The dynamics of Hamiltonians with non-integrable normal form

Ferdinand Verhulst

Mathematisch Instituut, University of Utrecht, The Netherlands
(E-mail: f.verhulst@uu.nl)

Abstract. In general Hamiltonian systems are non-integrable but their dynamics varies considerably depending on the question whether the corresponding normal form is integrable or not. We will explore this issue for two and three degrees of freedom systems; additional remarks on Hamiltonian chains can be found in [9]. A special device, the quadratic part of the Hamiltonian $H_2(p, q)$ is used to illustrate the results.

Keywords: Hamiltonian Chaos, normal forms, Hamiltonian time series.

1 Integrability versus non-integrability

We will consider time-independent Hamiltonian systems, Hamiltonian $H(p, q)$, $p, q \in \mathbb{R}^n$ with $n \geq 2$ degrees of freedom (DOF). A more detailed study is found in [9]. Regarding mechanics, or more generally dynamical systems, Hamiltonian systems are non-generic.

In addition we have that the existence of an extra independent integral besides the energy for two or more degrees of freedom is again non-generic for Hamiltonian systems (shown by Poincaré in 1892, [4] vol. 1).

So the following question is relevant: why would we bother about the integrability of Hamiltonian systems?

We give a few reasons, leaving out the esthetic arguments:

- Symmetries play a large part in mathematical physics models. Symmetries may sometimes induce integrability but more often integrability of the normal forms. An example is discrete (or mirror) symmetry.
- Near-integrability plays a part in many models of mathematical physics where the integrability, although degenerate, can be a good starting point to analyze the dynamics. Integrals of normal forms may help.
- Non-integrability is too crude a category, it takes many different forms. A first crude characterization is to distinguish non-integrable Hamiltonian systems with integrable or non-integrable normal form.
2 How to pinpoint (non-)integrability?

Looking for a smoking gun indicating integrability there are a few approaches:

1. Poincaré [4] vol. 1:
   A periodic solution of a time-independent Hamiltonian system has two characteristic exponents zero. A second integral adds two characteristic exponents zero except in singular cases. This can be observed (for an explicit Hamiltonian system) as a continuous family of periodic solutions on the energy manifold. Finding such a continuous family can be either a special degeneration of the system or a sign of the existence of an extra integral.

2. Symmetries of course; strong symmetries like spherical or axial symmetry induce extra integrals. Weaker symmetries may or may not induce an integral. An example is studied in [6] where discrete symmetry is explored in two degrees of freedom systems. It is shown for instance that the spring-pendulum displays many degenerations depending on the resonance studied.

3. Degenerations in variational equations or bifurcations are degenerations that often suggest the presence of integrals.

3 Normal forms

There are many papers and books on normalization. A rather complete introduction is [5]. One considers $k$-jets of Hamiltonians:

$$H(p, q) = H_2 + H_3 + \ldots + H_k,$$

usually in the neighbourhood of stable equilibrium $(p, q) = (0, 0)$. The $H_m$ are homogeneous polynomials in the $p, q$ variables.

An important feature is that $H_2(p, q)(t)$ is an independent normal form integral, see [5]; its physical interpretation is that $H_2$ is the energy of the linearized equations of motion. The implication of the existence of this integral is that near stable equilibrium, for two DOF, the normal form is for all resonance ratios integrable so that chaos has for two DOF near stable equilibrium generally a smaller than algebraic measure. This explains a lot of analytic and numerical results in the literature (see again [5]).

In general, for more than two DOF, integrability of the normal form can not be expected without additional assumptions. If we find integrability, it restricts the amount of chaos and also of Arnold diffusion.

**Example:** Braun’s parameter family

Two DOF normal forms are integrable but it is still instructive to consider them. An exemple is Braun’s family of Hamiltonians:

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + \omega^2 q_2^2) - \frac{a_1}{3} q_1^3 - a_2 q_1 q_2^2.$$
The analysis is given in [8] and summarized in [5]; consider for instance \( \omega = 1 \), \( a_1 \) and \( a_2 \neq 0 \) are parameters. The normal form to cubic terms produces two normal modes and, depending on the parameters, two families of in-phase periodic solutions, two families of out-phase periodic solutions and for specific parameter values two continuous families of periodic solutions on the energy manifold; see fig. 1. Normalizing to quartic terms the continuous family at \( a_1/(3a_2) = -1/3 \) (the Hénon-Heiles Hamiltonian) breaks up into separate periodic solutions; the continuous family at \( a_1/(3a_2) = 1/3 \) persists, this Hamiltonian before normalization is already integrable.

4 Three degrees of freedom

Genuine first-order resonances are characterized by its normal form. Apart from the three actions, this contains at least two independent combination angles. We have for three DOF:

- \( 1:2:1 \) resonance
- \( 1:2:2 \) resonance
- \( 1:2:3 \) resonance
- \( 1:2:4 \) resonance

A basic analysis of the normal forms to cubic order \( \tilde{H} = H_2 + \tilde{H}_3 \) yields short-periodic solutions and integrals. The use of integrals gives insight in the geometry of the flow, enables possible application of the KAM-theorem and may produce measure-theoretic restrictions on chaos.

5 Integrability of normal forms

The normal form has two integrals, \( H_2 \) and \( \tilde{H} \) (or \( \tilde{H}_3 \)). Is there a third integral? To establish (non-)integrability we have:
• Ingenious inspection of the normal form or obvious signs of integrability, see van der Aa and F.V. [7].
• Extension into the complex domain and analysis of singularities, see Duistermaat [2].
• Applying Shilnikov-Devaney theory to establish the existence of a transverse homoclinic orbit on the energy manifold, see Hoveijn and F.V. [3].
• Using Ziglin-Morales-Ramis theory to study the monodromy group of a particular nontrivial solution; this study may lead to non-integrability. This involves the variational equation and the characteristic exponents in the spirit of Poincaré. In an extension one introduces the differential Galois group associated with a particular solution; if it is non-commutative, the system is non-integrable. See Christov [1].

5.1 The genuine first-order resonances

A remarkable result is that the normal form to cubic terms of the 1 : 2 : 2 resonance is integrable with quadratic third integral, see [7]. We have that \( p_1 = q_1 = 0 \) corresponds with an invariant manifold of the normal form; the manifold consists of a continuous set of periodic solutions and is a degeneration according to Poincaré with 4 characteristic exponents zero. The calculation of the normal form to quartic terms produces a break-up of this continuous set into six periodic solutions on the energy manifold.

It was shown in [2] that the normal form to cubic terms of the 1 : 2 : 1 resonance is non-integrable. This was shown by singularity analysis in the complex domain. A different approach was used in [1] where it was shown that for a particular solution the monodromy group is not Abelian; this precludes that the normal form is integrable by meromorphic integrals.

Non-integrability was shown in [1] for the 1 : 2 : 4 resonance in a similar way. One identifies a particular solution in the \((p_1, q_1) = (0, 0)\) submanifold; the local monodromy group is not Abelian which precludes integrability.

The case of the 1 : 2 : 3 resonance is different. The analysis in [3] shows that a complex unstable normal mode \((p_2, q_2)\) is present. The normal form contains an invariant manifold \(N\) defined by \(H_2 = E_0, \bar{H}_3 = 0\). \(N\) contains an invariant ellipsoid, also homoclinic and heteroclinic solutions. They are forming an organizing center producing a horseshoe map and chaos in \(H_2 + \bar{H}_3 + \bar{H}_4\). So, the normal form contains only two integrals.

Later, Christov [1] showed by algebraic methods that \(H_2 + \bar{H}_3\) is already non-integrable, but the consequences for the dynamics are not yet clear. Technically, this is his most complicated case.

6 Discussion and consequences

In two DOF the Hamiltonian normal form is integrable to any order; this restricts the chaos near stable equilibrium to exponentially small sets between the invariant tori.
For three and more DOF, the situation is more complicated. If the normal form is integrable, chaos is restricted to sets that are algebraically small with respect to the small parameter that scales the energy with respect to stable equilibrium. We would like to distinguish between various kinds of non-integrability near equilibrium. The chaos is usually localized near homoclinic intersections of stable and unstable manifolds.

The phenomenon is most striking after a Hamiltonian-Hopf bifurcation of a periodic solution has taken place; see for the bifurcation diagram fig. 2.

![Hamiltonian–Hopf bifurcation](image)

**Fig. 2.** As a parameter varies, eigenvalues on the imaginary axis become coincident and then move into the complex plane.

Consider the following explicit examples of the $1 : 2 : 3$ resonance:

$$H(p, q) = \frac{1}{2}(p_1^2 + q_1^2) + (p_2^2 + q_2^2) + \frac{3}{2}(p_3^2 + q_3^2) + H_3(p, q),$$

$$H_3(p, q) = -q_1^2(a_2 q_2 + a_3 q_3) - q_2^2(c_1 q_1 + c_3 q_3) - b q_1 q_2 q_3.$$

We will consider two cases. If $a_2 > b$, analysis of the normal form shows that the $(p_2, q_2)$ normal mode is unstable of type HH (hyperbolic-hyperbolic or 4 real eigenvalues). If $a_2 < b$ the $(p_2, q_2)$ normal mode is unstable of type C (complex eigenvalues). The $H_2(p, q)(t)$ time series is shown in figs. 3 and fig 4. Both time series display chaotic behavior, but the case of instability C involves the Devaney-Shilnikov bifurcation producing strong chaotic behaviour; for more information see [3].

