# Comprehensive Chaotic Description of Heartbeat Dynamics Using Scale Index and Lyapunov Exponent

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**Abstract.** Since cardiac system is capable of exhibiting chaotic behaviors, many works have been carried out to study its dynamics. In this paper, based on Grudziński and Żebrowski's model, the impacts of external periodic stimuli on cardiac impulses were studied using the scale index and Lyapunov exponent. Obtained results revealed that the scale index can detect special behaviors in the action potential whereas the Lyapunov exponent is not capable of uncovering them. Furthermore, it was found that the non-periodicity of pacemaker rhythms in the presence of external factors is not high, but the restoration of the heart to normal conditions requires medical attentions.

**Keywords:** Chaos, Scale index, Lyapunov exponent, Cardiac system, Action potential.

## 1 Introduction

One of the pioneering mathematical models describing heartbeat dynamics has been established by Van der Pol and Van der Mark [1]. Important similarities between their oscillator behavior and cardiac impulses such as oscillation at rates, without effecting the amplitude of oscillation motivated other researches to extend this topic [2,3]. Recently, Grudziński and Żebrowski [4] have proposed a modification of the original Van der Pol oscillator as a more complete model of pacemaker rhythms by considering effective biological factors in generation of an action potential.

There are several researches focusing on the evidences of nonlinear characteristics and chaotic characteristics in cardiac system dynamics [5,6]. Two basic indicators of chaotic motion is: sensitive dependence on initial conditions and non-periodic long-term behavior [7]. So, if a given dynamical deterministic nonlinear system exhibit two above mentioned characteristics, then it is said to be chaotic. The Lyapunov exponent can be an indicator of sensitive dependence on initial conditions, albeit it cannot specify non-periodicity [8]. Hence, for proper investigation of the chaos, there is a need to a new scale to determine non-periodicity. The scale index proposed in [8], can meet our requirement. It can complement the Lyapunov exponent to an exact discussion about the chaotic behavior. Our innovation is the use of the scale index to comprehensive

study about cardiac impulses.
After the introduction, the mathematical model of the heart pacemaker [4] is reviewed in Section 2. Then a brief discussion about the Lyapunov exponent and the scale index are presented in Section 3. Section 4 includes the obtained results. Finally, conclusions are discussed in Section 5.

## 2 Mathematical Model

In this paper the model proposed by Grudziński and Żebrowski [4] was employed. Periodic and chaotic behaviors of their model correspond to the normal and pathological functioning of the cardiac conducting system, respectively [9]. Their model is as follow:

$$\frac{d^2x}{dt^2} + \alpha(x - v_1)(x - v_2)\frac{dx}{dt} + \frac{x(x+e)(x+d)}{ed} = F(t)$$

$$A, \alpha, e, d > 0, \quad v_1 v_2 < 0.$$

where $\alpha$ changes the refractory time, the $(v_1, v_2)$ pair modify the frequency of the action potential or the value of the resting potential, $e$ together with $d$ control the diastolic period [4] and $F(t) = Asin(\omega t)$ is an external driving which is available for external adjustments [9].
For the sake of simplicity the above equation was transformed to a set of two coupled first-order ordinary differential equations (ODE) [9]:

$$\dot{x} = y$$

$$\dot{y} = F(t) - \alpha(x - v_1)(x - v_2)y - \frac{x(x+e)(x+d)}{ed}$$

Table I illustrates parameter values which was used in this paper to reproduce the normal action potential carrying the main properties of the natural action potential in the absence of external driving. Under this condition, the system can represent periodic behaviors as well as chaotic ones under different values of $A$ with the same initial condition: $[x_0, y_0] = [-0.1, 0.025]$.

| | |
|---|---|
| $A$ | $\in [0, 12]$ |
| $\alpha$ | 15 |
| $v_1$ | 0.5 |
| $v_2$ | -0.3 |
| $e$ | 1.4 |
| $d$ | 1.5 |
| $omega$ | 1.9 |

**Table 1.** Parameter Values

The aim of [4] was to reproduce the normal rhythms of the cardiac system. Moreover, in [9] the authors have demonstrated that different initial conditions may exhibit the various behaviors of the action potential which may be related to the abnormal functioning of the cardiac pacemaker. However, neither [4] nor [9] have investigated the effects of external periodic forcing on the pacemaker impulses.

