi_{1}} u(t)=u\left(\xi_{1}\right), \\
& \beta(u)=\max _{\xi_{1} \leq t \leq \xi_{n-2}} u(t)=u\left(\xi_{1}\right), \\
& \alpha(u)=\max _{0 \leq t \leq \xi_{n-2}} u(t)=u(0) .
\end{aligned}
$$

It is obvious that for each $u \in \mathcal{P}$,

$$
\gamma(u) \leq \beta(u) \leq \alpha(u)
$$

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey

In addition, from by Lemma 1 , for each $u \in \mathcal{P}$,

$$
\|u\| \leq \frac{1}{M} \min _{0 \leq t \leq 1} u(t) \leq \frac{1}{M} \min _{0 \leq t \leq \xi_{1}} u(t)=\frac{1}{M} \gamma(u)
$$

Thus,

$$
\|u\| \leq \frac{1}{M} \gamma(u), \quad \forall u \in \mathcal{P}
$$

For the convenience, we denote

$$
\begin{aligned}
K & =\left(1-\xi_{1}\right) \phi^{-1}\left(\int_{0}^{\xi_{1}} q(\tau) d \tau\right) \\
L & =\frac{1}{1-\sum_{i=1}^{m-2} b_{i}} \phi^{-1}\left(\frac{1}{1-\sum_{i=1}^{m-2} a_{i}} \int_{0}^{\xi_{1}} q(\tau) d \tau\right)
\end{aligned}
$$

Theorem 1. Suppose that assumptions (A1), (A2) are satisfied. Let there exist positive numbers $a<b<c$ such that

$$
0<a<\frac{K}{L} b<\frac{K M}{L} c
$$

and assume that $f$ satisfies the following conditions
(A3) $f(t, u)>\phi\left(\frac{c}{K}\right)$, for all $(t, u) \in\left[0, \xi_{1}\right] \times\left[c, \frac{1}{M} c\right]$,
(A4) $f(t, u)<\phi\left(\frac{b}{L}\right)$, for all $(t, u) \in[0,1] \times\left[0, \frac{1}{M} b\right]$,
(A5) $f(t, u)>\phi\left(\frac{a}{K}\right)$, for all $(t, u) \in[0,1] \times[0, a]$.
Then the boundary value problem (1) has at least two positive solutions $u_{1}$ and $u_{2}$ satisfying

$$
a<\alpha\left(u_{1}\right) \text { with } \beta\left(u_{1}\right)<b, \quad b<\beta\left(u_{2}\right) \text { with } \gamma\left(u_{2}\right)<c .
$$

Proof. We define the completely continuous operator $T$ by (11). So, it is easy to check that $T: \overline{\mathcal{P}(\gamma, c)} \rightarrow \mathcal{P}$. We now show that all the conditions of Lemma 4 are satisfied. In order to show that condition $(i)$ of Lemma 4, we choose $u \in \partial \mathcal{P}(\gamma, c)$. Then $\gamma(u)=\min _{0 \leq t \leq \xi_{1}} u(t)=u\left(\xi_{1}\right)=c$, this implies that $c \leq u(t)$ for $t \in\left[0, \xi_{1}\right]$. Recalling that $\|u\| \leq \frac{1}{M} \gamma(u)=\frac{1}{M} c$, we get

$$
c \leq u(t) \leq \frac{1}{M} c, \quad t \in\left[0, \xi_{1}\right]
$$

Then assumption (A3) implies $f(t, u)>\phi\left(\frac{c}{A}\right)$, for all $(t, u) \in\left[0, \xi_{1}\right] \times$ $\left[c, \frac{1}{M} c\right]$. Therefore,

$$
\begin{aligned}
\gamma(T u) & =\min _{t \in\left[0, \xi_{1}\right]}(T u)(t)=(T u)\left(\xi_{1}\right) \\
& \geq \int_{\xi_{1}}^{1} \phi^{-1}\left(\int_{0}^{\xi_{1}} q(\tau) f(r, u(r)) d \tau\right) d s>\frac{c}{K}\left(1-\xi_{1}\right) \phi^{-1}\left(\int_{0}^{\xi_{1}} q(\tau) d \tau\right) \\
& =c
\end{aligned}
$$

Hence, condition $(i)$ is satisfied.
Secondly, we show that (ii) of Lemma 4 is satisfied. For this, we select $u \in \partial \mathcal{P}(\beta, b)$. Then, $\beta(u)=\max _{t \in\left[\xi_{1}, \xi_{n-2}\right]} u(t)=u\left(\xi_{1}\right)=b$, this means $0 \leq u(t) \leq b$, for all $t \in\left[\xi_{1}, 1\right]$. Noticing that $\|u\| \leq \frac{1}{M} \gamma(u)=\frac{1}{M} \beta(u)=\frac{1}{M} b$, we get

$$
0 \leq u(t) \leq \frac{1}{M} b, \text { for } 0 \leq t \leq 1 .
$$

Then, assumption (A4) implies $f(t, u)<\phi\left(\frac{b}{L}\right)$. Therefore

$$
\begin{aligned}
\beta(T u) & =\max _{t \in\left[\xi_{1}, \xi_{m-2}\right]}(T u)(t)=(T u)\left(\xi_{1}\right) \\
& \leq \frac{1}{1-\sum_{i=1}^{m-2} b_{i}} \phi^{-1}\left(\frac{1}{1-\sum_{i=1}^{m-2} a_{i}} \int_{0}^{1} q(\tau) f(\tau, u(\tau)) d \tau\right) \\
& <\frac{b}{L} \frac{1}{1-\sum_{i=1}^{m-2} b_{i}} \phi^{-1}\left(\frac{1}{1-\sum_{i=1}^{m-2} a_{i}} \int_{0}^{1} q(\tau) d \tau\right)=b .
\end{aligned}
$$