**References**

Fig. 3. The $H_2(p,q)(t)$ time series in the case of normal mode instability HH.

Fig. 4. The $H_2(p,q)(t)$ time series in the case of normal mode instability C.

Nonlinear Impacting Oscillations of a Simply Supported Pipe Conveying Pulsating Fluid Subjected to Distributed Motion Constraints

Yikun Wang\(^1\), Qiao Ni\(^1\), Min Tang\(^1\), Yangyang Luo\(^1\), Hao Yan\(^1\), Lin Wang\(^1\)

1. Department of Mechanics, Huazhong University of Science and Technology, Wuhan 430074, China
2. Hubei Key Laboratory for Engineering Structural Analysis and Safety Assessment, Wuhan 430074, China

Abstract

In this paper, the nonlinear dynamics of simply supported pipe conveying pulsating fluid is investigated, by introducing the effect of distributed motion constraint along the pipe axis modeled as trilinear springs. The internal fluid is assumed to be a harmonic component of flow velocity superposed on a constant mean value. Attention is concentrated on the possible motions of the system with various mean values of flow velocity, pulsating amplitude and frequency. As for the impact force term, the damping effect during impacting process is considered in analyzing. The partial differential equations are then transformed into a set of ordinary differential equations (ODEs) using the Galerkin’s method. The nonlinear dynamical responses are presented in the form of bifurcation diagrams, time histories, phase portraits and power spectrums. Some interesting results have been observed with different parameters.

Keywords

Nonlinear dynamics; chaotic motion; pipe conveying pulsating fluid; distributed motion constraints; bifurcation

1. Introduction

The dynamics of pipes conveying fluid is an important academic topic with broad industrial application, e.g., pump discharge lines, oil pipelines, propellant lines, reactor system components and so forth. They were also the most troublesome elements in these fields. These industries utilize high thermal efficiency shell and heat exchanger designs to avoid failure. Performance requirements often require the devices to be able to work under high coolant velocities and to be flexible tubes, which in turn would cause pipes to experience excessive flow-induced vibrations. A large number of studies have been made on flow-induced vibrations due to the corresponding significance [1-4]. Focuses of flow-induced vibrations were put on these fields in understanding the mechanisms of pipes conveying fluid. Works on this topic appear to have started in the 1960s. Many studies have investigated the stabilities and nonlinear dynamics of pipes conveying fluid both theoretically and experimentally. A very comprehensive introduction to vibrations induced by fluid flow and the associated linear stability problems can be found in the work of Chen [5]. The nonlinear behavior of slender structures subjected to axial fluid flows was discussed in detail in the monograph by Paidoussis [6]. The system exhibits a wide range of interesting dynamical behavior under different boundary conditions and motion constraints. These conditions cover a number of factors, such as parametric excitation in the form of flow fluctuation, external excitations, various support conditions, articulated or continuous...
configuration, additional system configurations like lumped mass, attached nozzles, elastic foundations, elastic constraints, and different forms of nonlinearities in the system arising from various sources.

In a survey of this subject, many researches were conducted either with pulsating fluid flow or motion constraint on one or finite number of locations on the pipe. Paidoussis and Sundararajan [7] explored the dynamics of a pipe conveying fluid when the flow velocity is harmonically perturbed about a mean value. Cantilevered pipe and clamped-clamped pipe models are investigated and parametric and combination resonances are analyzed. Namchchivaya [8] and Chen S S [9] examined the nonlinear dynamics of supported pipes conveying pulsating fluid in the vicinity of subharmonic resonances using the method of averaging. Some other detailed investigations based on linear analytical models of these parametric instability problems for simply supported pipes were conducted [10, 11, 12]. They have studied the parametric and combination resonances and evaluated instability regions using Bolotin’s method and numerical Floquet analysis. Various other authors considered nonlinear pipes conveying pulsating fluid referring to Sri Namchchivaya N et al. [13], Jayaraman et al. [14], Chang et al. [15], YOSHIZAWA et al. [16]. From these and several other studies, it is clear that the basic system of a pipe conveying pulsating fluid can lose stability when flow velocity becomes sufficiently high. Thus, the analysis of subharmonic and combination resonances was the main interest for simply supported pipes conveying pulsating fluid, yielding the stability boundaries in the parameter space. For a perspective on the whole field of pipes conveying pulsating fluid, the reader is referred to the book by Paidoussis [6].

In several recent papers, a simply supported pipe conveying pulsating fluid were analyzed, in which a non-linear force considered is associated with the axial extension of the pipe [17, 18]. The combination and principal parametric and internal resonances of a supported pipe were investigated. Paidoussis et al. [19] conducted a nonlinear analysis of a cantilevered pipe conveying fluid with a motion constraint on the pipe under steady flow velocity. Two contact models, cubic springs and trilinear springs, were introduced in the nonlinear equations of motion and results were obtained numerically. Another study performed by Hassan et al. [20] provided a means of representing contact as a combination of edge and segmental contact. The contact segment was unknown and determined artificially according to the researcher’s interests. The selection of the location of the segment could affect the performance of the system. Wang L. [21] further studied the nonlinear dynamics of a simply supported pipe conveying pulsating fluid by considering the effect of motion constraints modeled as cubic springs. Quasi-periodic and chaotic motions are obtained by using the Galerkin method with \( N=2 \). W. Xia et al. [22] developed an improved model with the consideration of the nonlinearity associated with the mean axial extension of the tube array. Cross flow and motion constraints are the main points of this study. The restraining forces were modeled cubic and trilinear springs too. Tang M. et al. [23] developed an improved model aimed at analyzing the fluidelastic vibration of a single flexible curved pipe that is surrounded by rigid cylinders and subjected to cross-flow and loose support.

In this study, it is investigated with both internal pulsating fluid flow and motion constraints imposed on the fluid conveying simply supported pipe systems. The simply supported pipe would impact the constraints once the motion becomes sufficiently large. The constraints are modeled as trilinear springs and further improved as distributed constraints acting on the pipe along its axis. Damping effects during impacting process are considered. The internal flow has a time-dependent harmonic component superposed on steady flow, such that \( u = u_0 (1+\sigma \sin\omega t) \), where \( \sigma \) is generally small and \( u_0 \) is called the mean flow velocity; \( \omega \) is defined as the pulsating frequency of the pulsating fluid flow. Attention is concentrated on the possible behaviors of the system with various values of
pulsating amplitude and frequency associated with the unsteady internal fluid. Some interesting results will be represented. Thus, bifurcation diagram, phase portraits and power spectral density diagrams will be constructed to represent the dynamical motions of the pipe system.

2. Equations of Motion

In the current work, the simply supported pipe conveying pulsating fluid interacting with distributed motion constraints is of length $L$, cross section of the pipe wall $A$, flexural rigidity $EI$, density $\rho_p$, mass per unit length $m$ and coefficient of viscoelastic damping $E^*$. The internal flowing fluid is of density $\rho_f$ and mass per unit length $M$, with flow velocity $U$. The impacting component is modeled as trilinear springs distributed along the pipe axis, as depicted in Fig. 1. The equation for unconstrained motions without taking into account the effect of motion constraints has been obtained before [17, 18, 24]. The equation of motions without motion constraints are modified here to describe this impacting oscillation with pulsating fluid. The equation of motion is given by

$$\frac{EI}{\partial x^4} + E^* I \frac{\partial^4 w}{\partial t \partial x^4} + 2MU \frac{\partial^2 w}{\partial t^2 \partial x} + (M + m) \frac{\partial^2 w}{\partial t^2} + \left[ MU^2 + M \frac{\partial U}{\partial t} (I - x) - \left( E + E^* \frac{\partial}{\partial t} \right) \frac{A}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \right] \frac{\partial^2 w}{\partial x^2} + F(w) = 0$$

in which, $F(\omega)$ represents the effect of the nonlinear motion constraint on the pipe. Here in this paper, impacting force $F_{spr}$ and damping force $F_{dmp}$ are included in calculation. Description of the nonlinear force is depicted by [19, 25]

$$F(w) = F_{spr} + F_{dmp}$$

Where,

$$F_{spr} = K_{spr} \left[ w - \frac{1}{2} \left( |w + w_0| - |w - w_0| \right) \right]^3$$

$$F_{dmp} = 1.5C_{dmp} F_{spr} \frac{\partial w}{\partial t} = 1.5C_{dmp} K_{spr} \frac{\partial w}{\partial t} \left[ w - \frac{1}{2} \left( |w + w_0| - |w - w_0| \right) \right]^3$$

