# Height Field Representation and Compression Using Fractal Interpolation Surfaces

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**Abstract.** Height fields provide efficient means for representing surface elevation data which can be used for rendering 3D terrains or landscapes. In this paper, a novel method for representing height fields using fractal interpolation techniques is presented. The proposed methodology allows describing natural surfaces with an intrinsic fractal structure in a more convenient manner. Specifically, fractal interpolation surfaces constructed on rectangular domains have been used. Results indicate the feasibility and advantages of the proposed method in terms of quality of representation as well as compression ratios.
**Keywords:** Fractals, IFS, Interpolation, Surfaces.

## 1 Introduction

Height fields provide an efficient tool for representing surface elevation data and are often used, among other applications, in 3D computer graphics for rendering 3D terrains or landscapes. A height field is essentially a 2D array of height values and is usually stored as a raster image; the pixel intensity corresponds to the height at the location defined by the pixel coordinates. An example of landscape rendering based on a height field is shown in Figure 1.

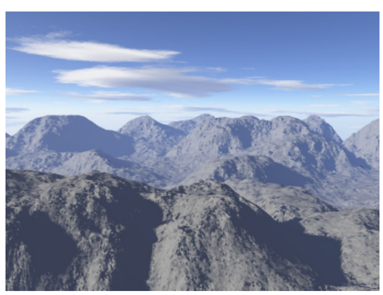


**Fig. 1.** Rendering of a landscape defined by a height field.

Fractal interpolation as defined by Barnsley[1] and other researchers is based on the theory of *iterated function systems*. It provides an alternative to traditional interpolation techniques, aiming mainly at data which present detail at different scales or possess some degree of self-similarity. These properties define an irregular, non-smooth, structure which is inconvenient to be described by using elementary functions such as polynomials. A *fractal interpolation function* is a continuous function whose graph is the attractor of an appropriately chosen iterated function system. In case this graph, usually of non-integral dimension, belongs to the three-dimensional space and has *Hausdorff - Besicovitch dimension* between 2 and 3, the resulting attractor is called *fractal interpolation surface*.

In this paper, a novel methodology for representing height fields using fractal interpolation techniques is introduced. Our motivation stems from the fact that natural surfaces, such as earth terrains, present an intrinsic fractal structure, i.e. detail at multiple scales and some degree of self-similarity. The most important and non-trivial part for constructing fractal interpolation surfaces on rectangular domains involves ensuring their continuity. We also present the application of the proposed methodology to terrain data, indicating its feasibility and advantages in terms of quality of representation as well as compression ratios.

The paper is structured as follows. In Section 2 we briefly review the theory of iterated function systems. The necessary concepts of fractal interpolation surfaces, focusing on the proposed construction on rectangular domains, are introduced in Section 3. In Section 4, we present the proposed methodology for height field representation and compression using the surfaces of the previous section. Section 5 contains the result of the application of our method to terrain data, in terms of quality of representation as well as compression ratios. Finally, Section 6 summarizes our conclusions and indicates areas of further work.

## 2 Iterated function systems

Let $X, Y \subset \mathbb{R}^n$. A function $f\colon X \to Y$ is called a *Hölder function of exponent* $a$ if

$$|f(x) - f(y)| \leq c\,|x - y|^a$$

for $x, y \in X$, $a \geq 0$ and for some constant $c$. Note that, if $a > 1$, the functions are constants. Obviously, $c \geq 0$. The function $f$ is called a *Lipschitz function* if $a$ may be taken to be equal to 1. A Lipschitz function is a *contraction* with *contractivity factor* $c$, if $c < 1$. An *iterated function system*, or *IFS* for short, is a collection of a complete metric space $(X, \rho)$ together with a finite set of continuous mappings $w_n\colon X \to X$, $n = 1, 2, \ldots, N$, where $\rho$ is a distance between elements of $X$. It is often convenient to write an IFS formally as $\{X; w_1, w_2, \ldots, w_N\}$ or, somewhat more briefly, as $\{X; w_{1-N}\}$.