So, we get $\beta(T u)<b$. Hence, condition (ii) is satisfied.
Finally, we show that the condition (iii) of Lemma 4 is satisfied. We note that $u(t)=a\left(\frac{\sum_{i=1}^{m-2} b_{i}-1}{1-\sum_{i=1}^{m-2} b_{i} \eta_{i}} t+1\right), 0 \leq t \leq 1$ is a member of $\mathcal{P}(\alpha, a)$, and so $\mathcal{P}(\alpha, a) \neq \emptyset$. Now, let $u \in \partial \mathcal{P}(\alpha, a)$. Then $\alpha(u)=\max _{t \in\left[0, \xi_{n-2}\right]} u(t)=u(0)=a$. This implies

$$
0 \leq u(t) \leq a, t \in[0,1] .
$$

By assumption (A5), $f(t, u)>\phi\left(\frac{a}{A}\right)$. Then,

$$
\begin{aligned}
\alpha(T u) & =\max _{t \in\left[0, \xi_{n-2}\right]}(T u)(t)=(T u)(0) \\
& \geq \int_{\xi_{1}}^{1} \phi^{-1}\left(\int_{0}^{s} q(\tau) f(\tau, u(\tau)) d \tau\right) d s \\
& >\left(1-\xi_{1}\right) \frac{a}{A} \phi^{-1}\left(\int_{0}^{\xi_{1}} q(\tau) d \tau\right)=a .
\end{aligned}
$$

So, we get $\alpha(T u)>a$. Thus, (iii) of Lemma 4 is satisfied. Hence, the boundary value problem (1) has at least two positive solutions $u_{1}$ and $u_{2}$ satisfying

$$
a<\alpha\left(u_{1}\right) \text { with } \beta\left(u_{1}\right)<b, \text { and } b<\beta\left(u_{2}\right) \text { with } \gamma\left(u_{2}\right)<c \text {. }
$$

## References

1.V. A. Il'in and E. I. Moiseer. Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator. Differential Equations, 23:979-987, 1987.
2.C. Bai and J. Fang. Existence of multiple positive solutions for nonlinear m-point boundary value problems. J. Math. Anal. Appl., 281:76-85, 2003.
3.C. Bai and J. Fang. Existence of multiple positive solutions for nonlinear m-point boundary value problems. Appl. Math. Comput., 140:297-305, 2003.
4.K. Q. Lan. Multiple positive solutions of semilinear differential equations with singularities. J. London Math. Soc., 63:690-704, 2005.
5.R. Ma. Existence theorem for a second order m-point boundary value problem. J. Math. Anal. Appl. 211:545-555, 1997.
6.J. Y. Wang and W. J. Guo. A singular boundary value problem for the onedimensional p-Laplacian. J. Math. Anal. Appl. 201:851-866, 1996.
7.Y. Wang and C. Hou. Existence of multiple positive solutions for one-dimensional p-Laplacian. J. Math. Anal. Appl. 315:144-153, 2006.
8.D. Ji, M. Feng and W. Ge. Multiple positive solutions for multipoint boundary value problems with sign changing nonlinearity. Appl. Math. Comput., 196:511520, 2008.
9.D. Ma, Z. Du and W. Ge. Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator. Comput. Math. Appl., 50:729-739, 2005.
10.R. I. Avery and J. Henderson. Two positive fixed points of nonlinear operators on ordered Banach spaces. Comm. Appl. Nonlin. Anal., 8:27-36, 2000.

# Adaptive Backstepping Controller Design for an Electro Hydraulic Servo System 

Touati Brahim A.<br>Department of automation, University of Boumerdes Boumerdes, Algeria<br>E-mail: ammar.touati@yahoo.fr

Kidouche M.<br>Department of automation, University of Boumerdes Boumerdes, Algeria<br>E-mail: kidouche_m@hotmail.com


#### Abstract

In this paper, an adaptive backstepping controller with tuning functions is designed to enhance tracking performance of electro hydraulic servo system (EHSS). A complete fifth-order nonlinear model of EHSS is presented, in addition to the use of valve dynamics, the friction force considered is nonlinear and the adaptive controller handles with viscous friction and the external disturbance. Simulation results are presented verifying the effectiveness of the developed controller.


Keywords: adaptive backstepping control, tuning functions, electro-hydraulic servo system, nonlinear friction force

## 1. Introduction

Electro hydraulic servo systems (EHSS) have been widely used in industrial applications by virtue of their small size-to-power ratios and the ability to apply very large forces and torques with fast response times. However, hydraulic systems also have a number of characteristics which complicate the development of high performance closed-loop controllers such as the highly nonlinear dynamics of hydraulic systems [1]. The system may be subjected to non-smooth and discontinuous nonlinearities due to control input saturation, directional change of valve opening, friction, and valve overlap. Valves also contain non-measurable states (position and velocity). Aside from the nonlinear nature of hydraulic dynamics, EHSS also have large extent of model uncertainties, such as the external disturbances and leakage that cannot be modelled exactly; and the nonlinear functions that describe them may not be known. In the past, much of the work in the control of hydraulic systems uses linear control theory $[2,3,4]$ and feedback linearization techniques [5, 6]. In [7], nonlinear adaptive control is applied to the force control of an active suspension driven by a double-rod cylinder where only the parametric uncertainties of the cylinder are considered in [8] an adaptive sliding mode controller combined with novel-type Lyapunov function has been developed to compensate nonlinear uncertain parameters caused by the various original control volumes. In [9] novel approach has decomposed the EHSS into two subsystems using graph theoretic decomposition then back integrating to construct the Lyapunov function .