In which, $w_0$ is the gap between the pipe axis and the edge of the motion constraints; $K_{spr}$ and $C_{dmp}$ are the trilinear spring stiffness and material damping coefficient, respectively. The nonlinear spring force $F_{spr}$ agrees with experimental test well according to Paidoussis et al. [19]. The damping force $F_{dmp}$ illustrated the opposite direction of the force and velocity. Introducing next the non-dimensional quantities
\[ \eta = \frac{w}{L}, \quad \zeta = \frac{x}{L}, \quad d = \frac{w_0}{L}, \quad \tau = \sqrt{\frac{EI}{m + M} \frac{t}{L^2}}, \quad u = \sqrt{\frac{M}{EI} L U}, \quad k = \frac{K_{sp} L^5}{EI}, \quad c = \frac{C_{dep} L^2}{EI} \]

\[ \alpha = \sqrt{\frac{EI}{m + M} \frac{L^2}{L^2}}, \quad \beta = \frac{M}{m + M}, \quad \kappa = \frac{AL^2}{2I} \]

Eq. (1) can be rewritten in a dimensionless form as follows:

\[ \alpha \frac{\partial^3 \eta}{\partial \tau^3} + \frac{\partial^4 \eta}{\partial \zeta^4} + \left[ u^2 + \sqrt{\beta u (1 - \xi)} \right] \frac{\partial^2 \eta}{\partial \xi^2} + \frac{2}{\sqrt{\beta}} \frac{\partial \eta}{\partial \tau} + \frac{\partial^2 \eta}{\partial \tau^2} - \kappa \int_0^1 \left( \frac{\partial \eta}{\partial \xi} \right)^2 d \xi - 2 \alpha \kappa \int_0^1 \frac{\partial \eta}{\partial \xi} \frac{\partial^2 \eta}{\partial \tau \partial \xi} d \xi + f (\eta) = 0 \] (5)

Nondimensional impact force is expressed as follows:

\[ f (\eta) = k \left[ \eta - \frac{1}{2} \left( |\eta + d| - |\eta - d| \right) \right] + \frac{3}{2} c k \frac{\partial \eta}{\partial \tau} \left[ \eta - \frac{1}{2} \left( |\eta + d| - |\eta - d| \right) \right]^3 \] (6)

The non-dimensional pulsating fluid velocity is described by a sinusoidal fluctuation depend on \( \tau \),

\[ u = u_0 \left( 1 + \sigma \sin \omega \tau \right) \] (7)

In Eq. (7), \( u_0 \) is the mean flow velocity and \( \sigma \) and \( \omega \) is the pulsating magnitude and pulsating frequency, respectively.

The infinite dimensional modal can be discretized by the Galerkin’s technique, with the simply supported beam eigenfunctions \( \phi_j(\xi) \). These eigenfunctions are used as a suitable set of base functions with \( q_j(\tau) \) being the corresponding generalized coordinates; thus,

\[ \eta(\xi, \tau) = \sum_{j=1}^N \phi_j(\xi) q_j(\tau) \] (8)

where, \( N \) is the number of modes taken into calculations. Substituting Eq. (8) into Eq. (5), multiplying by \( \phi_j(\xi) \) and integrating from 0 to 1 leads to

\[ \{ \ddot{q} \} + [C] \{ q \} + [K] \{ q \} + \{ f(q) \} + \{ g(q, \dot{q}) \} = \{ 0 \} \] (9)

\([C], [K], \{ f(q) \} \) and \( \{ g(q, \dot{q}) \} \) represent the stationary damping, stiffness matrices, nonlinear constraint force vector and nonlinear vector, respectively. The elements of \([C], [K], \{ f(q) \} \) and \( \{ g(q, \dot{q}) \} \) are given by

\[ C_{ij} = \alpha c_{ij}^1 + 2 \sqrt{\beta u_0} (1 + \sigma \sin \omega \tau) c_{ij}^2 \]

\[ K_{ij} = k_{ij}^1 + u_0^2 (1 + \sigma \sin \omega \tau)^2 k_{ij}^2 + \sqrt{\beta u_0 \sigma \omega \cos \omega \tau} k_{ij}^3 \]

\[ f_i = \int_0^1 \phi_i(\xi) f \left( \sum_{j=1}^N \phi_j(\xi) q_j \right) d \xi \]

\[ g_i = g_{jkl}^1 q_j q_k q_l + g_{jkl}^2 q_j q_k \dot{q}_l \] (10)
Where, the coefficients in these quantities can be written in the following form:

\[ c_{ij}^1 = \lambda_i^4 \delta_{ij}, \quad c_{ij}^2 = \int_0^1 \varphi_i \varphi_j' d\xi \]

\[ k_{ij}^1 = \lambda_i \delta_{ij}, \quad k_{ij}^2 = -\lambda_i^2 \delta_{ij}, \quad k_{ij}^3 = \int_0^1 (1 - \xi) \varphi_i \varphi_j' d\xi \]

\[ g_{ijkl}^1 = g_{ijkl}^2 = \int_0^1 \int_0^1 \varphi_i' \varphi_j' \varphi_k' \varphi_l' d\xi d\xi \]

(11)

For the purpose of numerical computation, define \( \{p\} = \{\dot{q}\} \) and \( \{z\} = \{\{q\}; \{p\}\} \); Eq. (9) is then reduced to its first-order form:

\[ \{\ddot{z}\} = [A]\{z\} + \{F(z)\} + \{G(z)\} \]

(12)

Where,

\[ [A] = \begin{bmatrix} [0] & [I] \\ -[K] & -[C] \end{bmatrix}, \quad \{G\} = \begin{bmatrix} \{0\} \\ -\{g\} \end{bmatrix}, \quad \{F\} = \begin{bmatrix} \{0\} \\ -\{f\} \end{bmatrix} \]

They are 2N×2N, 2N×1 and 2N×1 matrices, respectively. Solutions of \( \{q\} \) and \( \{p\} \) consist of the displacement and velocity at any point \( \xi \) along the pipe.

3. Results and Discussion

In the current work, the dynamical behaviors of the simply supported pipe conveying pulsating fluid with distributed motion constraints will be investigated numerically. To the author's knowledge, it is found that the non-linear responses in supported pipes conveying pulsating fluid with reasonably high mean flow velocity and for the case with distributed motion constraints have not yet been explored so far. Therefore, we analyze the non-linear vibrations of hinged–hinged pipes conveying fluid on the two topics. As it is well known that, for a simply-supported pipe conveying fluid with steady flow velocity, divergence in the first mode occurs at a dimensionless critical flow velocity \( u = \pi \) [6]. The main aim of this paper is to explore the effect of the pulsating amplitude \( \sigma \) and frequency \( \omega \) with higher mean flow velocity \( u_0 \) and distributed motion constraints on the dynamics of this pipe system. For that reason, solutions of Eq. (10) are obtained by using the fourth order Runge-Kutta method, with the following initial conditions employed, \( z(1) = \cdots = z(N) = -0.001 \) and \( z(N+1) = \cdots = z(2N) = 0 \).

In this project, the results to be presented have been obtained with \( N = 4 \) since reasonably converged frequencies are accomplished and identical to the theoretical results presented by Ni Q et al. [26], when \( N = 4 \). Some of the physical parameters are chosen to be

\[ \alpha = 0.005, \quad \beta = 0.2, \quad \kappa = 5000, \quad k = 5.6 \times 10^6, \quad c = 0.2, \quad d = 0.044 \]

(13)

3.1 Responses for various mean flow velocity

In this subsection, we consider the case of \( u_0 = 6 \) and \( u_0 = 8 \), respectively. The purpose is to explore how the responses of the pipe conveying pulsating fluid with distributed motion constraints would appear under low, middle and high mean flow velocities. In the calculations to construct the bifurcation diagram, whenever the midpoint velocity was zero, the midpoint displacement \( \eta(0.5, \tau) \) was recorded. The bifurcation diagrams for the midpoint displacement of the pipe are shown in Fig. 2, as forcing frequency \( \omega \) is varied.
It is obviously found that a great change takes place on the dynamical behavior of the simply supported pipe when mean flow velocity is under consideration, as seen in Figs. 2-6. As shown in Fig 2 (a) and (b), for mean flow velocity is at $u_0 = 8$, a large region of chaotic motion exists for $33.6 < \omega < 70$. However, when the pulsating frequency is in this region at $u_0 = 6$, the pipe exhibits chaotic motions and periodic motions, respectively. At the case of low pulsating frequencies ($0 < \omega < 12.2$ for $u_0 = 6$ and $0 < \omega < 18.3$ for $u_0 = 8$), dynamical behaviors of the two cases present periodic oscillations. As the forcing frequency increases, the pipe undergoes periodic motions, quasi periodic motions and chaotic motions. From Fig. 2 (a) and (b), both of the two systems exhibit chaotic motions and regain stability oscillating periodically. For example, for system of $u_0 = 6$, in the regions of $0 < \omega < 7.7$, $9 < \omega < 11.8$ and $20.8 < \omega < 27.9$, period-1 motion occurs; in the region of $7.8 < \omega < 8.9$, period-3 motion occurs; in the regions of $12 < \omega < 14.3$, $17.7 < \omega < 20.7$ and $28 < \omega < 34$, the system exhibits chaotic motions and transitions to chaotic motions. The system undergoes loosing stability and regaining stability as the forcing frequency increases.

Fig. 2 Bifurcation diagrams of the midpoint displacement of the pipe. (a) $u_0 = 6$; (b) $u_0 = 8$

Fig. 3 Phase portraits of the motions of the midpoint of the simply supported system at $u_0 = 6$;
Changing our view to the system of $u_0 = 8$, the pipe oscillates periodically at low pulsating frequencies and experiences chaotic motions and regains stability as the frequency takes a proper range of values. The regaining stability process is similar to a Hopf bifurcation to some extent.

