Variable Elasticity of Substitution in the Diamond Model: Dynamics and Comparisons

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Abstract. We study the dynamics shown by the discrete time Diamond overlapping-generations model with the VES production function in the form given by Revankar\textsuperscript{10} and compare our results with those obtained by Brianzoni \textit{et al.}\textsuperscript{2} in the Solow model. We prove that, as in Brianzoni \textit{et al.}\textsuperscript{2}, unbounded endogenous growth can emerge if the elasticity of substitution is greater than one; moreover, differently from Brianzoni \textit{et al.}\textsuperscript{2}, the Diamond model can admit two positive steady states. We also prove that complex dynamics occur if the elasticity of substitution between production factors is less than one, confirming the results obtained by Brianzoni \textit{et al.}\textsuperscript{2}. Numerical simulations support the analysis.

Keywords: Variable Elasticity of Substitution, Diamond Growth Model, Fluctuations and Chaos, Bifurcation in Piecewise Smooth Dynamical Systems.

1 Introduction

The elasticity of substitution between production factors plays a crucial role in the theory of economic growth, it being one of the determinants of the economic growth level (see Klump and de La Grandville\textsuperscript{6}).

Within the Solow model (see Solow\textsuperscript{11}, and Swan\textsuperscript{12}) it was found that a country exhibiting a higher elasticity of substitution experiences greater capital (and output) per capita levels in the equilibrium state (see Klump and de La Grandville\textsuperscript{6}, Klump and Preißler\textsuperscript{7}, and Masanjala and Papageorgiou\textsuperscript{8}). More recently, the role of the elasticity of substitution between production factors in the long run dynamics of the Solow model was investigated both considering the Constant Elasticity of Substitution production function (CES) (see Brianzoni \textit{et al.}\textsuperscript{1}) and the Variable Elasticity of Substitution production function (VES) (see Brianzoni \textit{et al.}\textsuperscript{2}). The results obtained demonstrate
that fluctuations may arise if the elasticity of substitution between production factors falls below one.

Miyagiwa and Papageorgiou [9] moved the attention to the Diamond overlapping-generations model (Diamond [4]) while proving that, differently from the Solow setup, “if capital and labor are relatively substitutable, a country with a greater elasticity of substitution exhibits lower per capita output growth in both transient and steady state”. To reach this conclusion they considered the normalized CES production function.

In the present work we consider the Diamond overlapping-generations model with the VES production function in the form given by Revankar [10] (see also Karagiannis et al. [5]). Our main goal is to study the local and global dynamics of the model to verify if the main result obtained by Brianzoni et al. [2] in the Solow model, i.e. cycles and complex dynamics may emerge if the elasticity of substitution between production factors is sufficiently low, still holds in the Diamond framework.

To summarize, the qualitative and quantitative long run dynamics of the Diamond growth model with VES production function are studied, to show that complex features can be observed and to compare the results obtained with the ones reached while considering the CES technology or the Solow framework.

2 The economic setup

Consider a discrete time setup, \( t \in \mathbb{N} \), and let \( y_t = f(k_t) \) be the production function in intensive form, mapping capital per worker \( k_t \) into output per worker \( y_t \). Following Karagiannis et al. [5] we consider the Variable Elasticity of Substitution (VES) production function in intensive form with constant return to scale, as given by Revankar [10]:

\[
y_t = f(k_t) = Ak_t^a[1 + bak_t]^{1-a}, \quad k_t \geq 0
\]

(1)

where \( A > 0, \ b < a < 1, \ b \leq -1 \); furthermore \( 1/k_t \geq -b \), in order to assure that \( f(k_t) > 0 \), \( f'(k_t) > 0 \) and \( f''(k_t) < 0 \), \( \forall k_t > 0 \), where

\[
f'(k_t) = Aak_t^a(1 + abk_t)^{1-a}[k_t^{-1}(1-a)b(1+abk_t)^{-1}]
\]

and

\[
f''(k_t) = A\frac{a(a-1)(1+abk_t)^{-a-1}k_t^2}{k_t^{-a}}.
\]

The elasticity of substitution between production factors is then given by

\[
\sigma(k_t) = 1 + bk_t
\]

hence \( \sigma \geq (\leq)1 \) iff \( b \geq (\leq)0 \). Thus the elasticity of substitution varies with the level of capital per capita, representing an index of economic development. Observe that, while the elasticity of substitution for the CES is constant along an isoquant, in the case of the VES it is constant only along a ray through the origin.
In the Diamond[4] overlapping-generations model a new generation is born at the beginning of every period. Agents are identical and live for two periods. In the first period each agent supplies a unit of labor inelastically and receives a competitive wage:

\[ w_t = f(k_t) - k_t f'(k_t), \]

thus, taking into account the specification of the production function in (1), we obtain

\[ w_t = A k_t^a \frac{(1 + 2abk)(1 - a)}{(1 + abk)^a}, \quad (2) \]

As in Miyagiwa and Papageorgiou[9] we assume that agents save a fixed proportion \( s \in (0, 1) \) of the wage income to finance consumption in the second period of their lives. All savings are invested as capital to be used in the next period’s production, so that the evolution of capital per capita is described by the following map

\[ k_{t+1} = \phi(k_t) = \frac{s}{1 + n} - w_t = \frac{s A}{1 + n} k_t^a \frac{(1 + 2abk)(1 - a)}{(1 + abk)^a}, \quad (3) \]

where \( n > 0 \) is the exogenous labor growth rate and capital depreciates fully.

As in Brianzoni et al.[2] we distinguish between the following two cases.

(a) If \( b > 0 \) the elasticity of substitution between production factors is greater than one and the standard properties of the production function are verified \( \forall k_t > 0 \); in this case \( k_t \) evolves according to (3). We do not consider the case \( b = 0 \) as \( \sigma(k_t) \) becomes constant and equal to one, \( \forall k_t \geq 0 \), thus obtaining a particular case of the CES production function.

(b) If \( b \in [-1, 0) \) the elasticity of substitution between production factors is less than one and the standard properties of the production function are verified for all \( 0 < k_t < -\frac{1}{b} \); in this case \( k_t \) evolves according to (3) iff \( k_t \in [0, -1/b] \) while, following Karagiannis et al.[5] and Brianzoni et al.[2], if \( k_t > -1/b \) then \( k_t = \phi(-1/b) \).