During the last decade, backstepping based design have emerged as powerful tools for stabilizing nonlinear systems for tracking and regulation purposes [10]. In [11] integrator backstepping is used to construct a controller which includes the friction compensation. In $[12,13]$ an adaptive backstepping control of hydraulic manipulators with friction compensation is presented. A third-order nonlinear dynamic model is used for the controller design while LuGre dynamic friction model characterizes the friction forces. In [14], Choux and Hovland developed an adaptive backstepping controller for a nonlinear hydraulicmechanical system considering valve dynamics and linear friction force.
In this paper, we develop an adaptive backstepping controller for a complete fifth order dynamic model of an electro-hydraulic servo system. In addition to the use of valve dynamics, the friction force considered is nonlinear and the adaptive controller handles with viscous friction and the external disturbance. Simulation results are presented verifying the effectiveness of the developed controller.

## 2. System dynamics

In this paper the hydraulic-mechanical system shown in Fig. 1 is considered [1]. The goal of the controller is to make the mass position xL track the reference.
The system is given by the following state equations


Fig. 1. A hydraulic actuator with four-way valve configuration.

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=x_{3}-\frac{1}{m} f_{f}\left(x_{1}, x_{2}\right) \\
& \dot{x}_{3}=-\frac{4 \beta_{e}}{V_{t}} C \sqrt{P_{s}-\operatorname{sgn}\left(x_{4}\right) \frac{m}{S} x_{3} x_{4}}-\frac{4 \beta_{e}}{V_{t}} S x_{2}-\frac{4 \beta_{e}}{V_{t}} \frac{m}{S} C_{t m} x_{3} \\
& \dot{x}_{4}=x_{5} \\
& \dot{x}_{5}=w_{e}^{2} u-w_{e}^{2} x_{4}-2 \zeta_{e} w_{e} x_{5} . \tag{6}
\end{align*}
$$

Certain particular assumptions are also considered as follows:
Assumption 1: The nonlinear function $f_{f}$ is defined by the inequality $f_{f} \leq \Delta\left(x_{1}^{2}, x_{2}^{2}\right)$, and $\quad \Delta\left(x_{1}^{2}, x_{2}^{2}\right)=\zeta_{1} x_{1}^{2}+\zeta_{2} x_{2}^{2} \quad([7,15])$. As maximum value we can consider $f_{f \text { max }}=\Delta\left(x_{1}^{2}, x_{2}^{2}\right)$.

Assumption 2: Assuming a symmetrical valve, where only positive spool displacement ( ${ }^{X_{v}}$ ) can be studied the valve flow equation can now be simplified as $Q_{L}=C x_{v} \sqrt{P_{S}-P_{L}}$
Under the above assumption the system (6) can be written as

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=x_{3}+\varphi_{2}^{T}\left(x_{1}, x_{2}\right) \theta \\
& \dot{x}_{3}=b_{3} \beta(x) x_{4}+\varphi_{3}^{T}\left(x_{2}, x_{3}\right) \theta \\
& \dot{x}_{4}=x_{5} \\
& \dot{x}_{5}=b_{5} u+\varphi_{5}^{T}\left(x_{4}, x_{5}\right) \theta . \tag{7}
\end{align*}
$$

with $\beta(x)=\sqrt{P_{s}-\frac{m}{s} x_{3}}, b_{3}=-\frac{4 \beta_{e}}{V_{t}} C$, and
$\theta=\left[-\frac{1}{m} \zeta_{1},-\frac{1}{m} \zeta_{2},-\frac{4 \beta_{e}}{V_{t}} S, \frac{4 \beta_{e}}{V_{t}} \frac{m}{S} C_{t m},-w_{e}^{2},-2 \zeta_{e} w_{e}\right]$

## 3. Adaptive backstepping controller design

System in (7) is in strict feedback form with an unknown non constant virtual control coefficient $b_{3} \beta(x)$. An extension of the tuning functions design presented in [10] for the above system is described as follows:
Step 1. Introducing the first two error variables

$$
\begin{align*}
& z_{1}=x_{1}-x_{r} .  \tag{8}\\
& z_{2}=x_{2}-\alpha_{1}-x_{r}^{(1)} \tag{9}
\end{align*}
$$

We rewrite $\dot{x}_{1}=x_{2}+\varphi_{1}^{T}\left(x_{1}\right) \theta$, the first equation of (7), as

$$
\begin{equation*}
\dot{z}_{1}=z_{2}+\alpha_{1}+w_{1}^{T}\left(x_{1}\right) \theta \tag{10}
\end{equation*}
$$

where, for uniformity with subsequent steps, we have defined the first regressor vector as

$$
\begin{equation*}
w_{1}=\varphi_{1}^{T}\left(x_{1}\right) \theta=0 . \tag{11}
\end{equation*}
$$

Our task in this step is to stabilize (10) with respect to the Lyapunov function

$$
\begin{equation*}
V_{1}=\frac{1}{2} z_{1}^{2}+\frac{1}{2} \theta^{T} \Gamma^{-1} \theta \tag{12}
\end{equation*}
$$

where $\Gamma$ is a positive definite matrix and $\tilde{\theta}=\theta-\hat{\theta}$.
The derivative of (12) along the solutions of (10) is

$$
\begin{equation*}
\dot{V}_{1}==-c_{1} z_{1}^{2}+z_{1} z_{2} . \tag{13}
\end{equation*}
$$