The associated map of subsets $W: \mathcal{H}(X) \to \mathcal{H}(X)$ is given by

$$W(E) = \bigcup_{n=1}^{N} w_n(E) \text{ for all } E \in \mathcal{H}(X),$$

where $\mathcal{H}(X)$ is the metric space of all non-empty, compact subsets of $X$ with respect to some metric, e.g. the Hausdorff metric. The map $W$ is called the *collage map* to alert us to the fact that $W(E)$ is formed as a union or 'collage' of sets. Sometimes $\mathcal{H}(X)$ is referred to as the "space of fractals in $X$" (but note that not all members of $\mathcal{H}(X)$ are fractals).

If $w_n$ are contractions with corresponding contractivity factors $s_n$ for $n = 1, 2, \ldots, N$, the IFS is termed *hyperbolic* and the map $W$ itself is then a contraction with contractivity factor $s = \max\{s_1, s_2, \ldots, s_N\}$ (Barnsley[1], Theorem 7.1, p. 81). In what follows we abbreviate by $f^k$ the $k$-fold composition $f \circ f \circ \cdots \circ f$. The *graph* of $f$ is the set of points $G(f) = \{(x, f(x)) : x \in X\}$.

The *attractor* of a hyperbolic IFS is the unique set $\mathcal{A}_\infty$ for which $\lim_{k\to\infty} W^k(E_0) = \mathcal{A}_\infty$ for every starting set $E_0$. The term attractor is chosen to suggest the movement of $E_0$ towards $\mathcal{A}_\infty$ under successive applications of $W$. $\mathcal{A}_\infty$ is also the unique set in $\mathcal{H}(X)$ which is not changed by $W$, so $W(\mathcal{A}_\infty) = \mathcal{A}_\infty$, and from this important perspective it is often called the *invariant set* of the IFS.

## 3 Rectangular subdomain fractal interpolation surfaces

Fractal interpolation surfaces constructed as attractors of iterated function systems were first proposed by Peter R. Massopust[4], where he considered the case of a triangular domain with coplanar boundary data. A slightly more general construction of such fractal surfaces was later presented by Jeffrey S. Geronimo and Douglas Hardin[3], where the domain used was a polygonal region with arbitrary interpolation points but same contractivity factors. Here, we focus on fractal interpolation surfaces constructed on rectangular domains with arbitrary boundary data and same contractivity factors.

Let $D$ be a closed nondegenerate rectangular region in $\mathbb{R}^2$ and let $S = \{q_0, q_1, \ldots, q_{m-1}\}$ be $m$, with $m > 4$, distinct points in $D$ such that $\{q_0, q_1, q_2, q_3\}$ are the vertices of $D$. Given real numbers $z_0, z_1, \ldots, z_{m-1}$ we wish to construct a function $f$ such that $f(q_j) = z_j$, $j = 0, 1, \ldots, m-1$ and whose graph is self-similar. Notice that the constructed FIS in the present work is not self-affine since it is resulting from bivariate functions. Let us denote by $C(D)$ the linear space of all real-valued continuous functions defined on $D$, i.e. $C(D) = \{f: D \to \mathbb{R} \mid f \text{ continuous}\}$. The basic idea is to decompose $D$ into $N$ subrectangles $R_1, R_2, \ldots, R_N$ with vertices chosen from $S$ and define affine maps $L_i: D \to R_i$ and contractions $F_i: D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, N$ such that $\Phi$ defined by

$$(\Phi f)(x, y) = F_i(L_i^{-1}(x, y), f(L_i^{-1}(x, y))) \qquad (1)$$

maps an appropriate subset of $C(D)$ onto itself. If $L_i$ is invertible, $G(f)$ is mapped onto $G(\Phi(f)|_{R_i})$ by $(L_i(x,y), F_i(x,y,f(x))$. Henceforth we assume that $\{R_i\}_{i=1}^N$ consists of nondegenerate rectangles whose interiors are nonintersecting, $L_i^{-1}(R_i) = D$ and that the set of vertices of $\{R_i\}_{i=1}^N$ equals $S$. Let $k: \{1,2,\ldots,N\} \times \{0,1,2,3\} \to \{0,1,\ldots,m-1\}$ be such that $\{q_{k(i,j)}\}_{j=0}^3$ gives the vertices of $\{R_i\}_{i=1}^N$.