3 Local and Global Dynamics

3.1 Elasticity of Substitution Greater than One

Let \( b > 0 \). Then the discrete time evolution of the capital per capita \( k_t \) is described by the continuous and differentiable map (3).

The establishment of the number of steady states is not trivial to solve, considering the high variety of parameters. As a general result, the map \( \phi \) always admits one fixed point characterized by zero capital per capita, i.e. \( k = 0 \) is a fixed point for any choice of parameter values. Anyway steady states which are economically interesting are those characterized by positive capital per worker. In order to determine the positive fixed points of \( \phi \), let us define the following function:

\[ G(k) = 1 - a \frac{1 + 2abk}{k^{1-a} (1 + abk)^a}, k > 0 \quad (4) \]
where
\[ G'(k) = \frac{(1-a)}{k^{2-a}(1+abk)^{1+a}}[(a-1) + (2a-1)abk], \tag{5} \]
then solutions of \( G(k) = \frac{1+n}{sA} \) are positive fixed points of \( \phi \).

The following proposition establishes the number of fixed points of map \( \phi \).

**Proposition 1** Let \( \phi \) given by (3).

(i) Assume \( b > 0 \) and \( a \leq \frac{1}{2} \). Then:
   (a) if \( \frac{1+n}{sA} > (ab)^{1-a}2(1-a) \), \( \phi \) has two fixed points given by \( k_t = 0 \) and \( k_s = k^* > 0 \);
   (b) if \( 0 < \frac{1+n}{sA} \leq (ab)^{1-a}2(1-a) \), \( \phi \) has a unique fixed point given by \( k_s = 0 \).

(ii) Assume \( b > 0 \) and \( a > \frac{1}{2} \) and let \( k_m = \frac{1-a}{a(2a-1)} \). Then:
   (a) if \( \frac{1+n}{sA} < (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) \), \( \phi \) has a unique fixed point given by \( k_t = 0 \);
   (b) if \( \frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) \), \( \phi \) has two fixed points given by \( k_t = 0 \) and \( k_s = k^* = k_m \);
   (c) if \( (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a) \), \( \phi \) has three fixed points given by \( k_t = 0 \), \( k_s = k_1 \) and \( k_s = k_2 \), where \( 0 < k_1 < k_2 < k_m \);
   (d) if \( \frac{1+n}{sA} \geq (ab)^{1-a}2(1-a) \), \( \phi \) has two fixed points given by \( k_t = 0 \) and \( k^* > 0 \), where \( 0 < k^* < k_m \).

**Proof.** \( k_t = 0 \) is a solution of equation \( k_t = \phi(k_t) \) for all parameter values hence it is a fixed point. Function (4) is such that \( G(k_t) > 0 \) for all \( k_t > 0 \), furthermore \( \lim_{k_t \to 0^+} G(k_t) = +\infty \) while \( \lim_{k_t \to +\infty} G(k_t) = (ab)^{1-a}2(1-a) \). (i) Observe that if \( b > 0 \) and \( a \leq \frac{1}{2} \), \( G(k) \) is strictly decreasing \( \forall k_t > 0 \) since \( G'(k_t) < 0 \). Hence \( G(k_t) \) intersects the positive and constant function \( g = \frac{1+n}{sA} \) in a unique positive value \( k_t = k^* \) iff \( \frac{1+n}{sA} > (ab)^{1-a}2(1-a) \).

(ii) Assume \( a > \frac{1}{2} \) and \( b > 0 \) then \( G \) has a unique minimum point \( k_m = \frac{1-a}{a(2a-1)} \) where \( G(k_m) = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) \). Hence, if \( (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a) \), then equation \( G(k_t) = \frac{1+n}{sA} \) admits two positive solutions. Similarly, it can be observed that if \( \frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) \) or \( \frac{1+n}{sA} \geq (ab)^{1-a}2(1-a) \) then \( \phi(k_t) \) admits a unique positive fixed point. Trivially, for the other parameter values, equation \( G(k_t) = \frac{1+n}{sA} \) has no positive solutions.

For what it concerns the local stability of the steady states the following proposition holds.

**Proposition 2** Let \( \phi \) be as given in (3).

(i) The equilibrium \( k_t = 0 \) is locally unstable for all parameter values.
(ii) If \( \phi \) admits two fixed points then the equilibrium \( k_t = k^* > 0 \) is locally stable.
(iii) If \( \phi \) admits three fixed points, then the equilibrium \( k_t = k_1 \) is locally stable while the equilibrium \( k_t = k_2 \) is locally unstable.
Proof. Firstly notice that function $\phi$ may be written in terms of function $G$ being:

$$\phi(k) = \frac{sA}{1+n}kG(k)$$

hence $\phi'(k) = \frac{sA}{1+n}[G(k) + kG'(k)]$.

(i) Since $\lim_{k_t \to 0^+} G(k_t) = +\infty$ and $\lim_{k_t \to 0^+} kG'(k_t) = +\infty$, then $\phi'(0) = +\infty$ and consequently the origin is a locally unstable fixed point for map $\phi$.

(ii) Assume that $\phi$ admits two fixed points. After some algebra it can be noticed that

$$\phi'(k) = \frac{a(1+a)sA (1 + abk)^{1-a}}{1+n} \left[2abk^2 + 2b(1 + a)k + 1\right] > 0 \quad \forall k > 0$$

hence $\phi(k)$ is strictly increasing and consequently $k^*$ is locally stable. In the particular case in which $\frac{1}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$ then $k^* = k_m$ is a non hyperbolic fixed point: a tangent bifurcation occurs at which $k^*$ is locally stable.

(iii) Assume that $\phi$ has three equilibria. Since $\phi'(k) > 0 \forall k > 0$ then point (iii) is easily proved.

The results concernig the existence and number of fixed points and their local stability when the elasticity of substitution between production factors is greater than one, are resumed in Fig. 1. We fix all the parameters but $s$ and we show that, as $s$ is increased, we pass from two to three and, finally, to one fixed point. Hence it can be observed that unbounded growth can emerge if the propensity to save in sufficiently high.