We can eliminate $\tilde{\theta}$ from $\dot{V}_{1}$, with the update law $\dot{\hat{\theta}}=\Gamma \tau_{1}$ Where

$$
\begin{equation*}
\tau_{1}=w_{1}\left(x_{1}\right) z_{1}=0 . \tag{14}
\end{equation*}
$$

If $x_{2}$ were our actual control, we would let $z_{2} \equiv 0$, that is, $x_{2} \equiv \alpha_{1}$. Then to make $\dot{V}_{1}=-c_{1} z_{1}^{2}$, we would choose

$$
\begin{equation*}
\alpha_{1}=-c_{1} z_{1} . \tag{15}
\end{equation*}
$$

But since $x_{2}$ is not our control we have $z_{2} \neq 0$, and we do not use $\dot{\hat{\theta}}=\Gamma \tau_{1}$, as an update law. Instead, we retain $\tau_{1}$ as our first tuning function and tolerate the presence of $\tilde{\theta}$ in $\dot{V}_{1}$ :

$$
\begin{equation*}
\dot{V}_{1}=-c_{1} z_{1}^{2}+z_{1} z_{2}-\hat{\theta}^{T}\left(\Gamma^{-1} \dot{\hat{\theta}}-\tau_{1}\right) \tag{16}
\end{equation*}
$$

The second term $z_{1} z_{2}$ in $\dot{V}_{1}$ will be cancelled at the next step. With $\alpha_{1}$ as in (14), the $Z_{1}$-system becomes

$$
\begin{equation*}
\dot{z}_{1}=-c_{1} z_{1}+z_{2} . \tag{17}
\end{equation*}
$$

Step 2. We now consider that $x_{3}$ is the control variable in the second equation of (7). Introducing

$$
\begin{equation*}
z_{3}=x_{3}-\alpha_{2}-x_{r}^{(2)} . \tag{18}
\end{equation*}
$$

We rewrite $\dot{x}_{2}=x_{3}+\varphi_{2}^{T}\left(x_{1}, x_{2}\right) \theta$ as

$$
\begin{equation*}
\dot{z}_{2}=-z_{3}+\alpha_{2}+w_{2}^{T} \theta-\frac{\partial \alpha_{1}}{\partial x_{1}} x_{2}-\frac{\partial \alpha_{1}}{\partial \hat{\theta}} \dot{\hat{\theta}}-\frac{\partial \alpha_{1}}{\partial x_{r}} x_{r}^{(1)} \tag{19}
\end{equation*}
$$

Where the second regressor vector $w_{2}$ is defined as

$$
\begin{equation*}
w_{2}=\varphi_{2}-\frac{\partial \alpha_{1}}{\partial x_{1}} \varphi_{1} . \tag{20}
\end{equation*}
$$

Our task in this step is to stabilize the $\left(z_{1}, z_{2}\right)$-system (17), (19) with respect to

$$
\begin{equation*}
V_{2}=V_{1}+\frac{1}{2} z_{1}^{2} . \tag{21}
\end{equation*}
$$

We get $\dot{V}_{2}=-C_{1} \mathrm{Z}_{1}^{2}-c_{2} \mathrm{Z}_{2}^{2}$, we have to choose $\alpha_{2}$ in such a way to make the bracketed term multiplying $z_{2}$ in (21) equal to $-c_{2} z_{2}$, namely

$$
\begin{equation*}
\alpha_{2}=-c_{1} z_{1}-z_{1}-\hat{\theta}^{T} w_{2}+\frac{\partial \alpha_{1}}{\partial x_{1}} x_{2}+\frac{\partial \alpha_{1}}{\partial \hat{\theta}} \dot{\hat{\theta}}+\frac{\partial \alpha_{1}}{\partial x_{r}} \Gamma \tau_{2} \tag{22}
\end{equation*}
$$

We retain $\alpha_{2}$ as our second tuning function in the term $\Gamma \tau_{2}$, which replaces $\dot{\hat{\theta}}$.
However, we do not use $\dot{\hat{\theta}}=\Gamma \tau_{2}$ as an update law, so that the resulting $\dot{V}_{2}$ is
$\dot{V}_{2}=-c_{1} z_{1}^{2}-c_{2} z_{2}^{2}+z_{2} z_{3}+z_{2} \frac{\partial \alpha_{1}}{\partial \hat{\theta}}\left(\Gamma \tau_{2}-\dot{\hat{\theta}}\right)+\tilde{\theta}^{T}\left(\tau_{2}-\Gamma^{-1} \dot{\hat{\theta}}\right)$.
the $\dot{Z}_{2}$ becomes

$$
\begin{equation*}
\dot{z}_{2}=-c_{2} z_{2}+z_{3}-z_{1}+w_{2}^{T} \tilde{\theta} \tag{24}
\end{equation*}
$$

Step 3. Proceeding to the third equation in (7) we introduce

$$
\begin{equation*}
z_{4}=x_{4}-\alpha_{3}-\frac{\hat{\rho}}{\beta(x)} x_{r}^{(3)} \tag{25}
\end{equation*}
$$

where $\hat{\rho}=\frac{1}{b_{3}}$,and rewrite $\dot{x}_{3}=b_{3} \beta(x) x_{4}+\varphi_{3}^{T}\left(x_{2}, x_{3}\right) \theta$ as
$\dot{z}_{3}=\dot{x}_{3}-\dot{\alpha}_{2}-x_{r}^{(3)}$
where the third regressor vector $w_{3}$ is defined as

$$
\begin{equation*}
w_{3}=\varphi_{3}-\sum_{j=1}^{2} \frac{\partial \alpha_{2}}{\partial x_{j}} \varphi_{j} \tag{27}
\end{equation*}
$$