Phase portraits and power spectral diagrams for $u_0 = 6$ and $u_0 = 8$ with several pulsating frequencies are shown in Figs. 3-6. It can be seen from the power spectral diagrams that period oscillations present several peaks and is a smooth curve. For transitions to chaotic motion, the PSD curve exhibits broadband characteristics with several peaks in it. While for chaotic motions, the PSD curve displays broadband characteristics only.
3.2 Responses for various pulsating amplitude

In this subsection, the effects of the pulsating amplitude to the dynamic behavior of the simply supported pipe with distributed motion constraints are studied. The pulsating amplitude is chosen to be $\sigma = 0.4$, compared to that in last subsection used as $\sigma = 0.2$. Interesting results have come out in this project. The bifurcation diagrams for the midpoint displacement of the pipe are shown in Fig. 7, as forcing frequency $\omega$ is varied. Compared to the bifurcations in Fig. 2, it can be seen from Fig. 7 (a) that the pipe become more stable for the case of $\sigma = 0.4$ when the mean flow velocity takes the value of $u_0$.

Fig. 5 Phase portraits of the motions of the midpoint of the simply supported system at $u_0 = 8$;
(a) $\omega = 15$ (period-1 motion); (b) $\omega = 18.3$ (phase transition);
(c) $\omega = 20.8$ (multi-period motion); (d) $\omega = 25.8$ (chaotic motion);

Fig. 6 Power spectral diagram of the midpoint of the simply supported system at $u_0 = 8$;
(a) $\omega = 15$ (period-1 motion); (b) $\omega = 18.3$ (phase transition);
(c) $\omega = 20.8$ (multi-period motion); (d) $\omega = 25.8$ (chaotic motion);
= 6, and regions of periodic motion is enlarged to a wide range of \( 0 < \omega < 44.2 \). From Fig. 7 (b), we can see that \( \sigma = 0.4 \) also expands the stable region of the system at \( u_0 = 8 \). This may be explained by the fact that, with large pulsating amplitude, the flowing fluid plays an important role of exciting to the system. The pulsating effect becomes more remarkable during the oscillating.

Fig. 7 Bifurcation diagrams of the midpoint displacement of the pipe. (a) \( u_0 = 6 \); (b) \( u_0 = 8 \)

4. Conclusions

In this paper, the nonlinear dynamics of simply supported pipe conveying pulsating fluid is investigated numerically, by introducing the effect of distributed motion constraint along the pipe axis modeled as trilinear springs. The distributed motion constraint’s parameters adopted here are in agreement with experiments. The internal pulsating fluid is assumed to be a harmonic component of flow velocity superposed on a constant mean value. The effects of the pulsating amplitude, forcing frequency and mean flow velocity to the dynamical behavior of the pipe are studied in this project. Interesting results have been obtained by exploring these parameters. It has been shown that the pipe is capable of displaying chaotic motions with various pulsating amplitudes and frequencies under the existing of distributed motion constraints. It ought to be remarked that, all the results obtained in the foregoing are based on the Galerkin method with \( N = 4 \). It has been proved effective for relatively low pulsating frequencies by Ni Q et al. [26]. However, it is expected that some such results for higher flow velocities and pulsating frequencies, especially for chaos, would also exist in different ranges of the parameter space if higher values of \( N \) had been used.

References
7 Paidoussis, M. P., & Sundararajan, C. Parametric and combination resonances of a pipe conveying pulsating
First observation of Quasi-Chaos in Erbium doped fiber ring laser

S. Zafar Ali

Department of Electrical Engineering, Air University, E-9 Islamabad, Pakistan
email zafarali@mail.au.edu.pk

Abstract

This work reports the first numerical evidence of existence of pseudo or quasi-chaos in a loss modulated EDFRL true Additive Chaos Modulation (ACM) scheme, thus possibly adding fifth region of operation in non-autonomous nonlinear systems. Quasi-chaos apparently looks like chaos but actually converges to same time and physical phase space trajectory, even with widely separated initial conditions, behaviour exactly opposite to the basic essence of chaos i.e. sensitive dependence on initial conditions (SDIC). Subject quasi-chaos was earlier believed to be pure chaos since the output passed qualitative visual tests of chaos like aperiodicity in time domain, rich spectral content in frequency domain, direct observation test in phase space, and fast decreasing autocorrelation function. Even quasi-chaos gives positive Lyapunov Exponent (LE), using TISEAN, in time delayed pseudo phase space built by time delaying lasing E field. Thus a complete knowledge of numerical model and driving conditions is a must to validate existence of a pure or quasi-chaos. EDFRL Chaos Message Masking(CMS) configuration is also shown here producing a pure chaos, for comparison, with desired sensitivity to initial conditions, besides passing all above mentioned visual tests of chaos and a positive LE spectrum. Emergence of quasi-chaos will have far reaching implications in chaos applications.

I Introduction

Chaos is the third most important discovery of 21st century being actively researched in multiple disciplines in theoretical and applied contexts and new aspects of chaos are still forthcoming [1]. An improved knowledge of chaos will help better understanding of various important phenomenon including heartbeat and neuron signals which are inherently chaotic. Optical chaos produced by different types of lasers is a well-researched field which met successful experimental demonstration in Athens [2]. EDFRL shows rich dynamical behaviour and have proven to be a useful platform to study nonlinear dynamics and chaos [3-10] in addition to other practical applications. There are three main schemes for generation of chaos in EDFRL i.e. loop nonlinearities [3], cavity loss modulation [4-9] and pump modulation [10]. The detailed study of chaos generation dynamics of EDFRL with five key parameters i.e. cavity loss, cavity gain, modulation index, pump power and modulating frequency variation was done earlier [4,6], showing how EDFRL switches, with the change of above mentioned control parameters, between four possible regions of operation i.e. periodic, quasi-periodic, stable and chaotic. It was next shown [7] that using square and other complex loss modulating signals the LE of EDFRL chaos increases thereby raising the degree of unpredictability and security. Later [8] it was found that pulsed chaos gives better LE than non-pulsed chaotic oscillations in EDFRL. It was pointed out later [9] that the original EDFRL model with chaos message masking (CMA/CMS) as proposed by Luo [4] and later studied in detail [6] has message signal also being added into the loop which makes it true ACM scheme instead of reported CMA / CMS. The detailed study of effect of message
parameters i.e. message frequency, amplitude and phase on EDFRL chaos dynamics therefore became necessary which was carried out in next work [9]. It was shown there [9] that chaos is produced only once the modulating and message frequencies are not integral multiple of each other which shows that the two frequencies interplay with each other to give shown results.

The detailed study carried out in this work was triggered by an unusual observation during simulations that different initial conditions did not produce different chaos as expected but all chaos seemed to be forced to single trajectory. This is a violation of SDIC which is main defining attribute of chaos and therefore this output needs to be termed as something other than chaos, say quasi-chaos, because apparently it behaves like chaos unless numerically subjected to different initial conditions. Negating SDIC in turn negates long term unpredictability. Previously, it was believed and reported [4-9] to be pure chaos because it mimicked all behaviour of chaos and qualifies qualitative tests as well as LE test in time delayed pseudo phase space using TISEAN routines [11]. Once message signal is removed from the loop, EDFRL starts producing pure chaos for same set of parameters and modulating signal.

The paper is organised as follows. Section I is introduction and literature review, followed by Section II which gives mathematical models and optical circuits of both configurations studied in this paper. Section III on simulations shows convergence of quasi-chaos of ACM to same trajectory irrespective of IC and divergence of trajectories in CMS configuration even for small IC deviations. Section IV discusses these simulation results and Section V concludes the results and indicates their implications.

II Mathematical model

In this section the optical circuits and corresponding mathematical models of loss modulated EDFRL true ACM and true CMA/CMS schemes respectively in Fig.1 and Eq. (1) and (2), ‘true’ emphasizing their corrected versions [9]. It is obvious from figures and equations that ACM has message sine wave being added into the loop modifying chaos dynamics, as studied in detail [6,9] while CMS is devoid of message and its effects on chaos dynamics.

\[
\begin{align*}
\dot{E}_{LA} &= -k_a (E_{LA} - c_a S_{in}) + g_a E_{LA} D_A + \xi_{LA} \\
\dot{D}_A &= -\frac{1}{\tau} [(1 + I_{PA} + E_{LA}^2)D_A - I_{PA} + 1] \\
k_a &= k_{a0} (1 + m_a \sin(\omega_a t)) \\
S_{in} &= S_0 (1 - m_s \sin(\omega_s t))
\end{align*}
\]

\[
\begin{align*}
\dot{E}_{LA} &= -k_a E_{LA} + g_a E_{LA} D_A + \xi_{LA} \\
\dot{D}_A &= -\frac{1}{\tau} [(1 + I_{PA} + E_{LA}^2)D_A - I_{PA} + 1] \\
k_a &= k_{a0} (1 + m_a \sin(\omega_a t))
\end{align*}
\]

where “.” denotes time derivative, \(E_{LA}\) is the lasing field strength, \(D_A\) is population inversion density, \(\tau\) is the of Erbium meta-stable state decay time, \(\xi_{LA}\) is the spontaneous emission factor, \(I_{PA}\) is the pump power, \(k_{a0}\) is the cavity loss (decay rate), \(g_a\) is the cavity gain, \(m_a\) is the modulation index, \(\omega_a\) is the angular loss modulating frequency, \(S_0\) is the
message amplitude and \( \omega_s \) is the message frequency. The various sets of parameters for which chaos is produced is extensively discussed earlier [6].