## 3 Methods

### 3.1 Lyapunov Exponent

The Lyapunov exponent measures the rate of the convergence/divergence of the trajectories of a given dynamical system in the phase space [10]. Indeed, it is a quantitative measure to indicate sensitive dependence on initial conditions. A typical $m$-dimensional dynamical system has $m$ Lyapunov exponent which can be negative, positive or zero. The existence of a positive Lyapunov exponent indicates chaos [10–12]. So, Lyapunov exponents are usually used to determine whether the system is chaotic or not.
Generally, the Lyapunov exponents can be estimated from either the differential equations that govern underlying dynamics or the observed time series [13,14]. In this paper, the Wolf's algorithm in which the Lyapunov exponent in the $i$th direction is computed as follow:

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\delta x_i(t)\|}{\|\delta x_i(0)\|}.$$

was used. Where $p_i(0)$ represents initial distance between two nearly orbits and $p_i(t)$ distance between them after time $t$ in the $i$th direction.
First, the Lyapunov exponent was computed for a given control parameter, $A$, then the amplitude of periodic force was increased by 0.01, and the new Lyapunov exponent was computed versus the new $A$. This process was continued until the whole range of the interval $[0, 12]$ was covered.

### 3.2 Scale Index

Non-periodic behavior means that there are trajectories that do not settle down to periodic or quasi periodic orbits as $t \to \infty$ [7]. Although the Fourier transforms can be used for studying the periodicity of a given signal in frequency domain, it has a limitation, namely, that the signal must be stationary. Therefore, to study the periodicity of a non-stationary signal, the wavelet transforms must be replaced. Recently, the scale index was proposed in the basis of the wavelet transforms as a measure to investigate the degree of non-periodicity [8].The summarized definition of the scale index is introduced as follows.
The CWT of the signal $f$ at time $u$ and scale $s$ is defined as:

$$W_\psi f(u, s) = \int_{-\infty}^{+\infty} f(t)\psi^*_{u,s}(t)dt.$$

where

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi(\frac{t-u}{s}).$$

The inner scalogram $\overline{S}^{inner}$ of $f$ is defined as:

$$\overline{S}^{inner}(s) = (\frac{\int_{c(s)}^{d(s)} |W_\psi f(u,s)|^2 du}{d(s) - c(s)})^{\frac{1}{2}}.$$

Indeed, the inner scalogram is the normalized energy of the CWT of $f$ on interval $[c(s), d(s)]$ at scale $s$. In practice, the inner scalogram is studied on a finite interval $[s_0, s_1]$. The scale $s_{max}$ is a scale for which the inner scalogram reaches its maximum and $s_{min}$ is the smallest ones, i.e., $\overline{S}^{inner}(s_{min}) \leq \overline{S}^{inner}(s)$ for all $s$ such that $s_{max} \leq s \leq s_1$. So, the scale index, $i_{scale}$, of $f$ on the scale interval $[s_0, s_1]$ is computed as follow:

$$i_{scale} = \frac{\overline{S}^{inner}(s_{min})}{\overline{S}^{inner}(s_{max})}.$$

The scale index, $i_{scale}$, is defined such that $0 \leq i_{scale} \leq 1$: Highly non-periodic orbits correspond to the values close to 1 and periodic orbits to the values close to 0. In contrast with the Lyapunov exponent, the bounded feature of the scale index makes it as a much sensible tool in the sense of chaoticity and non-periodicity measures.
In this paper the Daubechies eight-wavelet (db8) function and the scale parameter from $s_0 = 1$ to $s_1 = 512$, with $\Delta s = 1$ were considered in the computation of the $i_{scale}$ for a given control parameter. Increasing control parameter by 0.01, the new $i_{scale}$ was computed for new parameter and this procedure was continued until the whole range of the control parameter was covered.