Our task in this step is to stabilize the $\left(z_{1}, z_{2}, z_{3}\right)$-system with respect to

$$
\begin{equation*}
V_{3}=V_{2}+\frac{1}{2} z_{3}^{2}+\frac{1}{2 \gamma} \tilde{b}_{3}^{2}+\frac{\left|b_{3}\right|}{2} \tilde{\rho}^{2} \tag{28}
\end{equation*}
$$

Whose derivative $\dot{V}_{3}=-c_{1} z_{1}^{2}-c_{2} z_{2}^{2}+\tilde{\theta}^{T}\left(\tau_{2}+z_{3} w_{3}-\Gamma^{-1} \dot{\hat{\theta}}\right)$

$$
\begin{equation*}
+\hat{b}_{3} \beta(x) z_{3} z_{4}-\left(b_{3}\left(x_{r}^{(3)}+\bar{\alpha}_{3}\right) z_{3}+\frac{\left|b_{3}\right|}{\gamma} \dot{\bar{\rho}}\right) \tilde{\rho} \tag{29}
\end{equation*}
$$

We can eliminate $\hat{\rho}$ from $\dot{V}_{3}$ with the update law

$$
\begin{equation*}
\dot{\hat{\rho}}=-\gamma \operatorname{sign}\left(b_{3}\right)\left(x_{r}^{(3)}+\bar{\alpha}_{3}\right) z_{3} \tag{30}
\end{equation*}
$$

the $\dot{Z}_{3}$ becomes

$$
\begin{align*}
& \dot{z}_{3}=-c_{3} z_{3}-z_{2}+\hat{b}_{3} \beta(x) z_{4}+\tilde{\theta}^{T} w_{3} \\
&+b_{3}\left(x_{r}^{(3)}+\bar{\alpha}_{3}\right) \tilde{\rho}+\tilde{b}_{3} \beta(x) z_{4} \tag{31}
\end{align*}
$$

Step 4. we introduce

$$
\begin{equation*}
z_{5}=x_{5}-\alpha_{4}-\frac{\hat{\rho}}{\beta(x)} x_{r}^{(4)} \tag{32}
\end{equation*}
$$

and $\dot{X}_{4}=x_{5}+\varphi_{4}^{T} \theta$ rewrite as

$$
\begin{align*}
\dot{z}_{4}= & z_{5}+\alpha_{4}-\frac{\dot{\hat{\rho}}}{\beta(x)} x_{r}^{(3)}+\tilde{\theta}^{T} w_{4}+\varphi_{4}^{T} \hat{\theta}-\frac{\partial \alpha_{3}}{\partial \hat{\theta}} \dot{\hat{\theta}} \\
& +\hat{\rho} \frac{m}{s} \frac{x_{r}^{(3)}}{2 \beta(x)^{2}} b_{3} x_{4}+\hat{\rho} \frac{m}{s} \frac{x_{r}^{(3)}}{2 \beta(x)^{2}} \varphi_{3}^{T} \hat{\theta} \\
- & \sum_{k=1}^{3} \frac{\partial \alpha_{3}}{\partial x_{k}} x_{k+1}-\sum_{k=1}^{3} \frac{\partial \alpha_{3}}{\partial x_{r}^{(k-1)}} x_{r}^{(k)}-\sum_{k=1}^{3} \frac{\partial \alpha_{3}}{\partial x_{k}} \varphi_{k}^{T} \theta+\frac{\partial \alpha_{3}}{\partial \hat{\rho}} \dot{\hat{\rho}} . \tag{33}
\end{align*}
$$

where the third regressor vector $W_{4}$ is defined as

$$
\begin{equation*}
w_{4}=-\frac{\partial \alpha_{3}}{\partial x_{2}} \varphi_{2}-\frac{\partial \alpha_{3}}{\partial x_{3}} \varphi_{3}+\frac{m}{2 s} \frac{\hat{\rho}}{\beta(x)^{3}} x_{r}^{(3)} \varphi_{3} . \tag{34}
\end{equation*}
$$

Our task in this step is to stabilize the $\left(Z_{3}, Z_{3}, Z_{3}\right)$-System with respect to

$$
\begin{equation*}
V_{4}=V_{3}+\frac{1}{2} z_{4}^{2} \tag{35}
\end{equation*}
$$

whose derivative along (17),(19),(26) and (33) is

$$
\begin{align*}
\dot{V}_{4} & =-c_{1} z_{1}^{2}-c_{2} z_{2}^{2}-c_{3} z_{3}^{2}-c_{4} z_{4}^{2} \\
\dot{V}_{4} & =-c_{1} z_{1}^{2}-c_{2} z_{2}^{2}-c_{3} z_{3}^{2}+\tilde{\theta}^{T}\left(\tau_{3}+z_{4} w_{4}-\Gamma^{-1} \dot{\hat{\theta}}\right) \\
& +z_{4}\left[z_{5}+\alpha_{5}-\frac{\hat{\rho}}{\beta(x)} x_{r}^{(3)}+\hat{\rho} \frac{m}{s} \frac{x_{r}^{(3)}}{2 \beta(x)^{3}} b_{3} \beta(x) x_{4}+\hat{\theta}^{T} w_{4}\right. \\
& -\sum_{k=1}^{3} \frac{\partial \alpha_{3}}{\partial x_{k}} x_{k+1}-\sum_{k=1}^{3} \frac{\partial \alpha_{3}}{\partial x_{r}^{(k-1)}} x_{r}^{(k)}-\frac{\partial \alpha_{3}}{\partial \hat{\theta}} \dot{\hat{\theta}} \\
& \left.+\frac{\partial \alpha_{3}}{\partial \hat{\rho}} \dot{\hat{\rho}}-\hat{b}_{3} \beta(x) \frac{\partial \alpha_{3}}{\partial x_{3}} x_{4}-\sum_{k=2}^{3} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma w_{4} z_{k}\right] \\
& +\hat{b}_{3} \beta(x) z_{3} z_{4}-\left(b_{3}\left(\bar{\alpha}_{3}+x_{r}^{(3)}\right) z_{3}+\frac{\left|b_{3}\right|}{\gamma} \dot{\hat{\rho}}\right) \tilde{\rho} . \tag{36}
\end{align*}
$$