![Cavity loss modulation in EDFRL](image)

**Fig.1** Cavity loss modulation in EDFRL
(a) True ACM producing quasi-chaos
(b) True CMA/CMS producing pure chaos

The values of model parameters used in simulations in following section are same as early [6] and given in Table.1 below. The only difference is there is no message \( S_0 \) in CMS model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
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<tbody>
<tr>
<td>Pump power</td>
<td>( I_{PA} )</td>
<td>10 mw</td>
</tr>
<tr>
<td>Modulation index</td>
<td>( m_a )</td>
<td>0.03</td>
</tr>
<tr>
<td>Decay rate</td>
<td>( k_{a0} )</td>
<td>3.3x10^7</td>
</tr>
<tr>
<td>Gain</td>
<td>( g_a )</td>
<td>6.6 x10^7</td>
</tr>
<tr>
<td>Message amplitude</td>
<td>( S_0 )</td>
<td>1</td>
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<tr>
<td>Modulating frequency</td>
<td>( \omega_a )</td>
<td>3.5x10^5</td>
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<tr>
<td>Message frequency</td>
<td>( \omega_s )</td>
<td>3.14x10^5</td>
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<tr>
<td>Message Coupling strength</td>
<td>( c_a )</td>
<td>0.01</td>
</tr>
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</table>

III **Simulations results**

Fig.2 shows the convergence to same trajectory of differently starting chaotic trajectories in time and phase plots for same set of EDFRL parameters as given in Table.1 as well as driving conditions in loss modulated ACM model. It may be noted that one set of initial conditions i.e. \( E_{LA0}=0 \) and \( D_{A0}=0.47 \) is taken as a reference to compare the convergence time of other initial conditions to the reference trajectory. It can be observed in time domain plots of Fig. 2(a) where the initial conditions are \( E_{LA0}=12 \) and \( D_{A0}=0.496 \) that it converges to the defined reference trajectory in approx. 0.75 msec. The arrows in time domain mark significantly different amplitudes in the chaotic pulses initially. However, it can be observed that amplitude difference is very small in next pulses after which the two chaotic trajectories converge to same value exactly overlapping in time and phase plots. In Fig. 2(b) once the initial conditions are \( E_{LA0}=5 \) and \( D_{A0}=0.496 \), resulting chaos converge to the reference trajectory in about 1.75 msec. It is to be noted that the distance of initial conditions from reference initial conditions is more in Fig. 2(a) than in Fig. 2(b), yet convergence time is
smaller in first figure than the second. Thus the convergence time is not proportional to the
distance of initial conditions from the reference initial condition. The arrows in phase plots
indicate that the initial conditions for the two waveforms are taken far away from each other
yet the red phase plot converges to the green one after few turns in phase plot.

![Phase plots](image1)

![Phase plots](image2)

**Fig. 2** ACM Quasi-chaos convergence for different initial conditions.
(a) $E_{LA}=0$, $D_A=0.47$ (green) and $E_{LA}=12$, $D_A=0.496$ (red)
(b) $E_{LA}=0$, $D_A=0.47$ (green) and $E_{LA}=5$, $D_A=0.496$ (red)
Once convergence of quasi-chaos to same trajectory is established above simulations of the ACM model, the CMS model is simulated next, using Table.1 parameter values again, with different initial conditions to dig for either converging or diverging behaviour with different initial conditions but same parameters eliminating message sine wave. It is found that even slightly varying initial conditions as labelled in Fig.3 time domain plots make the pure chaos diverge in longer run. Also the time of start of divergence of trajectories as marked by arrow in Fig.3 increases with the decrease in difference in initial conditions. The later behaviour is quite as expected for a pure chaos which is SDIC as well as long-term unpredictable.

![Time domain plots showing pure chaos with different initial conditions.](image_url)

Fig.3 CMS Pure chaos showing SDIC for different initial conditions.
- (a) $E_{L0}=0.1$, $D_{A0}=0.4$ (red) and $E_{L0}=0.11$, $D_{A0}=0.4$ (blue)
- (b) $E_{L0}=0.1$, $D_{A0}=0.4$ (blue) and $E_{L0}=0.101$, $D_{A0}=0.4$ (red)
- (c) $E_{L0}=0.1$, $D_{A0}=0.4$ (blue) and $E_{L0}=0.1001$, $D_{A0}=0.4$ (red)
- (d) $E_{L0}=0.1$, $D_{A0}=0.4$ (blue) and $E_{L0}=0.10001$, $D_{A0}=0.4$ (red)
- (e) $E_{L0}=0.1$, $D_{A0}=0.4$ (blue) and $E_{L0}=0.100001$, $D_{A0}=0.4$ (red)
The various plots (time, phase space, frequency, and autocorrelation) are shown in Fig.4 to compare quasi and pure chaos, in left and right columns, as produced by ACM and CMS configurations respectively with system parameters kept same except elimination of message in later. The quasi-chaos time plots in Fig.4(a) (left) has pulse time period equal to time period of modulating sine wave while pure chaos plot (on right) is chaotic in time interval as well as pulses amplitude. The pulses in quasi-chaos are bunched while pure chaos has no bunching, while the dynamic range of pure chaos appears better than its counterpart. Fig.4(b) shows the phase space of both chaos and it can be seen that both phase space plots are strange attractors indicating chaos as per direct observation method [6]. The apparent crossing of phase space lines will not be there if time is added as third dimension because the third assumed differential equation will be simply $t'=1$. It can be seen in Fig.4(c) that the phase space of quasi-chaos is less fractal as compared to that of pure chaos. Also quasi-chaos is denser at lower pulse amplitudes while pure chaos is denser at higher pulse amplitudes making phase space plots denser on inner and outer sides respectively. Fig.4(c) shows frequency spectrum of quasi and pure chaos with latter being richer and more random in spectral lines as compared to former. Also the modulating frequency and its harmonics are visible only in the case of quasi-chaos frequency spectrum because here the pulse repetition time is being decided by modulating signal itself as reported earlier [6] also. The autocorrelation diagrams of both chaos are compared in Fig.4(d) and it is found that pure chaos has a sharper decay of autocorrelation than its counterpart while the autocorrelation function has nonzero lower values in quasi-chaos due to humps beneath pulses in time domain. But above all the most important observation is that quasi-chaos is in fact difficult to detect from the diagrams as shown in Fig.4 till it is discovered by actually testing initial conditions as in this work. Now once it is discovered the clues of quasi-chaos can be outlined, the most important being the fixed time period instead of chaotic time period and visibility of modulating frequency and harmonics in frequency domain.
Fig. 4  Comparison of quasi-chaos (left) with pure chaos (right) for loss modulation EDFRL
(a) Time domain  (b) Phase space (c)  Frequency domain (d) Autocorrelation
The LE spectrums of quasi chaos generated from ACM and reported earlier for sine[6] and square [7] modulating signals were based on time series analysis of pseudo phase space generated by time delaying $E_{LA}$ lasing. The reason why quasi-chaos also showed positive LE there, is that SDIC is not violated in pseudo time delayed phase space of lasing field. This fact is shown here in Fig. 5 for two such values of time delay (in samples) i.e. $\tau=2$ and $\tau=5$ samples, that these attractors are fractal in nature. Fractal nature of phase space means a strange attractor which shall give positive LE result with TISEAN. It implies that LE calculation using TISEAN is not a sufficient test to differentiate pure chaos from quasi-chaos and subjecting the system to different initial conditions numerically or experimentally is a must.

![Fig.5](image)

**Fig.5** Time delayed ($\tau$) pseudo phase plots of quasi-chaos
(a) $\tau=2$  
(b) $\tau=5$

IV. Discussion

The simulations were carried out to establish the presence of quasi-chaos in ACM and pure chaos in CMS. First it is shown that ACM loss modulated EDFRL produces an apparent chaos which converges to single trajectory making it a quasi- instead of a pure chaos. It is also observed for quasi-chaos that convergence time is not strictly dependent on deviation of initial conditions. As soon as the message is removed to reconfigure it as CMS, keeping all other parameters same including loss modulating sine wave, the convergence is immediately replaced by divergence of pure chaos trajectories. This divergence of trajectories is observed even for very small deviations in initial conditions as is expected for a pure chaos. It is noted that the time for the start of this divergence decreases with the increase in deviation of initial conditions, also as expected. Hence it is indicated that message signal is adding an extra periodic perturbation in the cavity which is interacting with the multiplicative perturbation of loss modulating sine wave, thus producing quasi instead of pure chaos. Next the LE spectrums of quasi and pure chaos are calculated using time series analysis with well-known TISEAN routines and plotted side by side. Most surprisingly, quasi-chaos gives positive LE since TISEAN uses time delayed pseudo phase space of lasing E field. The time delayed phase space is found fractal for two time delays. Thus LE calculation using time series analysis by TISEAN is not a sufficient test to identify pure chaos as shown in this study, because it misjudges quasi-chaos in EDFRL also as pure chaos. The most valid test of chaos is physically subjecting the system or its numerical model to different initial conditions and look for either convergence or divergence of trajectories for quasi and pure chaos respectively. This observation of pseudo or quasi chaos adds fifth region of operation in
nonlinear systems; the other well-known four regions being periodic, quasi-periodic, stable and chaotic already reported and studied in detail for loss modulated EDFRL [6].