As in Brianzoni et al. [2], if the elasticity of substitution between the two factors is greater than one ($b > 0$), then unbounded endogenous growth can be observed but only simple dynamics can be produced. Anyway, differently from Brianzoni et al. [2], the growth model can exhibit two positive steady states so that the final outcome of the economy depends on the initial condition (in fact if $k_0 \in (0, k_2)$ then the convergence toward $k_1$ is observed while if $k_0 > k_2$ then unbounded endogenous growth is exhibited).

3.2 Elasticity of Substitution Less than One

Let $b \in [-1, 0)$. Then the discrete time evolution of the capital per capita $k_t$ is described by the following continuous and piecewise smooth map:

$$k_{t+1} = F(k_t) = \begin{cases} \phi(k_t) & \forall k_t \in [0, \frac{1}{b}] \\ \phi(-\frac{1}{b}) & \forall k_t > -\frac{1}{b} \end{cases}.$$
Proposition 3 Let $F$ be as given in (8) and $b \in [-1,0)$.

(i) Assume $a > \frac{1}{2}$. Then $F$ has two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{2ab})$.

(ii) Assume $a \leq \frac{1}{2}$ and $M = \frac{(-b)^{1-n}(1-2a)}{1-a}$. Then:
   (a) if $\frac{1+n}{sA} \geq M$ there exist two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{b})$;
   (b) if $\frac{1+n}{sA} < M$ there exist two fixed points given by $k = 0$ and $k^* = F(-\frac{1}{b})$.

Proof. It is easy to see that $k = 0$ is a fixed point for any choice of the parameter values.

(i) Firstly notice that $\phi \geq 0$ iff $k \in [0, -\frac{1}{2ab}]$ and $\phi(0) = \phi(-\frac{1}{2ab}) = 0$, so values of $k > -\frac{1}{2ab}$ are not economically significant. Moreover $\phi$ has a unique maximum point given by $k_M = \frac{-1-a+\sqrt{1+a^2}}{2ab}$ with $\phi(k_M) = \frac{sA}{1+n} \left( \frac{\sqrt{1+a^2}+1-a}{1+a^2+1-a} \right) (1-a)(\sqrt{1+a^2} - a)$. Finally $\lim_{k \to 0^+} \phi'(k) = \infty$. Hence equation $\phi(k) = k$ has always a unique positive solution given by $k^* \in (0, -\frac{1}{2ab})$.

(ii) The positive fixed points of $F$ such that $k \leq -\frac{1}{b}$ are given by the solutions of equation $G(k) = \frac{1+n}{sA}$ with $G(k)$ as given in (4) and $G > 0$ defined in $(0, -\frac{1}{b})$. Being $G'(k) = \frac{k-a-1}{k^2(1-abk)^{1+a}} [(a-1) + (2a-1)abk]$, $G$ is strictly decreasing $\forall k \in (0, -\frac{1}{b}]$ with minimum point in $k_m = -\frac{1}{b}$ and $G(k_m) = G(-\frac{1}{b}) = \frac{(-b)^{1-n}(1-2a)}{(1-a)^{1-2a}} = M$. Hence $G(k) = \frac{1+n}{sA}$ has a unique positive
solution \( k^* \in (0, -\frac{1}{b}] \) iff \( \frac{1+n}{sA} \geq \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} \). Differently, the unique fixed point of \( F \) such that \( k > -\frac{1}{b} \) is defined by \( k^* = F(-\frac{1}{b}) = \phi(-\frac{1}{b}) \) and it exists iff \( F(-\frac{1}{b}) > -\frac{1}{b} \), which is equivalent to require \( \frac{1+n}{sA} < \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} \).

Let us move to the study of the local stability of the fixed points. Since

\[
\phi'(k) = \frac{a(1+a)sA(1+abk)^{-1-a}}{1+n} \frac{k^{1-a}}{[2abk^{2} + 2b(1+a)k + 1]}
\]

then \( \lim_{k \to 0^+} \phi'(k) = +\infty \), so that the equilibrium characterized by zero capital-per capita is always locally unstable.

We firstly focus on the case with \( a > \frac{1}{2} \). As it has been discussed, the related one dimensional map is continuous and differentiable in its domain \([0, -\frac{1}{2aB}]\). Furthermore, \( \phi(0) = \phi(-\frac{1}{2aB}) = 0 \) and \( \phi''(k) < 0 \ \forall k \in (0, -\frac{1}{2aB}) \), i.e. it is strictly concave. As a consequence map \( \phi \) behaves as the logistic map, that is it exhibits the standard period doubling bifurcation cascade as one parameter is moved (see Devaney [3]).

The period doubling bifurcation cascade is observed, for instance, if \( A \) is increased. In fact it can be easily observed that \( \phi(k_M) \) increases as \( A \) increases so that \( \exists A \) such that \( \phi(k_M) > -\frac{1}{2d} \ \forall A > A \), i.e. almost all trajectories are unfeasible. At \( A = A \) a final bifurcation occurs (the origin is a pre-periodic fixed point and \( \phi \) is chaotic in a Cantor set) while \( \forall A \in (0, A) \) the period doubling bifurcation cascade is observed (see Fig. 2 (a), (b) and (c)). Notice also that the situation presented in panel (b) becomes simpler if a greater value of \( b \) is considered (see Fig. 2 (c)), proving that in order to have complex dynamics \( b \) must be sufficiently low (as also showed in panel (d)).

In order to study the local stability of the positive fixed point when \( a \leq \frac{1}{2} \) and \( b \in [-1, 0) \) we observe that function \( F \) has a non differentiable point given by

\[
P = \left( \frac{1}{b}, F(-\frac{1}{b}) \right),
\]

where \( F(-\frac{1}{b}) = \frac{sA}{1+n}(-b)^{-a}(1-a)^{1-a}(1-2a) \).

Notice that if \( P \) is above the main diagonal, the fixed point \( k^* \) is superstable being \( F'(k^*) = 0 \) and no complex dynamics can be exhibited. This case occurs, for instance, if \( A \) is great enough and the related situation is presented in Fig. 3 (a). If \( \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} \leq \frac{1+n}{sA} \) we get that \( k^* = -\frac{1}{b} \), then a border collision of the superstable fixed point occurs.