## 4 Results

In this section the effects of changing the amplitude of external stimuli on action potential are discussed.
In order to investigate different states of the model, the bifurcation analysis method was applied. For do, the amplitude of external forcing was taken as a control parameter. To show the global structure of the model, the amplitude of external forcing $A$, was restricted to vary on a finite interval [0,12]. Figure 1 shows the bifurcation diagram. Both the local maximum and minimum values of the $x$-variable were plotted in the bifurcation diagram. It is clear that both kinds of orbits (periodic and non-periodic) can be achieved with changing $A$.
Accordingly, for the each value of control parameter, the action potential was generated. Figure 1 shows the action potential and related phase spaces in the absence ($A = 0$) and presence of the external forcing. In the absence of any external stimuli, it can be recognized that the heart behaves as a periodic oscillator [15]. Pan et al. [16] have showed that the external stimuli may accelerate the action potential generation in cardiac pacemaker cells. Accordingly, generated action potentials in the presence of external force (e.g. $A = 1.42$ and $A = 4.49$) depicted increase in the heartbeat rhythms generation. This acceleration of the rhythms generation can lead to tachycardia [17]. In addition,

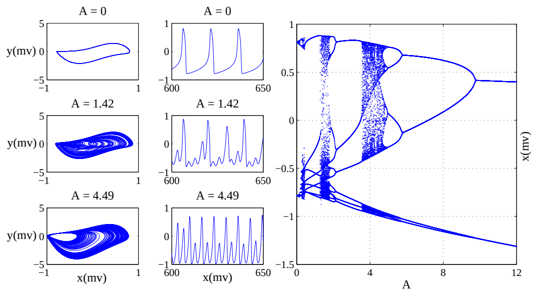


**Fig. 1.** Bifurcation diagram and phase spaces for different values of $A$ and related action potentials.

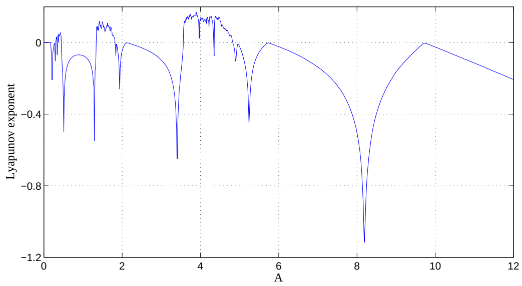


**Fig. 2.** Lyapunov exponent vs. $A$.

these figures showed abnormalities in the action potential. Abnormalities may be caused by strong interrelation between inward and outward ion currents. Especially, it is evident from Figure 1 that changing the amplitude of external periodic driving has been resulted in appearing delayed after depolarization in the action potential. Briefly, these diagrams demonstrated that:

1. The parameter sensitivity of the action potential compared to the amplitude of external forcing was very high [18]. This parameter sensitivity of the action potential confirmed the previous works [19,20] proving the existence of the chaos in the cardiac system.
2. External stimuli may accelerate action potential generation in cardiac pacemaker cells [16].
3. There were three distinct regions defined as $A \in [0.31, 0.44]$,$[1.34, 1.81]$ and $[3.57, 4.82]$ for which the behavior of the system was non-periodic. The region defined as $A \in [1.34, 1.81]$ was more non-periodic in comparison with two others, since the space between two branches of the bifurcation diagram had been filled thoroughly. In this region the functioning of the cardiac pacemaker was more abnormal and no regular patterns in the action potential were observed.