Let $i \in \{1,2,\ldots,N\}$. Since $D$ and $R_i$ are nondegenerate, there is a unique invertible affine map $L_i: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$L_i(q_j) = q_{k(i,j)}, \quad j = 0,1,2,3. \tag{2}$$

Let $s_i$ be given such that $|s_i| < 1$ and $F_i: \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$F_i(x,y,z) = a_i x + b_i y + g_i xy + s_i z + c_i, \tag{3}$$

where $a_i$, $b_i$ and $c_i$ are uniquely determined by

$$F_i(q_j, z_j) = z_{k(i,j)}, \quad j = 0,1,2,3. \tag{4}$$

With these definitions for $L_i$ and $F_i$ we have $\Phi(f)|_{R_i} \in C(R_i)$ and $(\Phi f)(q_{k(i,j)}) = z_{k(i,j)}$, $j = 0,1,2,3$, whenever $f \in C(D)$ and $f(q_j) = z_j$, $j = 0,1,2,3$. If $R_i$ and $R_{i'}$ are adjacent rectangles with common edge $\overline{q_j q_{j'}}$, it remains to be determined whether $\Phi f$ is well-defined along $\overline{q_j q_{j'}}$, i.e., whether $\Phi f$ satisfies the "join-up" condition

$$F_i(L_i^{-1}(x,y), f(L_i^{-1}(x,y))) = F_{i'}(L_{i'}^{-1}(x,y), f(L_{i'}^{-1}(x,y))),$$

for all $(x,y) \in \overline{q_j q_{j'}}$. We consider the case where the graph associated with the tesselation $\{R_i\}_{i=1}^N$ has chromatic number 4. The *chromatic number* of a graph is the least number of symbols required to label the vertices of the graph so that any two adjacent vertices (i.e., joined by an edge) have distinct labels. Since each edge is part of some $R_i$ this implies the vertices $\{q_j\}_{j=0}^{m-1}$ can be labelled with $l = l(j) \in \{0,1,2,3\}$ such that the vertices of each $R_i$ have distinct labels. For $i = 1,2,\ldots,N$ and $j = 0,1,2,3$ let $k(i,j)$ be determined by the condition

$$k(i, l(j')) = j' \quad \text{for all vertices } q_{j'} \text{ of } R_i.$$

Then, Eqs. (2) and (4) become

$$L_i(q_{l(j)}) = q_j, \quad F_i(q_{l(j)}, z_{l(j)}) = z_j \tag{5}$$

for each of the vertices $q_j$ of $R_i$.

Let $C_B(D)$ denote the collection of continuous functions $f$ such that $f(q_j) = z_j$, $q_j \in \partial D$.

**Theorem 31** *Suppose the graph associated with $\{R_i\}_{i=1}^N$ has chromatic number 4. Let $L_i$ and $F_i$, $i = 1,2,\ldots,N$, be determined by (3) and (5) with $s_i = s(|s| < 1)$. Let $\Phi$ be defined by (1). Then $\Phi: C_B(D) \to C_B(D)$ is well-defined and contractive in the sup-norm with contractivity $s$. Furthermore $(\Phi f)(q_j) = z_j$, $j = 0,1,\ldots,m-1$ and $f \in C_B(D)$.*

*Proof.* See Drakopoulos and Manousopoulos[2] □

Then the corresponding IFS is of the form $\{\mathbb{R}^3; w_{1-N}\}$, where

$$w_i(x, y, z) = (L_i(x, y), F_i(x, y, z)).$$

An illustration of this is shown in Fig. 2, where the left part indicates the vertices and connecting edges of $D$ and the middle and right parts of the figure indicate where these vertices are mapped by the domain contractions. For larger data sets, this pattern is repeated as necessary.