If \( P \) is below the main diagonal then \( k^* \) may be locally stable or unstable and complex dynamics may arise.

The following Proposition states a sufficient condition for the existence of a stable 2-period cycle \( \{k_1, k_2\} \) such that \( k_i \in R_i, (i = 1, 2) \).

**Proposition 4** Let \( b \in [-1, 0) \). For all \( b \) in the region defined as

\[
\Omega = \left\{ b : F^2\left(-\frac{1}{b}\right) > -\frac{1}{b} \cap \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} < \frac{1+n}{sA} \right\}
\]

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Fig. 2. $a = 0.6, n = 0.1, s = 0.7$. (a) If $b = -0.7$ and $A = 9$ a stable two period cycle is presented, while (b) if $A = 10$ complexity emerges. (c) Locally stable fixed point for $A = 10$ and $b = -0.3$. (d) Bifurcation diagram w.r.t. $b$. (e) Bifurcation diagram w.r.t. $A.$

map $F$ admits a superstable 2-period cycle defined as $C_2 = \{F(-\frac{1}{b}), F^2(-\frac{1}{b})\}$.

Proof. A 2-cycle for map $F$ is given by $\{k_1, k_2\}$ with $F(k_1) = k_2$ and $F(k_2) = k_1$. Let $k_0 > -\frac{1}{b}$ with $k_0 \in R_2$, then $k_1 = F(-\frac{1}{b})$ belongs to $R_1$ (being the point $P$ below the main diagonal) and $k_2 = F(k_1) = F(F(-\frac{1}{b})) = F^2(-\frac{1}{b})$. If $F^2(-\frac{1}{b}) > -\frac{1}{b}$, then $F^2(-\frac{1}{b}) \in R_2$ and consequently $F(F^2(-\frac{1}{b})) = F(k_2) = F(-\frac{1}{b}) = k_1$. This proves the existence of a two period cycle. Moreover, the eigenvalue of such cycle is zero, since $F''(k_2) = 0$, therefore it is a superstable two period cycle.
Notice that in $F^2(\frac{-1}{b}) = \frac{-1}{b}$ a border collision bifurcation of the superstable 2-period cycle occurs. The superstable two period cycle is depicted in Fig. 3 (b).

![Fig. 3. a = 0.4, n = 0.1, s = 0.7. (a) If $b = -0.3$ and $A = 30$ the positive steady state is superstable. (b) The superstable two period cycle for $b = -0.3$ and $A = 15.$](image)

In order to describe how complex dynamics may emerge if $a \leq \frac{1}{2}$, we recall that $F$ is unimodal and $k_M = \frac{1-a+\sqrt{1+a^2}}{2ab}$ is its maximum point.

If $k^* \in (0, k_M)$ (i.e. point $(k_M, F(k_M))$ is below the main diagonal), then $k^*$ is globally stable $\forall k_0 \neq 0$. On the contrary, if $F(k_M) > k_M$ (i.e. point $(k_M, F(k_M))$ is above the main diagonal), then its eigenvalue is negative and $k^*$ may lose stability only via a period-doubling bifurcation. Therefore, a necessary condition for a flip bifurcation is that that point $(k_M, F(k_M))$ is above the main diagonal.

To recap, as in Brianzoni et al.[2], our model can exhibit cycles or more complex dynamics iff $P$ is below the main diagonal while the maximum point $k_M$ is above the main diagonal. In this case all positive initial conditions produce trajectories converging to an attractor belonging to a trapping interval $J$ defined in the following proposition.

**Proposition 5** Let $\frac{b^{\frac{1-a}{(1-a)^{\frac{1-a}{1-a}}}}} {a^{\frac{1-a}{(1-a)^{\frac{1-a}{1-a}}}}} < \frac{1+n}{sA}$ and $F(k_M) > k_M$. Then the one-dimensional map $F$ admits a trapping interval $J$, where $J$ is defined as follows:

1. $J = [F(\frac{-1}{b}), F(k_M)]$ if $F(k_M) \geq -\frac{1}{b}$.
2. $J = [F^2(k_M), F(k_M)]$ if $F(k_M) < -\frac{1}{b}$.

**Proof.** If the one-dimensional map $F$ has a maximum point $k_M$ above the main diagonal and point $P$ is below the main diagonal, then through the graphical analysis it is possible to see that when the image of $k_M$ belongs to $R_2 \cup \{-\frac{1}{b}\}$, then $J = [F(\frac{-1}{b}), F(k_M)]$ is mapped into itself; otherwise $J = [F^2(k_M), F(k_M)]$ is mapped into itself by $F$.

Since every initial condition $k_0 \neq 0$ creates bounded trajectories converging to an attractor included into the trapping interval $J$, it can be noticed that if $F(k_M) \geq -\frac{1}{b}$, the flat branch of map $F$ is involved. Moreover, since all
the points mapped in $R_2$ have the same trajectory of point $F(-\frac{1}{b})$, then the attractor will be a cycle. The transition from $F(k_M) \geq -\frac{1}{b}$ to $F(k_M) < -\frac{1}{b}$ corresponds to a border collision bifurcation.

In order to describe the qualitative dynamics occurring on set $J$, we consider the situation in which $k^* \in R_1$ is locally stable (as in Fig. 4 (a)), for instance $b$ is close to zero. Then, as $b$ decreases, $k^*$ becomes unstable via flip bifurcation and a period doubling route to chaos occurs till a border collision bifurcation emerges at $F(k_M) = -\frac{1}{b}$. This bifurcation occurs at $b = b_c$ and a point of the attractor of $F$ collides with point $P$. In Fig. 4 (b) and (c) the situations immediately before and after the border collision bifurcation occurring at $b_c \approx -0.315$ are presented. Notice that after this bifurcation the qualitative dynamics drastically changes, passing from a complex attractor to a locally stable 5-period cycle. The related bifurcation diagram is presented in Fig. 4 (d).