Step 5. At the final step, we introduce

$$
\begin{equation*}
z_{5}=x_{5}-\alpha_{4}-\frac{\hat{\rho}}{\beta(x)} x_{r}^{(4)} \tag{37}
\end{equation*}
$$

and $\hat{\lambda}=\frac{1}{\hat{b}_{3}}$ where the e control $u$ is chosen as:

$$
\begin{equation*}
u=\hat{\lambda}\left(\alpha_{5}+\frac{\hat{\rho}}{\beta(x)} x_{r}^{(4)}\right) \tag{38}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\dot{V}=-\sum_{i=1}^{5} c_{i} z_{i}^{2} \tag{39}
\end{equation*}
$$

## 5. Simulation results

Results of simulations are presented in this section, the dynamics model of the valve is represented by a second order transfer function, the friction in the cylinder is nonlinear and moreover the compressibility of the fluid is not neglected inside the load and thus can the cylinder accumulate fluid. The values of the system parameters used in the model are in [17].


Fig. 1 Position tracking error of the system using a step reference


Fig 3. Tracking error of the system using a step reference


Fig 5. Position tracking using $0.1 \sin (t)$ as reference input


Fig 2. tracking error using of the system using reference input


Fig 4. Position tracking using 0.1 snn ( t ) as reference input


Fig 6. . Controller effort using $0.1 \sin (t)$ as reference input

## 5. Conclusion

A systematic methodology for the design of a nonlinear adaptive backstepping controller for single rod electro-hydraulic servo actuator has been presented in this work. The model used for the controller design is a nonlinear fifth-order system model which takes into account the valve dynamic system. The friction force is considered nonlinear which has enhanced the modelling and as result the transient performance. Finally, the simulations confirm that the new
proposed control law is effective and robust against parametric uncertainties and achieves satisfactory the tracking task in different reference inputs

## References

[1] H. E. Merritt, Hydraulic control systems. New York: Wiley, 1967.
[2] T. C. Tsao, M. Tomizuka, "Robust adaptive and repetitive digital control and application to hydraulic servo for noncircular machining," ASME J. Dynamic Systems, Measurement, and Control, vol. 116, pp. 24-32, 1994.
[3] P. M. Fitz Simons ,J. J. Palazzolo, " Part i: Modeling of a one-degree of freedom active hydraulic mount; part ii: Control," ASME J. Dynamic Systems, Measurement, and Control, vol. 118, no. 4, pp. 439-448, 1996.
[4] A. R. Plummer and N. D. Vaughan, "Robust adaptive control for hydraulic servo systems," ASME J. Dynamic System, Measurement and Control, vol. 118,pp. 237244, 1996.
[5] R. Vossoughi , M. Donath, "Dynamic feedback linearization for electrohydraulically actuated control systems," ASME J. Dynamic Systems, Measurement, and Control, vol. 117, no. 4, pp. 468-477, 1995.
[6] L. D. Re and A. Isidori, "Performance enhancement of nonlinear drives by feedback linearization of linear-bilinear cascade models," IEEE Trans. on Control Systems Technology, vol. 3, no. 3, pp. 299-308, 1995.
[7] A. Alleyne and J. K. Hedrick, "Nonlinear adaptive control of active suspension," IEEE Trans. on Control Systems Technology, vol. 3, no. 1, pp. 94-101, 1995.
[8] G. Cheng and Z. Shan , "Derivative and integral sliding mode adaptive control for a class of nonlinear system and its application to an electro-hydraulic servo system," Automatica doi: CNKI:SUN:ZGDC.0.2005-04-00I (2005)
[9] M. kidouche; H. Habbi "On Lyapunov stability of interconnected nonlinear systems recursive integration methodology" Nonlinear Dynamics (Springer) DOI: 1007/s11071-009-9588-7, No.60, pp.183-191, 2010
[10] Krstic, M., Kanellakopoulos, I., Kokotovic, P.V.: "Nonlinear and Adaptive Control Design". Wiley, New York (1995)
[11] G. A. Sohl, J. E. Bobrow, "Experiments and simulations on the nonlinear control of a hydraulic servo system," IEEE Trans. on Control Systems Technology, vol. 7, no. 2, pp. 238-247, 1999.
[12] Zeng, H., Sepehri, N. "Adaptive backstepping control of hydraulic manipulators with friction compensation using LuGre model". In Proc. American Control Conference. pp. 3164-3169, 2006.
[13] Zeng, H., Sepehri, N. "Tracking Control of Hydraulic Actuators Using a LuGre Friction Model Compensation" .Journal of Dynamic Systems, Measurement, and Control, 130(1): 0145021 \{0145027. doi: 10.1115/1.2807181. 2008
[14] M, Choux, G, Hovland. "Adaptive Backstepping, Control of Nonlinear HydraulicMechanical System Including Valve Dynamics". Modelling, Identification and Control, Vol. 31, No. 1, pp. 35-44, 2010
[15] Y. Hong, F. Z. Jing, W, X, Young.: "Nonlinear control for a class of hydraulic servo system" Journal of Zhejiang University 5(11), pp. 1413-1417, 2004
[16] . M. Fitz Simons and J. J. Palazzolo, "Part i: Modeling of a one-degree of-freedom active hydraulic mount; part ii: Control," ASME J. Dynamic Systems, Measurement, and Control, vol. 118, no. 4, pp. 439-448, 1996.
[17] A. Touati Brahim "Adaptive backstepping controller design using tuning functions approach: application to electro hydraulic servo system" Master's thesis, University of Boumerdes, 2012