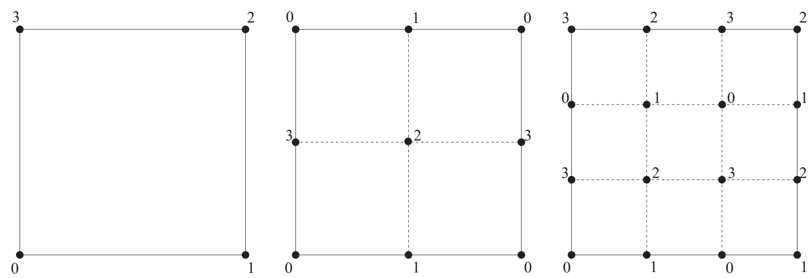



**Fig. 2.** Rectangular domain contractions to satisfy join-up conditions.

## 4 Height fields

A *height field* or *relief map* (see e.g. Theoharis *et al.*[5], p. 505) is defined as a 2D array of height values:

$$H = \{(x_i, y_j, z_{ij} : \ i = 0, 1, \ldots, M \text{ and } j = 0, 1, \ldots, N\},$$

where the $x$, $y$ coordinates define a rectangular grid on the plane and the $z$ coordinate defines the height. The underlying grid is usually regular, i.e.

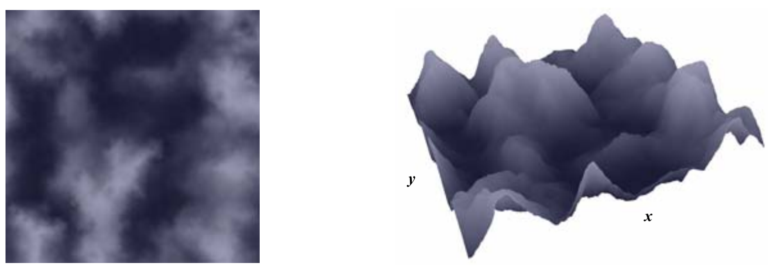
$$x_i = x_0 + i\Delta_x, \quad y_j = y_0 + j\Delta_y,$$

for every $i = 0, 1, \ldots, M$ and $j = 0, 1, \ldots, N$, where $\Delta_x = (x_M - x_0)/M$ and $\Delta_y = (y_N - y_0)/N$. From the above definition, it is clear that a height field can be directly represented by a fractal interpolation surface of Section 3. The only issue to be determined is whether all of the height field data will be used in the construction of the surface or only a subset of them. This can be achieved by regularly sampling the height field along the $x$, $y$ dimensions. The sampling frequency defines a trade-off between quality of representation and compression ratio.

The proposed representation is expected to be especially fruitful for height fields defining natural surfaces, such as terrains. These often possess an intrinsic fractal structure which is conveniently described by fractal interpolation models. The idea of representing natural surfaces using fractal interpolation has also been suggested in Xie *et al.*[6], where the generation of rock fracture surfaces using fractal interpolation was examined.

## 5 Results

A height field of resolution $257 \times 257$ is presented in Figures 3(a) and 3(b). Specifically, the former figure depicts the height field as a 2D raster image where brighter areas indicate greater height; the latter figure contains its 3D depiction. This height field, which was created using Terragen$^{\mathrm{TM}}$ Classic, contains a total of $257 \times 257 = 66049$ points.

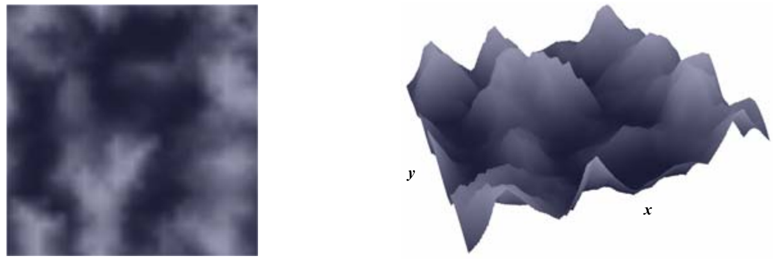



(a) (b)

**Fig. 3.** The original height field depicted (a) as a 2D image and (b) as a 3D surface.

Figures 4(a) and 4(b) show the 2D and 3D representation of this height field, respectively, using the proposed method. Specifically, it has been represented by a fractal interpolation surface constructed on a subset of the original data with $s = 0.02$; every 8th point along each dimension of the height field has been chosen as interpolation point. This results in a rectangular grid of resolution $33 \times 33$, containing 1089 points in total, i.e. about 1.65% of the original points. Despite the significant sparsity of the interpolation points, the quality of the reconstructed height field is satisfactory.