As in Brianzoni et al.[2] if elasticity of substitution between production factors in less then one, then the system becomes more complex as $b$ decreases since cycles or more complex features may be exhibited.
4 Conclusions

In this paper we considered the Diamond overlapping-generations model with the VES production function in the form given by Revankar[10]. We examined existence and stability conditions for steady state and the results of our analysis show that fluctuation or even chaotic patterns can be exhibited. As in Brianzoni et al.[2], cycles or complex dynamics can emerge if the elasticity of substitution between production factors is low enough. Moreover, unbounded endogenous growth can be observed. A new feature is due to the fact that, if elasticity of substitution is greater than one, then up to three fixed point can be exhibited.

References

Self-organization and fractality created by gluconeogenesis in the metabolic process

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Abstract. Within a mathematical model, the process of interaction of the metabolic processes such as glycolysis and gluconeogenesis is studied. As a result of the running of two opposite processes in a cell, the conditions for their interaction and the self-organization in a single dissipative system are created. The reasons for the appearance of autocatalysis in the given system and autocatalytic oscillations are studied. With the help of a phase-parametric diagram, the scenario of their appearance is analyzed. The bifurcations of the doubling of a period and the transition to chaotic oscillations according to the Feigenbaum scenario and the intermittence are determined. The obtained strange attractors are created as a result of the formation of a mixing funnel. Their complete spectra of Lyapunov indices, KS-entropies, “horizons of predictability,” and the Lyapunov dimensions of strange attractors are calculated. The conclusions about the reasons for variations of the cyclicity in the given metabolic process, its stability, and the physiological state of a cell are made.

Keywords: Gluconeogenesis, glycolysis, metabolic process, self-organization, fractality, strange attractor, Feigenbaum scenario.

1 Introduction

Gluconeogenesis is a biochemical process of formation of glucose from hydrocarbonless predecessors such as pyruvates, aminoacids, and glycerin. The biosynthesis of glucose runs analogously to glycolysis, but in the reverse direction. Gluconeogenesis is realized by means of the inversion of seven invertible stages of glycolysis. Three remaining stages of glycolysis are exergenous and, therefore, irreversible. They are replaced by three “by-pass reactions” that are thermodynamically gained for the synthesis of glucose. Since gluconeogenesis uses the same invertible reactions, as glycolysis does, its biochemical evolution occurred, apparently, jointly with glycolysis. Maybe, the symbiosis of these biochemical processes arose else in protobionts 3.5 bln years in Earth’s oxygenless atmosphere. It can be considered as one of the primary open nonlinear biochemical systems, being far from the equilibrium. The self-organization of the given biochemical system resulted in the appearance of a stable dissipative system independent of other biochemical processes of a primary broth. The directedness of the running of a reaction in it was determined by the energy-gained balance. The organic molecule ATP, which was formed as a result of glycolysis, became the principal carrier of the energy consumed in all other biochemical processes. This created the conditions of self-

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organization of other biochemical processes that used \textit{ATP} as the input product of a reaction. But if the need in glucose arose in other biochemical processes, the directedness of biochemical reactions in the system was changed by the opposite one. In the course of the subsequent biochemical evolution, the given dissipative system is conserved and is present in cells of all types, which indicates their common prehistory.

Thus, the studies of the reactions of gluconeogenesis are determined in many aspects by the results of studies of glycolysis. The direct sequence of reactions with the known input and output products is studied easier than the reverse one.

The experimental studies of glycolysis discovered autooscillations [1]. In order to explain their origin, a number of mathematical models were developed [2-4]. Sel’kov explained the appearance of those oscillations by the activation of phosphofructokinase by its product. In the Goldbeter--Lefever model, the origin of autooscillations was explained by the allosteric nature of the enzyme. Some other models are available in [5-7].

The present work is based on the mathematical model of glycolysis and gluconeogenesis, which was developed by Professor V.P. Gachok and his coauthors [8-10]. The peculiarity of his model consists in the consideration of the influence of the adeninenucleotide cycle and gluconeogenesis on the phosphofructokinase complex of the given allosteric enzyme. This allowed one to study the effect of these factors as the reason for the appearance of oscillations in glycolysis.

At the present time, this model is improved and studied with the purpose to investigate gluconeogenesis. Some equations were added, and some equations were modified in order to describe the complete closed chain of the metabolic process of glycolysis-gluconeogenesis under anaerobic conditions. The developed complete model allows us to consider glycolysis-gluconeogenesis as a united integral dissipative structure with a positive feedback formed by the transfer of charges with the help of NAD. Glycolysis with gluconeogenesis is considered as an open section of the biosystem, which is self-organized by itself at the expense of input and output products of the reaction in a cell, which is a condition of its survival and the evolution. The appearance of an autocatalytic process in the given dissipative structure can be a cause of oscillatory modes in the metabolic process of the whole cell.

Gluconeogenesis occurs in animals, plants, fungi, and microorganisms. Its reactions are identical in all tissues and biological species. Phototrophs transform the products of the own photosynthesis in glucose with the help of gluconeogenesis. Many microorganisms use this process for the production of glucose from a medium, where they live.

The conditions modeled in the present work are established in muscles after an intense physical load and the formation of a large amount of lactic acid in them. As a result of the running of the reverse reaction of gluconeogenesis, it is transformed again in glucose.
2 Mathematical Model

The given mathematical model describes glycolysis-gluconeogenesis under anaerobic conditions, whose output product is lactate. At a sufficient level of glucose, the process runs in the direct way. At the deficit of glucose, it runs in the reverse one: lactate is transformed in glucose.

The general scheme of the process of glycolysis-gluconeogenesis is presented in Fig.1. According to it, the mathematical model (1) - (16) is constructed with regard for the mass balance and the enzymatic kinetics.