# Approche probabiliste de la fonction conduite d'un aquifère hétérogène. Exemple de la plaine d'Annaba-Algérie 

TOUBAL Ahmed Chérif<br>Université des Sciences et de la Technologie USTHB Bab-Ezzouar, Alger, Algérie<br>E-mail : toubal@hotmail.com

Résumé : Encaissée dans un bassin subsident, la plaine d'Annaba renferme un système aquifère multicouche à structure complexe. Les alluvions grossières s'y concentrent en chenaux extrêmement productifs mais dans les interfluves, les formations marneuses forment de hauts fonds à potentiel hydraulique médiocre. Bien qu’il existe une continuité hydraulique au sein des sédiments, la forte hétérogénéité qui les caractérise, induit une répartition chaotique de leurs propriétés hydrodynamiques. Face à un tel désordre, la fonction de transfert de l'aquifère est identifiée par une approche probabiliste, basée sur la théorie des variables régionalisée [1]. L’approche choisie porte sur la quantification des lois spatiales de ce paramètre hydrodynamique et l'estimation de ses valeurs moyennes par la technique du krigeage ordinaire. L'évaluation de ce paramètre est ensuite affinée grâce à la géostatistique multivariable (cokrigeage, méthode régressive) qui permet l'implication de procédés géophysiques fiables et peu onéreux. La recherche fournit dans un cadre probabiliste, les éléments cartographiques indispensables à une implantation judicieuse d’ouvrages de captage.

Mots clés: $\quad$ Seybouse - Théorie des variables régionalisées - Transmissivité- Résistance Transversale.

## 1. Un aquifère hétérogène

L'étude géologique révèle la nature complexe des matériaux comblant la plaine d'Annaba-centre (Fig.4d). On y reconnaît, des niveaux discontinus de graviers et sables d’âge plio-quaternaire, séparés parfois par de faibles épaisseurs d'argile lenticulaire [2]. Exploités par une centaine de forages, ils constituent un réservoir économiquement intéressant [2]. L’hétérogénéité du milieu justifie le recours à la théorie des fonctions aléatoires.

## 2. Méthodologie

2.1. Le krigeage ordinaire : La valeur estimée d'une variable régionalisée est donnée par une moyenne pondérée de valeurs mesurées, selon la formule [3 \& 4]: $\mathrm{Z}_{0}^{*}=\mathrm{Z}^{*}\left(\mathrm{x}_{0}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{0}^{\mathrm{i}} \mathrm{Z}_{\mathrm{i}}$
$\mathrm{Z}_{0}^{*}$ est l'estimation de la valeur exacte $\mathrm{Z}_{0}$ au point $\mathrm{x}_{0}$. Le problème consiste à trouver les poids $\lambda^{\mathrm{i}}$ qui donneront la meilleure estimation possible.


Fig. 1- Variogramme expérimental et ajustement théorique- $\gamma(\mathrm{h})=0.15 \operatorname{Lin}(16)+0.075$

La reconnaissance optimale du champ des transmissivités requiert les étapes successives suivantes [5]: 1) -Transformation logarithmique des transmissivités, en vertu d'une corrélation plus forte de $\mathrm{Z}=\mathrm{LogT}$; 2) - Estimation du variogramme expérimental à partir d'un échantillonnage discret du paramètre étudié (Fig. 1); 3) - ajustement d'une fonction théorique au variogramme brut (Fig. 1) ; 4) - contrôle par «validation croisée» de la pertinence des modèles retenus. [7, 8] ; 5)- cartographie des Transmissivités par krigeage ordinaire, avec un « voisinage glissant » [6].
2.2. La Géostatistique multivariable : Il s’agira ici de régénérer le champ des transmissivités dans des secteurs où l'information hydrodynamique fait défaut par absence de forages, en ayant recours à la prospection électrique (Résistance Transversale) (F.ig. 4a, b et c).

Krigeage associé à une régression linéaire : La méthode est appliquée dès lors qu'on aie pu dégager des relations linéaires évidentes entre la transmissivité et la résistance transversale (Fig. 2 \& 4ab )


Fig. 2- Droite d'ajustement par régression linéaire $\log T=0.719$ LogRt -4.875 -

Le cokrigeage permet, lui, d'estimer une variable régionalisée en utilisant en même temps les mesures de plusieurs variables (Fig.3). Dans le cas de K variables, l'estimation pour la $\mathrm{r}^{\text {ième }}$ variable s'écrit [7 \& 8]:

$$
\mathrm{Z}_{\mathrm{r}}^{*}\left(\mathrm{x}_{0}\right)=\sum_{\mathrm{p}=1}^{\mathrm{K}} \sum_{\mathrm{i}=1}^{\mathrm{n}_{\mathrm{p}}} \lambda_{\mathrm{i}}^{\mathrm{p}} \mathrm{Z}_{\mathrm{p}}\left(\mathrm{x}_{\mathrm{i}}\right)
$$



Fig.3-Variogramme croisé.