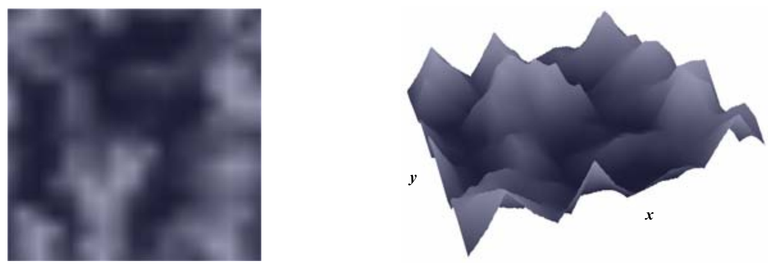
Another example is given in Figures 5(a) and 5(b), where the same height field has been represented by a fractal interpolation surface using even fewer interpolation points. Specifically, every 16th point along each dimension has been chosen as interpolation point. This results in a rectangular grid of resolution $17 \times 17$, containing 289 points in total, i.e. about 0.44% of the original points. Also in this case, the results are satisfactory despite the even smaller number of interpolation points. These results indicate that fractal interpolation surfaces are indeed capable of describing natural surfaces, such




(a) (b)

**Fig. 4.** The reconstructed height field, using every 8th data point as interpolation point, depicted (a) as a 2D image and (b) as a 3D surface.

as terrains, with considerable quality even when high compression ratios are involved.




(a) (b)

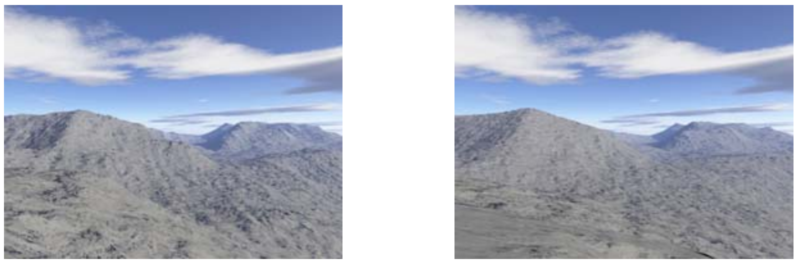
**Fig. 5.** The reconstructed height field, using every 16th data point as interpolation point, depicted (a) as a 2D image and (b) as a 3D surface.

Figures 6–7 depict an artistic rendering of the original height field as well as its two aforementioned reconstructions; these figures were created using Terragen$^{TM}$ Classic. As shown in the figures, the reconstructed height fields produce equivalent results compared to the original one, even though the significant sparsity of the interpolation points.

## 6 Conclusions and future work

We have presented a novel method for the representation and compression of height fields using fractal interpolation techniques. Specifically, we have represented a height field as a fractal interpolation surface constructed on the rectangular domain defined by a subset of the original data. The results indicate that the proposed methodology is feasible, while achieving satisfactory results in terms of quality of representation as well as compression ratios.

**Fig. 6.** Artistic rendering of the original height field.



(a) (b)

**Fig. 7.** Artistic rendering of the reconstructed height field, using as interpolation point (a) every 8th data point and (b) every 16th data point.

Further work will focus on the calculation of the optimal values of the vertical scaling factors in order to achieve increased localized accuracy, as well as on the exploration of alternative fractal interpolation surface models, affine or bivariate, including recurrent fractal interpolation surfaces.

## References


1. M. F. Barnsley. *Fractals everywhere*, 2nd ed., San Diego, 1993. Academic Press Professional.
2. V. Drakopoulos and P. Manousopoulos. Bivariate fractal interpolation surfaces: Theory and applications. *International Journal of Bifurcation and Chaos*, to appear, 2012.
3. J. S. Geronimo and D. Hardin. Fractal interpolation surfaces and a related 2-D multiresolution analysis. *Journal of Mathematical Analysis and Applications*, 176:561-586, 1993.
4. P. R. Massopust. Fractal surfaces. *Journal of Mathematical Analysis and Applications*, 151:275-290, 1990.
5. T. Theoharis, G. Papaioannou, N. Platis, and N. M. Patrikalakis. *Graphics & Visualization: Principles and Algorithms*, Massachusetts, 2008. AK Peters.
6. H. Xie, H. Sun, Y. Ju and Z. Feng. Study on generation of rock fracture surfaces by using fractal interpolation. *International Journal of Solids and Structures*, 38:5765-5787, 2001.