The equations describe variations in the concentrations of the corresponding metabolites: (1) – lactate \( L \); (2) – pyruvate \( P \); (3) - 2-phosphoglycerate \( \psi_3 \); (4) – 3-phosphoglycerate \( \psi_2 \); (5) - 1,3-diphosphoglycerate \( \psi_1 \); (6) - fructose-1,6-diphosphate \( F_2 \); (7) – fructose-6-phosphate \( F_1 \); (8) – glucose \( G \); (9), (10), and (11) - ATP, ADP, and AMP, respectively, form the adeninenucleotide cycle at the phosphorylation; (12) - \( R_1 \) and (13) - \( R_2 \) (two active forms of the allosteric enzyme phosphofructokinase; (14) - \( T_1 \) and (15) - \( T_2 \) (two inactive forms of the allosteric enzyme phosphofructokinase; (16) - \( NAD - H \) (where \( NAD - H(t) + NAD^+(t) = M \)).

Variations of the concentrations of omitted metabolites have no significant influence on the self-organization of the system and are taken into account in the equations generically. Since glycolysis and gluconeogenesis on seven sections of the metabolic chain are mutually reverse processes, only the coefficients are changed, whereas the system of equations describing glycolysis is conserved [8-10]. The model involves the running of gluconeogenesis on the section: glucose - glucose-6-phosphate. Here in the direct way with the help of the enzyme hexokinase, the catabolism of glucose to glucose-6-phosphate occurs. In the reverse direction with the help of the enzyme glucose-6-phosphatase, glucose is synthesized from glucose-6-phosphate. Thus, the positive feedback is formed on this section.
Fig. 1. General scheme of the metabolic process of glycolysis-gluconeogenesis.
\[
\frac{dL}{dt} = l_1 V(N)V(P) - m_3 \frac{L}{S},
\]
\( (1) \)

\[
\frac{dP}{dt} = l_2 V(\psi_3)V(D) - m_6 \frac{P}{S} - l_7 V(N)V(P),
\]
\( (2) \)

\[
\frac{d\psi_3}{dt} = \frac{\psi_2}{S} - \frac{m_3}{S} l_2 V(\psi_3)V(D) - m_4 \frac{\psi_3}{S},
\]
\( (3) \)

\[
\frac{d\psi_2}{dt} = l_3 V(\psi_1)V(D) - m_5 \frac{\psi_2}{S},
\]
\( (4) \)

\[
\frac{d\psi_1}{dt} = \frac{m_5 (F_2 \lambda / S)}{S_1 + m_5 (F_2 \lambda / S)} - l_2 V(\psi_1)V(D) + m_3 V(M - N)V(P),
\]
\( (5) \)

\[
\frac{dF_2}{dt} = l_4 V(R_1)V(F_1)V(T) - l_5 \frac{1}{1 + \gamma A} V(F_2) - m_5 \frac{F_2}{S},
\]
\( (6) \)

\[
\frac{dF_1}{dt} = l_6 V(G)V(T) - l_7 V(R_1)V(F_1)V(T) + l_8 \frac{1}{1 + \gamma A} V(F_2) - m_3 \frac{F_1}{S},
\]
\( (7) \)

\[
\frac{dG}{dt} = \frac{G_0}{S} \frac{m_1}{m_1 + F_1} - l_8 V(G)V(T),
\]
\( (8) \)

\[
\frac{dT}{dt} = l_9 V(\psi_3)V(D) - l_1 V(R_1)V(F_1)V(T) + l_3 \frac{A}{\delta + A} V(T) - l_4 \frac{T^4}{\beta + T^4} + l_7 V(\psi_1)V(D) - l_6 V(G)V(T),
\]
\( (9) \)

\[
\frac{dD}{dt} = l_4 V(R_1)V(F_1)V(T) - l_2 V(\psi_1)V(D) + 2 \cdot l_3 \frac{A}{\delta + A} V(T) - l_9 V(\psi_1)V(D) + l_6 V(G)V(T),
\]
\( (10) \)

\[
\frac{dA}{dt} = l_1 \frac{T^4}{\beta + T^4} - l_3 \frac{A}{\delta + A} V(T),
\]
\( (11) \)

\[
\frac{dR_1}{dt} = k_1 T V(F_1^2) + k_3 R_2 V(D^2) - k_2 R_1 \frac{T}{1 + T + \alpha A} - k_2 R_1 V(T^2),
\]
\( (12) \)
\[
\frac{dR_3}{dt} = k_3R_1 \frac{T}{1+T+\alpha A} - k_3R_2V(D^2) + k_2T_2V(F_1^2) - k_8R_2V(T^2), \tag{13}
\]

\[
\frac{dT_1}{dt} = k_7R_1 \frac{T}{1+T+\alpha A} + k_2T_2V(D^2) - k_1T_1V(F_1^2), \tag{14}
\]

\[
\frac{dT_2}{dt} = k_3T_1 \frac{T}{1+T+\alpha A} - k_4T_2V(D^2) - k_2T_2V(F_1^2) + k_8R_2V(T^2), \tag{15}
\]

\[
\frac{dN}{dt} = -l_2V(N)\dot{V}(P) + l_1V(M - N)\dot{V}(\psi_1). \tag{16}
\]

Here, \( V(X) = X/(1 + X) \) is the function that describes the adsorption of the enzyme in the region of a local coupling. The variables of the system are dimensionless [8-10]. We take

\[
l_1 = 0.046; \quad l_2 = 0.0017; \quad l_3 = 0.01334; \quad l_4 = 0.3; \quad l_5 = 0.001; \quad l_6 = 0.01;
\]

\[
l_7 = 0.0535; \quad l_8 = 0.001; \quad k_1 = 0.07; \quad k_2 = 0.01; \quad k_3 = 0.0015; \quad k_4 = 0.0005;
\]

\[
k_5 = 0.05; \quad k_6 = 0.005; \quad k_7 = 0.03; \quad k_8 = 0.005; \quad m_1 = 0.3; \quad m_2 = 0.15; \quad m_3 = 1.6;
\]

\[
m_4 = 0.0005; \quad m_5 = 0.007; \quad m_6 = 10; \quad m_7 = 0.0001; \quad m_8 = 0.0000171; \quad m_9 = 0.5;
\]

\[
G_0 = 18.4; \quad L = 0.005; \quad S = 1000; \quad A = 0.6779; \quad M = 0.005; \quad S_1 = 150;
\]

\[
\alpha = 184.5; \quad \beta = 250; \quad \delta = 0.3; \quad \gamma = 79.7.
\]

In the study of the given mathematical model (1)-(16), we have applied the theories of dissipative structures [11] and nonlinear differential equations [12,13], as well as the methods of mathematical modeling used in author’s works [14-34]. In the numerical solution, we applied the Runge–Kutta–Merson method. The accuracy of calculations is \(10^{-8}\). The duration for the system to asymptotically approach an attractor is \(10^{6}\).