## 3. Discussion

Krigeage ordinaire: La carte krigée relative à la plaine d'Annaba (non reproduite ici) est peu nuancée. On y distingue néanmoins une zone de bonnes transmissivités le long de la vallée de la Seybouse.

Méthode régressive : La carte des transmissivités relative à la plaine de la Seybouse (Fig.5a) parait plus nuancée, avec des valeurs fluctuant dans une gamme plus large ( $0.810^{-3}$ à $8.5 \cdot 10^{-3} \mathrm{~m}^{2} / \mathrm{s}$ ). On notera l'apparition d'une anomalie à fortes valeurs $\left(8.10^{-3} \mathrm{~m}^{2} / \mathrm{s}\right)$ au sud-ouest des Salines tandis que toute la partie orientale du domaine, est caractérisée par des valeurs plus basses ( 2 à $4.10^{-3} \mathrm{~m}^{2} / \mathrm{s}$ ).

Le cokrigeage : La carte en courbes isovaleurs, (Fig. 5b) se révèle nettement plus différenciée que les précédentes. La carte montre trois secteurs distincts qui coïncident avec des zones de subsidence bien précises.

- Le secteur occidental qui englobe la vallée de l'oued Seybouse, concorde avec l'axe du bassin d'effondrement de Ben-Ahmed. Il se caractérise par les valeurs de transmissivité les plus élevées ( $710^{-3}$ à $10,6.10^{-3} \mathrm{~m}^{2} / \mathrm{s}$ ). Dans ce secteur fortement subsident, l'oued Seybouse a creusé de véritables canyons dans les argiles de la plaine. Ces canyons ont été, par la suite, remblayés de dépôts grossiers à forte perméabilité. Erosion et sédimentation ont, au cours du quaternaire, aboutit à la formation d'un paléo-chenal allongé, à écoulement préférentiel.
- Le secteur central correspond au haut fond qui sépare le fossé d'effondrement de Ben Ahmed à l'ouest, du fossé d'effondrement de Ben M'Hidi à l'est. Ce horst qui est représenté à l'affleurement par la butte numidienne de Daroussa tend à s'ennoyer sous les sédiments à mesure que l'on se rapproche de la mer. Dans ce secteur, partiellement épargné par les phases d'érosion et de comblement, les forages ont une productivité relativement faible.
- Le secteur oriental, enfin, correspond au fossé d'effondrement de Ben-M'Hidi dont l'axe est orienté selon la direction NE-SW. Les apports détritiques grossiers sont ici de nouveau abondants mais leur perméabilité moins importante, ce qui contribue à accroître de façon modérée la transmissivité du système aquifère.
On remarquera en conclusion, que la structure des nappes profondes est assez bien rendue et les nombreuses anomalies concentriques prouvent bien l'existence de poches graveleuses très perméables au sein d'un encaissant plus stérile.


## 4. Conclusion

La géostatistique, utilisée ici sous différentes formes, s’avère un outil intéressant pour décrire la variabilité spatiale des transmissivités. Le cokrigeage qui intègre des informations géophysiques bien réparties, offre semble-t-il les estimations les plus nuancées et les plus réalistes.

## Bibliographie

[1] MATHERON, G. 1971. Théorie des Variables Régionalisées et ses Applications. Les cahiers du Centre de Morphologie Mathématique, fasc. 5, ensmp édit., 212 p.
[2] TOUBAL A. C., 1998. Apports de la Géophysique à l'étude des problèmes d'hydrodynamique et d'invasion marine en milieu souterrain. Exemple des plaines d'Annaba de la Mitidja et de la Baie d'Alger. Thèse doctorat ès Sciences, I.S.T.-U.S.T.H.B.,1998, 226 p.
[3] DELHOMME, J. P. 1978. Kriging in the hydrosciences. Advances in Water Resources, vol. 1, nº 5, pp. 251-266.
[4] DELHOMME, J. P. 1979. Spatial variability and uncertainty in groundwater flow parameters: a geostatistical approach. Water Res. Res., 15, 269-280.
[5] DARRICAU-BEUCHER, H. 1981. Approche géostatistique du passage des données de terrain aux paramètres des modèles en hydrogéologie. Thèse de doct-Ing. en Sciences et Techniques Minières, ENSMP, 118 p .
[6] MARSILY G. de. \& S. AHMED. 1987. Application of kriging techniques in ground water hydrology, Jour. Geol. Soc. India, vol. 29 (1), 51-82.
[7] MYERS, D. E. 1984b. Co-Kriging - New Developtments, G. Verly et al. Eds., Geostatistics for Natural Resources Characterization, part 1, 205-305.
[8] ARMSTRONG, M. 1984. Improving the estimation and modelling of the variogram. In G. Verly et al. (Eds.), Geostatistics for Natural Res
[9] BOUFASSA, A. ARMSTRONG, M. Comparison betwween different kriging estimators. Math. Geol., vol. 21 n³, April 1989, 331-345.[D-2441].

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey


## Légende

Points de mesure

- a) - de la transmissivité
- b) - de la résistance transversale
+ c) - du débit spécifique
- Limite de la plaine d'Annaba-centre


Fig. 4- Réseau de mesures et coupe géologique simplifiée.


Fig. 5- Carte des transmissivités obtenues par la méthode régressive (a) et par cokrigeage (b).

Proceedings, 6th Chaotic Modeling and Simulation International Conference 11-14 June 2013 Istanbul, Turkey


[^0]:    *Electronic address: mabdul_10@yahoo.com
    $\dagger$ Electronic address: fsaif@camp.nust.edu.pk

[^1]:    ${ }^{1}$ Corresponding author: Email:mrpsuresh83@gmail.com, Phone: 91-044-24355648,24334845,Fax:24357591.