V. Conclusions
This work was motivated by an unusual observation of converging behaviour in temporal and physical phase space of apparently chaotic trajectories, for different initial conditions, in ACM loss modulated EDFRL. Such behaviour is totally new and cannot be categorised as any of four known modes of operation in nonlinear systems i.e. periodic, quasi-periodic, stable and chaos. Therefore, it is termed as quasi-chaos, as it looks like chaos, but violates the basic definition of chaos i.e. sensitive dependence on initial conditions. The findings here confirm the presence of convergence to same quasi-chaotic trajectories even for very widely separated initial conditions. Previously, this output from this EDFRL configuration was considered to be pure chaos since the output passed all qualitative visual tests of chaos; like aperiodicity in time domain, rich spectral content in frequency domain, direct observation test in phase space, and fast decreasing autocorrelation function. In this work all above plots are placed side by side for making comparisons between pure and quasi-chaos. However, EDFRL quasi-chaos surprisingly gives positive lyapunov exponents with TISEAN; as TISEAN routines are based on time delayed pseudo phase space of observed time series data, which is found fractal in this work, for the lasing E field. At the same time, it in no case implies that loss modulation scheme in EDFRL is not capable of producing a pure chaos. For comparison purposes, EDFRL is also shown to be able to produce a pure chaos, just by eliminating the message from the loop (CMS scheme), exhibiting desired sensitivity to minute changes in initial conditions. Pure chaos, as expected, passes all above mentioned visual tests of chaos and also giving positive LE using TISEAN. The only main test of quasi-chaos is thus numerically subjecting the system model to different initial conditions.

This is an important discovery which has five main implications on chaos theory and its engineering applications. Firstly, a complete knowledge of numerical model and driving conditions is a must to validate existence of either pure or quasi-chaos. Secondly, finding quasi-chaos would not be possible in an experimental work on EDFRL, because of inaccessibility of population inversion initial condition. Secondly, fractal nature of time delayed pseudo phase space is responsible for positive lyapunov exponent calculations, using TISEAN, misinterpreting quasi-chaos as pure chaos in earlier works. Fourth, chaos synchronisation of quasi-chaotic systems is also artificial i.e. ACM EDFRL receiver is in fact not synchronised to corresponding transmitter because of any seed being fed, since both outputs readily get converged to same trajectory, independent of their initial conditions, in reality, due to their quasi-chaotic nature. Fifth, pure chaos will always prove as chaos in qualitative and quantitative tests, but quasi-chaos will spoof itself as pure chaos until complete model is available for trying different initial conditions; output time series shall not be the only thing available. Other possible implications of this discovery are presently under study and will be reported shortly.
References


Second Observation of Quasi-Chaos in Erbium doped fiber ring laser

S. Zafar Ali

Department of Electrical Engineering, Air University, E-9 Islamabad, Pakistan
email zafarali@mail.au.edu.pk

Abstract
This numerical investigation is motivated by the exciting recent discovery of quasi-chaos, in loss modulated erbium doped fiber ring laser (EDFRL), which looks like chaos but converges to single trajectory for widely separate initial conditions in physical phase space. Both pure and quasi-chaos are generated in pump modulated EDFRL using chaos message masking and additive chaos modulation configurations respectively, for comparison in different domains. Quasi-chaos has chaotic amplitude in time domain, rich spectral content in frequency domain, fractal physical phase space, and fast decreasing autocorrelation function. Sensitive dependence on initial conditions is numerically tested for both these chaos, with pure chaos diverging even for minute deviations while quasi-chaos converging even for extreme values of initial conditions. Lyapunov exponent of quasi-chaos, calculated with TISEAN, however, are still positive, as TISEAN works on time delay embedded phase space of single variable, which is shown fractal here. Quasi-chaotic pulses are periodic in time and chaotic in amplitude, with bunching of sub-pulses into super pulses with respective fixed periods. Quasi-chaos cannot be used for secure communication and experimental outputs in forced chaotic oscillators under noisy conditions need careful analysis. This evidence marks the confirmation of existence of fifth region of operation in nonlinear systems.

Keywords:
Quasi-chaos, Nonlinear dynamics of fiber laser, Pump modulation.

I Introduction
Chaos, quantum mechanics and theory of relativity are widely accepted as the three most important discoveries of 21st century. Chaos, which is mainly identified by its sensitive dependence on initial conditions, is a ubiquitous phenomenon in many nonlinear systems fulfilling Poincare Bendixon’s criteria. The application of optical chaos in secure optical communication has reached successful results in Athens experiment [1] and yet new exciting aspects of optical chaos are being discovered [2, 3]. Optical chaos in EDFRL can be produced both in autonomous manner [4] and by periodic perturbations as in loss modulation [5-6] and pump modulation [7-14]. Recently quasi-chaos was discovered [3] in loss modulated EDFRL, which looked like chaos in time, frequency, phase space and autocorrelation signatures but violated the basic criteria of chaos i.e. sensitive dependence on initial conditions (SDIC) once the system, is numerically subjected to different initial conditions. Until first discovery [3] of quasi-chaos, nonlinear systems were earlier known to exhibit only four regions of operation i.e. chaos, periodic, stable and quasi-periodic with switching of regions being determined by the tuning of parameters and driving conditions.

There are three message encoding configurations possible in each of three mentioned schemes in EDFRL i.e. additive chaos modulation (ACM) where message gets entered into
laser rate equations, chaos message masking (CMS) where message is not part of rate equations and is added in the last and chaos shift keying (CSK) where one parameter is switched with data. The message encoding scheme of loss modulated EDFRL reported earlier [5] as chaos message masking (CMS) was latter [6] corrected to be additive chaos modulation (ACM) scheme. The effect of message parameters (message frequency, amplitude and phase) on EDFRL chaos dynamics was studied in detail [6] and it was shown that one necessary condition of chaos generation is that message and modulating sine waves frequencies are not integral multiples of each other. However, the recent paper [3] has proven that the behaviour earlier identified as pure chaos in EDFRL ACM scheme [5,6], was in fact quasi-chaos, because it straightaway violates sensitive dependence on initial conditions. However, surprisingly it still looks like chaos in several domains and even renders positive lyapunov exponent using TISEAN [15]. The reason for this anomaly was identified [3] was the fractal nature of time-delayed embedded phase space of lasing field intensity. It was also shown [3] that in order to produce pure chaos, message had to be completely eliminated from EDFRL. Therefore, the factor responsible for generation of quasi chaos was the interplay of message and loss modulating frequencies within the laser cavity.

EDFRL pump modulation scheme shall be the next logical candidate for tracing the signs of quasi-chaos, because it is the next forced configuration of EDFRL which produces chaos. We will investigate both message encoding schemes i.e. chaos message masking and additive chaos modulation for generation of pure and quasi-chaos respectively, in the same stepwise manner as done earlier [3] in loss modulation scheme. We will specifically inspect the convergence of trajectories to same path in quasi-chaos for widely separated initial conditions. We shall also carry out all qualitative tests and quantitative test of lyapunov exponent calculation on both chaos and evaluate the results. Once quasi-chaos is proven discovered second time here, it will be safe to assume that it is a ubiquitous phenomenon in all forced chaotic generators under similar conditions as identified earlier [3] and revalidated here. The paper is organised as follows. Section I provides introduction and literature review, followed by Section II mentioning the mathematical models and optical circuits of both configurations studied in this paper. Section III shows simulations and section IV discusses these results. Section V concludes the results and indicates their implications in research.
II Mathematical model

In this section the optical circuits and corresponding mathematical models of pump modulated EDFRL CMS and ACM schemes in Fig.1 (a)-(b) and Eq. 1(a)-(c) and Eq. 2 (a)-(d) respectively, the basic model adapted from Luo[14]. It is obvious from Fig.1 (a) and Eq.(1) that message is not part of loop dynamics and is added in the last to the chaos generated by loop in CMS. However, it can be seen in Fig. 1(a) and Eq.(2) that ACM has message sine wave is being added into the loop thus modifying chaos dynamics.

\[
\begin{align*}
\dot{E}_{LA} &= -k_a E_{LA} + g_a E_{LA} D_A + \xi_{LA} \quad (1a) \\
D_A &= \frac{1}{\tau} [(1 + I_{PA} + E_{LA}^2)D_A - I_{PA} + 1] \quad (1b) \\
I_{PA} &= I_{PA0} (1 + m_a \sin(\omega_a t)) \quad (1c) \\
\dot{E}_{LA} &= -k_a (E_{LA} - c_a S_m) + g_a E_{LA} D_A + \xi_{LA} \quad (2a) \\
D_A &= \frac{1}{\tau} [(1 + I_{PA} + E_{LA}^2)D_A - I_{PA} + 1] \quad (2b) \\
I_{PA} &= I_{PA0} (1 + m_a \sin(\omega_a t)) \quad (2c) \\
S_m &= S_0 (1 - m_s \sin(\omega_s t)) \quad (2d)
\end{align*}
\]

where “.” denotes time derivative, \( E_{LA} \) is the lasing field strength, \( D_A \) is population inversion density, \( \tau \) is the of Erbium meta-stable state decay time, \( \xi_{LA} \) is the spontaneous emission factor, \( I_{PA} \) is the pump power, \( k_a0 \) is the cavity loss (decay rate), \( g_a \) is the cavity gain, \( m_a \) is the modulation index, \( \omega_a \) is the angular loss modulating frequency, \( S_0 \) is the message amplitude, \( c_a \) is the coupling strength of message and \( \omega_s \) is the message frequency.