3 The Results of Studies

The mathematical model includes a system of nonlinear differential equations (1)-(16) and describes the open nonlinear biochemical system involving glycolysis and gluconeogenesis. In it, the input and output flows are glucose and lactate. Namely the concentrations of these substances form the direct or reverse way of the dynamics of the metabolic process. Both processes are irreversible and are running in the open nonlinear system, being far from the equilibrium. The presence of the reverse way of gluconeogenesis in the glycolytic system is the reason for the autocatalysis in it. In addition, the whole metabolic process of glycolysis is enveloped by the feedback formed by redox
reactions with the transfer of electrons with the help of NAD (16) and the presence of the adeninenucleotide cycle (9) – (11) (Fig.1).

We now study the dependence of the dynamics of the metabolic process of glycolysis-glucconeogenesis on the value of parameter $l_5$ characterizing the activity of glucconeogenesis. The calculations indicate that, as the value of this parameter increases to 0.234, the system passes to the stationary state. As this parameter increases further, the autooscillations of a 1-fold periodic cycle $1 \cdot 2^0$ arise and then, at $l_5 \approx 0.2369$, transit to chaotic ones $- \cdot 2^1$. The analogous behavior of the system is observed at larger values of $l_5$. As the parameter decreases to 0.43, the system stays in a stationary state. If the parameter $l_5$ decreases further, the system gradually transits in a 1-fold periodic cycle $1 \cdot 2^0$, and the region of oscillatory dynamics arises.

Let us consider the oscillatory dynamics of this process. We constructed the phase-parametric diagrams, while $l_5$ varies in the intervals 0.235 – 0.28 and 0.25 – 0.266 (Fig.2,a,b). The diagrams are presented for fructose-6-phosphate $F_1$. We emphasize that the choice of a diagram for the namely given variable is arbitrary. The diagrams of other components are analogous by bifurcations. We want to show that the oscillations on the section fructose-6-phosphate – fructose-1,6-biphosphate can be explained by the oscillations of fructose-6-phosphate caused by glucconeogenesis, rather than the allosteric property of the enzyme phosphofructokinase.

Fig. 2. Phase-parametric diagram of the system for the variable $F_1(t)$: a - $l_5 \in (0.235,0.28)$; b - $l_5 \in (0.25,0.266)$.

The phase-parametric diagrams were constructed with the help of the cutting method. In the phase space, we took the cutting plane at $R_2 = 1.0$. This choice is explained by the symmetry of oscillations $F_1(t)$ relative to this point. At the cross of this plane by the trajectory, we fix the value of each variable. If a multiple periodic limiting cycle arises, we will observe a number of points on the plane, which coincide in the period. If a deterministic chaos arises, the points, where the trajectory crosses the plane, are located chaotically.
Considering the diagram from right to left, we may indicate that, at 
$j_l^{278} = 0.278$, the first bifurcation of the period doubling arises. Then at 
$j_l^{265} = 0.265$ and $j_l^{262} = 0.262215$, we see the second and third bifurcations, 
respectively. Further, the autooscillations transit in the chaotic mode due to the 
intermittence. The obtained sequence of bifurcations satisfies the relation
\[
\lim_{i \to \infty} \frac{j_l^{i+1} - j_l^i}{j_l^{i+2} - j_l^{i+1}} \approx 4.668.
\]
This number is very close to the universal Feigenbaum constant. The transition to the chaos has happened by the Feigenbaum scenario [35].

It is seen from Fig.2,a,b that, for $l_5 = 0.25612$ and $l_5 = 0.2451$, the 
periodicity windows appear. Instead of the chaotic modes, the periodic and 
quasiperiodic modes are established. The same periodicity windows are 
observed on smaller scales of the diagram. The similarity of diagrams on small 
and large scales means the fractal nature of the obtained cascade of bifurcations 
in the metabolic process created by gluconeogenesis.

As examples of the sequential doubling of a period of autoperiodic modes 
of the system by the Feigenbaum scenario, we present projections of the phase 
portraits of the corresponding regular attractors in Fig.3,a-c. In Fig.3,d-f, we 
show some regular attractors arising in the periodicity windows. For $l_5 = 0.256$, 
the 3-fold periodic mode $3 \cdot 2^0$ is formed. For $l_5 = 0.2556$, we observe the 5-
fold mode. Then, as $l_5 = 0.245$, the 3-fold periodic cycle is formed again.

Fig.3. Projections of phase portraits of the regular attractors of the system:

a - $1 \cdot 2^1$, for $l_5 = 0.268$; b - $1 \cdot 2^2$, for $l_5 = 0.264$; c - $1 \cdot 2^4$, for $l_5 = 0.262$;
d - $3 \cdot 2^0$, for $l_5 = 0.256$; e - $5 \cdot 2^0$, for $l_5 = 0.2556$; and f - $3 \cdot 2^0$, for $l_5 = 0.245$. 
In Fig. 4,a,b, we give projections of the strange attractor $2^\perp$ for $l_5 = 0.25$. The obtained chaotic mode is a strange attractor. It appears as a result of the formation of a funnel. In the funnel, there occurs the mixing of trajectories. At an arbitrarily small fluctuation, the periodic process becomes unstable, and the deterministic chaos arises.

Fig.4. Projections of the phase portrait of the strange attractor $2^\perp$ for $l_5 = 0.25$: a – in the plane $(T_2, P)$, b – in the plane $(R_1, F_1)$.

In Fig.5,a,b, we present, as an example, the kinetics of autooscillations of some components of the metabolic process in a 1-fold mode for $l_5 = 0.3$ and in the chaotic mode for $l_5 = 0.25$. The synchronous autooscillations of fructose-6-phosphate and the inactive form $T_2$ of the allosteric enzyme phosphofructokinase are replaced by chaotic ones.