After these recognitions, the Lyapunov exponent need to be computed by considering the different values of control parameter for identifying chaotic regions. Figure 2 shows the Lyapunov exponent versus control parameter $A$. Regions

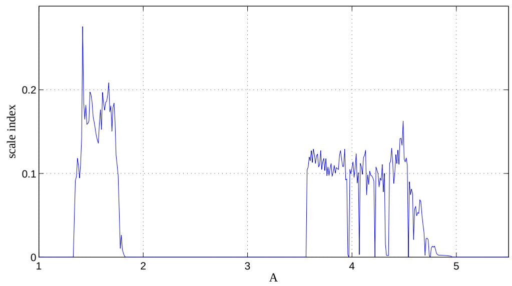


**Fig. 3.** Scale index vs. $A$.

associated with the negative values of the Lyapunov exponent indicated that the system was in stable states and cardiac pacemaker functioning was normal. It is clear from Figure 2 that the value of the Lyapunov exponent for $A \in [0.31, 0.44]$, $[1.34, 1.81]$ and $[3.57, 4.82]$ was positive and cardiac pacemaker functioning was abnormal. These results were in accordance with previous works [19,20] showing the pathological states have positive Lyapunov exponents and hence chaotic nature. Especially, it was found that The variation range of the Lyapunov exponent in the region defined as $A \in [1.34, 1.81]$ was low than the $A \in [3.57, 4.82]$. In terms of the Lyapunov exponent this means that the latter region was more chaotic, whereas the bifurcation diagram showed that the non-periodicity in the latter's was low. This perceived shortcoming of the Lyapunov exponent in addressing whether the system's behavior was more non-periodic or low demands a new measure for properly identifying chaos in the cardiac system. Accordingly, it was illustrated how the scale index $i_{scale}$ can detect the non-periodic orbits of cardiac pacemaker.
In order to show the effectiveness of the scale index, $i_{scale}$, its diagram was compared with the Lyapunov exponent and bifurcation. Figure 3 depicts the variation of $i_{scale}$ versus to $A$. Due to better resolution, $A$ was restricted to vary on a finite interval [1,5.5]. It was shown that:

1. There was a good agreement among the chaotic regions of the bifurcation diagram, regions where the Lyapunov exponent were positive and regions where $i_{scale}$ was much greater than 0. The values of $A$ for which the Lyapunov exponent was negative were also the values for which $i_{scale} \approx 0$. It is noteworthy that the high value of the scale index associates with the chaotic regions [8]. Since pacemaker cells generate periodic impulses in normal conditions and these impulses are responsible for controlling the rate and rhythm of the heartbeats, the high value of the scale index displayed disturbance in the periodicity of the action potential which has been caused by changes in the rate and rhythm of heartbeats.
2. It is significant to know that the relative maximum in the $i_{scale}$ and the sudden expansion of the size of the attractor at $A = 1.42$ were simultaneous. This point is in agreement with overlapping the main branches of the bifurcation diagram which was not detected by the Lyapunov exponent. Also, the same overlapping was occurred for the sudden contraction of the

size of the attractor at $A = 4.49$. As mentioned before, the appearance of two relative maximums indicates the acceleration in the rhythms generation and alternation in the amplitude of the action potential related to the abnormal functioning of the cardiac pacemaker, like arrhythmia.

3. The values range of the scale index on the region defined as $A \in [1.34, 1.81]$ was greater than the region defined as $A \in [3.57, 4.82]$, the result that had previously been expected from the bifurcation diagram.
4. It is also noticeable that the numerical computations resulted that $(i_{scale})_{max} \approx 0.28$, expressing the measure of chaoticity and non-periodicity in the cardiac action potential was not high, but the restoration of the heart to normal conditions requires medical attentions.

## 5 Conclusions

Focusing on the model of Grudziński and Żebrowski, the effects of changing the amplitude of external forcing on pacemaker rhythm were examined and the deficiency of the Lyapunov exponent in detecting non-periodic behaviors in the action potential was shown. Then the ability of the scale index in overcoming the shortcoming of the Lyapunov exponent was explored. Furthermore, it was shown that the measure of the chaos in the action potential was not high. Finally, it was found that the scale index can provide a non-invasive assessment of the heart in real-life conditions.
Next thing to be mentioned is that the results of this study can be improved by analyzing the models that can more accurately represent the natural conditions of the heart dynamics under external factors. Furthermore, the findings of this study provide new perspectives into the new methods should be applied to control of abnormal heartbeats in order to avoid chaotic behaviors in the action potential.

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