Fig.1 Pump modulation in EDFRL
(a) Chaos message Masking
(b) Additive Chaos Modulation
The values of model parameters used in simulations in following section are given in Table 1 below are same for CMS and ACM configurations except the last two are present in latter only. The numerical integration is performed using fourth order Runge-Kutta method with a step size of 10 nsec, in all simulations here, to ensure best accuracy of results. It may also be mentioned here that $E_{LA0}$ and $D_{A0}$ are the initial conditions for lasing field intensity and population inversion density here. The dynamic range is 0 to 150 a.u. for $E_{LA0}$ and -1 to 1 for $D_{A0}$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>Decay time metastable state</td>
<td>$T$</td>
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<tr>
<td>Spontaneous emission factor</td>
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<td>Pump power</td>
<td>$I_{P0}$</td>
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<td>Modulation index</td>
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<td>Message frequency</td>
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### III Simulations results

Initially the CMS model of Eq.1(a)-(c) is simulated using parameters as given in Table 1. The results are plotted in Fig. 2 for minutely varying initial conditions with Fig.2 (a) taken as reference i.e. $E_{LA0}=0$ and $D_{A0}=0.47$. $E_{LA0}$ is kept at zero and $D_{A0}$ is varied to 0.470001, 0.47001 and 0.4701 in Fig.2 (b)-(d) and the point of change in chaos signature with reference to Fig.2 (a) is marked by an arrow in last three diagrams. There are two important observations made here Firstly, the chaos outputs are different in all figures, even for the smallest change of $10^{-6}$ in $D_{A0}$ in Fig.2 (b) which is quite in line with SDIC and is as expected for a pure chaos. Secondly, the starting time of change in chaos as marked by arrow shifts to right with the gradual increase in difference of $D_{A0}$ from value of 0.47 which is again as expected. Both of these observations are well known for a pure chaos and are responsible for its long-term unpredictability.
The ACM model as given in Eq.2(a)-(d) is simulated using same parameters as for CMS except for the addition of message signal in the loop as per Table.1 Quasi-chaos is observed this time with different chaos like time domain plots converging to single trajectory for largely varied values of both $E_{LA0}$ and $D_{A0}$. $D_{A0}$ is kept at 0.5 and $E_{LA0}$ is changed at four different values i.e. 0, 40, 80 and 120 and result plotted at two time scales in Fig.3 (a) and (b). $E_{LA0}$ is kept at 0 and $D_{A0}$ is changed at four different values i.e. 0.5, 1.,-0.5 and -1 and results plotted at two time scales in Fig.3 (c) and (d). It can be observed in all these four figures that the output converges to same trajectory for all these values which is exactly opposite to pure chaos behaviour, which is well known and just seen in CMS simulation. It is to be noted here that chaos outputs are observed here to be more sensitive to $D_{A0}$ changes as compared to $E_{LA0}$ because latter has a smaller scale of variation. However, the output still converges despite using extreme possible values of both $E_{LA0}$ and $D_{A0}$ which proves convergence for all smaller values of initial conditions. Moreover, the results are shown till 2msec only for clarity of diagrams but the convergence is tested to persist till 20 msec once it is achieved. It is believed that convergence will persist forever once achieved. The time of convergence is maximum for $E_{LA0} = 80$ in Fig. 2(a) and (b) ; not for $E_{LA0} = 120$ as could be speculated. However, it increases with absolute increase in deviation of $D_{A0}$ from reference $D_{A0}=0.5$, with $D_{A0}=-1$ taking longest convergence time of 0.55 msec in Fig. 2(c) and (d).
Fig. 3 Convergence of quasi-chaos trajectories for different initial conditions.

(a) $E_{LA0}=0$, $E_{LA0}=40$, $E_{LA0}=80$, $E_{LA0}=120$ and $D_{A0}=0.5$ (2msec)
(b) $E_{LA0}=0$, $E_{LA0}=40$, $E_{LA0}=80$, $E_{LA0}=120$ and $D_{A0}=0.5$ (1msec)
(c) $D_{A0}=0.5$, $D_{A0}=1$, $D_{A0}=-1$, $D_{A0}=-0.5$ and $E_{LA0}=0$ (2msec)
(d) $D_{A0}=0.5$, $D_{A0}=1$, $D_{A0}=-1$, $D_{A0}=-0.5$ and $E_{LA0}=0$ (1msec)

The time, phase space, frequency, and autocorrelation plots are shown in Fig. 4 (a) to (d) for quasi and pure chaos, with system parameters kept same in ACM and CMS, except addition of message in former. The time domain plot of quasi-chaos here is having super pulses with bunches of sub pulses, both being Gaussian, with super pulses being periodic in time but chaotic in amplitude. The time period of sub-pulses is decided by the pump modulating frequency i.e. 9 kHz and there are three to four sub-pulses in every super pulse. Another way of looking at this chaos is considering them as periodic bunches of chaotic Gaussian pulses with humps underneath as reported earlier for loss modulation [5]. The pure chaos has independent Gaussian pulses with no humps underneath and sometimes two or three pulses seem getting merged together due to chaotic timing of pulses themselves. One important indicator of pure pulsed chaos is that it is chaotic in time as well as amplitude while quasi-chaos is chaotic in amplitude only and super and sub pulses are not chaotic in their respective time periods. Each loop in phase space corresponds to a gaussian pulse in time domain. The phase space of quasi-chaos is almost uniformly distributed as the pulses amplitude spreads over a bigger dynamic range. The phase space of pure chaos is spread uniformly on lower amplitudes and then on higher amplitudes with a gap owing to its temporal signature. Also the DC component and the modulating frequency of 9 kHz are prominent lines in both frequency spectrums, but harmonics of 9kHz are more prominent in quasi-chaos frequency spectrum, since the pulse repetition time of quasi-chaos sub-pulses is fixed and is being determined by modulating signal as reported earlier also [6]. The frequency of super pulse and its harmonics is also visible in quasi-chaos spectrum. The pure chaos spectrum is otherwise relatively flatter and richer due to variable chaotic time of pulses. The pure chaos autocorrelation diagrams of both chaos are has a sharper decay like early while the quasi-chaos autocorrelation function has nonzero lower values in due time domain. The fixed time period of chaotic pulses, super and subpulses or bunching and humps and visibility of modulating frequency harmonics in frequency domain are some clues of quasi-chaos revalidated here. There is no fixed linear relationship between delta of initial conditions and time of convergence of trajectories to same trajectory in quasi-chaos.
The time delayed embedded plots of $E_{LA}(t)$ vs $E_{LA}(t-\tau)$ are shown for different values of time delay $\tau$ for quasi-chaos and pure chaos in Fig.5 and 6 respectively with two important observations to be made. Firstly, both of these plots are fractal in nature and the degree of correlation between $E_{LA}(t)$ and $E_{LA}(t-\tau)$ decreases with the increase in $\tau$ in both figures. Secondly, the degree of correlation is lesser in pure chaos as compared to quasi-chaos for corresponding values of $\tau$ because randomness and degree of unpredictability is higher in pure chaos. The latter observation is further validated by LE spectrum calculation of both chaos using TISEAN, a well-known routines pack available. We get positive LE for both these types of chaos as shown in Fig.7 (a) and (b) with LE for pure chaos being significantly higher than quasi-chaos which is as expected. This proves that time delayed method of LE calculation will not differentiate between quasi and pure chaos. The only way to identify quasi-chaos is numerically simulating the system with different initial conditions and seeing the time domain plots or physical phase space or physically subjecting the system to different initial conditions if these are accessible in experimental works.
Fig. 6 Quasi-chaos time delayed phase plots
(a) $\tau = 0.5$ usec    (b) $\tau = 1$ usec    (c) $\tau = 2$ usec    (d) $\tau = 4$ usec

Fig. 7 Lyapunov exponent spectrum
(a) Pure chaos    (b) Quasi-chaos
IV. Discussion

This numerical investigation is done step-wise on same lines as done earlier \[3\] for loss modulation scheme and the results are also corresponding, proving the second appearance of quasi-chaos. In order to test the SDIC of pure chaos, the initial condition \(D_{A0}\) is varied by very minute differences and the output is still found to diverge even for the slightest of the difference as expected for pure chaos. However, the time of start of divergence of trajectories increases with decrease in difference of \(D_{A0}\) which is also anticipated behaviour. On the other hand, quasi-chaos proclaimed to converge instead of diverging, is tested with \(E_{LA0}\) and \(D_{A0}\) taken to their extreme limits. However, the differently starting trajectories still converge to same single trajectory instead of diverging with the time of convergence increasing this time with the deviation of initial conditions. The message signal adds an extra perturbation in the cavity which is interacting with the loss modulating sine wave, thus producing quasi instead of pure chaos.

The time, frequency, phase space and autocorrelation plots of quasi-chaos once seen independently give an impression of chaos. However, once pure and quasi-chaos plots are observed critically, some differences are observed in respective domains. One major difference is that quasi-chaos has periodic bunches of super pulses while pure chaos pulses, however, are chaotic both in time and in amplitude. Each super pulse has further four sub pulses and the frequency of sub-pulses is fixed at 9 kHz i.e. the modulating frequency. The frequency spectrum of quasi-chaos has all harmonics of both sub and super pulse frequencies while pure chaos spectrum is flatter and richer with better message masking capabilities. The pump modulating frequency is visible in both the spectrums but its harmonics are more vivid in quasi-chaos. The autocorrelation diagram of pure chaos is depicting more randomness due to chaotic timing of pulses. The LE of quasi-chaos calculated using TISEAN are still positive although the physical phase space converges to single trajectory. The reason of this anomaly is the fact that TISEAN calculates LE by creating a pseudo phase space by time delayed embedding of time series data of one physical variable i.e. \(E_{LA(t)}\) and \(E_{LA(t-\tau)}\) in this case. This pseudo phase space gives positive LE if it is fractal in nature; and it has been found to be fractal in this work not only for pure chaos but also for quasi-chaos. However, the LE values of pure chaos are order of magnitude higher than quasi-chaos. \(E_{LA(t)}\) vs \(E_{LA(t-\tau)}\) plotted for different values of \(\tau\) i.e. 0.5 to 4 usec, give fractal plots whose correlation decreases with the increase of \(\tau\). The correlation is however, higher in quasi-chaos than pure chaos as the pulses timing is chaotic in latter only.
V. Conclusions

Rediscovery of quasi-chaos in pump modulation scheme in this research, after its first discovery in loss modulation of EDFRL proves that quasi-chaos is also a ubiquitous phenomenon and is the fifth region of operation in nonlinear systems. This work proves that this phenomenon can be traced in all forced chaotic oscillators once the requisite conditions shown here are tuned i.e. one additional sine wave perturbation with frequency not integral multiple of forcing sine wave is added into the system dynamics. Quasi-chaos in this work spoofs itself as a pure chaos by passing all qualitative and visual tests i.e. it has rich spectral content and fast decreasing autocorrelation function and a strange attractor in pseudo phase space. The only visual indicator observed in this work is that pulses are chaotic in amplitude and not in time for quasi-chaos with bunching of pulses into super pulses. The most surprising result in this work is that quasi-chaos gives positive LEs’ spectrum using time delayed pseudo phase space analysis of lasing field intensity using TISEAN, a well-known package of LE calculation. Therefore, qualitative visual tests and LE calculations on time delayed series have proven to be a weak test of pure chaos and we need to find stronger quantitative measures of distinguishing pure chaos from quasi-chaos and be cautious with experimental results in forced chaotic systems. The strangely deceptive behaviour of quasi-chaos observed in this work raises many questions from application point of view e.g. can quasi-chaos be used for secure chaotic communication and whether to believe experimental data from forced chaotic systems to be pure chaos. Briefly, pure chaos is sensitively dependent on initial conditions both in physical phase space of all dynamic variables and time delayed pseudo phase space of any one physical variable while quasi-chaos is converging in physical phase space and sensitively dependent on initial conditions only in time delayed pseudo phase space.
VI. References


Striation model of the population growth of the Earth

Vladimir G. Zhulego¹, Artem A. Balyakin²

¹ Department of analysis and forecast, NRC "Kurchatov Institute", Moscow, Russia, (E-mail: zhulego@mail.ru)
² Department of scientific and technical programs and projects, NRC "Kurchatov Institute", Moscow, Russia, (E-mail: Balyakin_AA@nrcki.ru)

Abstract. On a base of a previous two-component model of population growth for one country the new striation model of population growth of the entire Earth is proposed. The obvious advantages compared with other models are studied. The methodological rules for finding the parameters included in the model are discussed as well as the possible application of the proposed model to international organizations such as UNESCO or the United Nations. Some results of numerical simulation are presented, socio-economic aspects of the model are discussed: the ability to control population size by adjusting the flow of people from the city to the countryside, and the trends in the urban and rural population.

Keywords: population growth, chaotic simulation, modeling of socio-economic processes, striation model.

In this work we discuss several related problems: the problem of constructing a model of population growth, the problem of predicting population growth, and the problem of numerical simulation of population dynamics. Those issues attract a lot of attention in both natural and social sciences. Initially it was caused by simplest modeling of such dynamics when constant increase of population was inevitably shown as leading to overpopulation of the planet (Malthusian crisis). Later, a precise date of overpopulation was esteemed as 2004. By now a number of theoretical works arose, clarifying model representation of such dynamics. Along with Malthus classic work, one should mention Verhulst model [1], Kapitsa model [2], Forrester world-system model [3] and many others. As it turned out, the rate of population growth crucially determines the growth rate of GDP, and this fact has largely spurred interest in the subject [4,5]. The most interesting dynamics is connected with so called "demographic transition point” when population growth sharply decreases and number of people achieves stable value. However, at the same time, all proposed model are unlikely to describe the entire dynamics of the population: the explosive growth of the initial time, saturation stage (demographic transition point) and almost stable behavior on later stage.

In our previous work [6] we proposed new approach to modeling the dynamics of population growth. We consider the population dynamics from the
physical viewpoint: in particular, we believe that the population should be considered as two-phase system - “rural population” and “urban population”, with each phase to behave under its own laws of growth, with having, however, a flow between two phases. This view of the population dynamics growth can decrease the number of arbitrary parameters in the model, and also give additional arguments and levers to control the pace of population growth that is important for countries experiencing problems with overcrowding and/or depopulation challenges. In our current work we outline the contours of the model of population growth throughout the Earth.

We assume that the entire population can be divided into two relatively independent groups (phases), focused respectively on the intensive and extensive ways of development - urban and rural areas. Note that these concepts are not geographical, and probably reflect the attitude of the population to the production of wealth and investing in future generations and lifestyles. The most important characteristic that allows extracting these two groups, apparently, is the population density per square kilometer. The problem to calculate/evaluate this value would be another interesting task that we will not consider in our work.

On a base of a previous two-component model of population growth for one country the new striation model of population growth of the entire Earth is proposed.

\[
\begin{align*}
\frac{dx_i}{dt} &= -a_i x_i(t) + \beta_i \frac{x_i(t - \tau_i) y_i(t - \tau_i)}{x_i(t) + a_i^2} + \sum_{k=1}^{N} \gamma_a \frac{x_i(t - \tau_i) y_i(t - \tau_i)}{x_i(t) + a_i^2 + a_k^2} \\
\frac{dy_i}{dt} &= h_i x_i(t) - c_i y_i(t) - \beta_i \frac{x_i(t - \tau_i) y_i(t - \tau_i)}{x_i(t) + a_i^2} 
\end{align*}
\]

Here \(i = 1, 2, 3, \ldots, N\) denotes the total number of countries in the world and \(x_i\) denote urban population, \(y_i\) denotes the rural population in the country \(i\). Totally this system consists of \(2N\) differential equations. The Earth’s population will be calculated as a simple sum of all \((x_i + y_i)\). Analysis of the system can be carried out numerically in the same way as it was done for the analysis of the model for a single country (see [6]).

We should mention that in our model there can be the difference in the behavior of various parts of the planet, and the whole planet is not considered as one large country. This approach could also work for a vast territories and/or countries.

In our opinion the socio-economic consequences of proposed strata model are of great interest. This mostly refers to GDP growth (so-called world-dynamics or the world-system dynamics). It is known that in such models economic growth is critically dependent on the rate of population growth. For example, in the Solow-Swan model [5,7] economy enters the steady growth of the national income at a constant rate equal to the rate of growth of the labor force. The stratification (or "splitting") of the population into two phases allows us to explain the economic growth in the country, even in the case where the
total population growth is missing or insufficient. The most striking examples of this growth is an economic growth in the USSR in the period of industrialization and economic growth of China under strict birth control nowadays. Thus in such models the economic growth becomes endogenous. Modelling of growth of the Russian economy on the basis of population stratification in eight groups was undertaken in [8,9], but it was done without reference to the modeling of population growth, which, of course, reduces the value of this approach.

The proposed model of the Earth's population growth has obvious advantages compared with other models and can be a good basis for the calculation of the realistic medium-term and long-term forecast population growth of the Earth, which is important to many international organizations such as UNESCO or the United Nations. It should be understood, however, that work on compiling such a forecast is very labor-intensive because of the need to analyze a large amount of statistical data on population growth in the countries and on this basis to determine the model parameters, such as the rate of flow of the villagers in the city and residents of one country to another. However, we hope that this work can be very useful and will provide, along with the forecast population growth, and long-term economic growth forecasts.

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References