Fig.5. Kinetic curves of the variables: $F_1(t)$ - a and $T_2(t)$ - b in the 1-fold periodic mode for $l_5 = 0.3$ (1) and in the chaotic mode for $l_5 = 0.25$ (2).

While studying the phase-parametric diagrams in Fig.2,a,b, it is impossible beforehand to determine, for which values of parameter $l_3$ a multiple stable (quasistable) autoperiodic cycle or a strange attractor is formed.

For the unique identification of the type of the obtained attractors and for the determination of their stability, we calculated the complete spectra of
Lyapunov indices $\lambda_1, \lambda_2, \ldots, \lambda_{16}$ for chosen points and their sum $\Lambda = \sum_{j=1}^{16} \lambda_j$.

The calculation was carried out by Benettin’s algorithm with the orthogonalization of the perturbation vectors by the Gram--Schmidt method [13].

As a specific feature of the calculation of these indices, we mention the difficulty to calculate the perturbation vectors represented by $16 \times 16$ matrices on a personal computer.

Below in Table 1, we give several results of calculations of the complete spectrum of Lyapunov indices, as an example. For the purpose of clearness, we show only three first indices $\lambda_1 - \lambda_3$. The values of $\lambda_4 - \lambda_{16}$ and $\Lambda$ are omitted, since their values are not essential in this case. The numbers are rounded to the fifth decimal digit. For the strange attractors, we calculated the following indices, by using the data from Table 1. With the use of the Pesin theorem [36], we calculated the KS-entropy (Kolmogorov-Sinai entropy) and the Lyapunov index of a “horizon of predictability” [37]. The Lyapunov dimension of the fractality of strange attractors was found by the Kaplan--Yorke formula [38,39]:

By the calculated indices, we may judge about the difference in the geometric structures of the given strange attractors. For $l_5 = 0.25$, the KS-entropy takes the largest value $h = 0.00014$. In Fig.4,a, we present the projection of the given strange attractor. For comparison, we constructed the strange attractors for $l_5 = 0.26$ (Fig.6,a) and $l_5 = 0.237$ (Fig.6,b). Their KS-entropies are, respectively, 0.00008 and 0.00005. The comparison of the plots of the given strange attractors is supported by calculations. The trajectory of a strange attractor (Fig.4,a) is the most chaotic. It fills uniformly the whole projecton plane of the attractor. Two other attractors (Fig.6,a,b) have the own relevant regions of attraction of trajectories. The phase space is divided into the regions, which are visited by the trajectory more or less.

![Fig.6. Projections of the phase portraits of the strange attractors $2^+$ in the plane $(T_2, P)$: a – for $l_5 = 0.26$ and b – for $l_5 = 0.237$.](image)

Table 1. Lyapunov indices, KS-entropy, “horizon of predictability,” and the Lyapunov dimension of the fractality of strange attractors calculated for various modes
The Lyapunov dimensions of the given strange attractors are changed analogously. We have, respectively: 4.6, 4, and 3.6. These values characterize generally the fractal dimension of the given attractors. If we separate small rectangular area on one of the phase curves in each of the given plots and increase them, we will see the geometric structures of the given strange attractors on small and large scales. Each arisen curve of the projection of a chaotic attractor is a source of formation of new curves. Moreover, the geometric regularity of construction of trajectories in the phase space is repeated for each strange attractor. In the given case, the best geometric self-similarity conserves in the presented strange attractors in the following sequence: Fig.4,a, Fig.6,a, and Fig.6,b.

The value of “horizon of predictability” \( t_{\text{min}} \) for the modes presented in the table is the largest for \( l_2 = 0.237 \) (Fig.6,b). The narrow regions of attraction of the projection of the strange attractor correspond to the most predictable kinetics of the running metabolic process. From all metabolic chaotic modes, this mode is the mostly functionally stable for a cell.
The above-described study of the process of glycolysis-gluconeogenesis with the help of a change of the coefficient of positive feedback \( I_5 \) indicates that, in the given metabolic process under definite conditions, the autocatalysis arises. The value of \( I_5 \) determines the activity of gluconeogenesis on the section of the transformation of glucose-6-phosphate in glucose. This reaction is catalyzed by the enzyme glucose-6-phosphatase. This phosphatase is magnesium-dependent. If the magnesium balance is violated or some other factors come in play, the rate of this reaction varies. In addition, the absence of some coenzymes in a cell affects essentially also the rates of other enzymatic reactions, which can lead to the desynchronization of metabolic processes. As a result, the autooscillations arise in the metabolic process of glycolysis-gluconeogenesis. The autooscillations can be autoperiodic with various multiplicities or chaotic. Their appearance can influence the kinetics of the metabolic process in the whole cell and its physiological state.

Conclusions

With the help of a mathematical model, we have studied the influence of gluconeogenesis on the metabolic process of glycolysis. The metabolic chain of glycolysis-gluconeogenesis is considered as a single dissipative system arisen as a result of the self-organization, i.e., as a product of the biochemical evolution in protobionts. The reasons for the appearance of autocatalysis in it are investigated. A phase-parametric diagram of autooscillatory modes depending on the activity of gluconeogenesis is constructed. We have determined the bifurcations of the doubling of a cycle according to the Feigenbaum scenario and have shown that, as a result of the intermittence, the aperiodic modes of strange attractors arise. The fractal nature of the calculated cascade of bifurcations is demonstrated. The strange attractors arising as a result of the formation of a mixing funnel are found. The complete spectra of Lyapunov indices for various modes are calculated. For strange attractors, we have calculated the KS-entropies, “horizons of predictability,” and the Lyapunov dimensions of the fractality of attractors. The structure of a chaos of the given attractors and its influence on the stability of the metabolic process, adaptation, and functionality of a cell are studied. It is shown that a change of the cyclicity in the metabolic process in a cell can be caused by the violation of the magnesium balance in it or the absence of some coenzymes. The obtained results allow one to study the influence of gluconeogenesis on the self-organization of the metabolic process in a cell and to find the reasons for a change of its physiological state.

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References


Two parametric bifurcation in LPA model

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Abstract. The structured population LPA model describes flour beetle population dynamics of four stage populations: eggs, larvae, pupae, and adults with cannibalism between these stages. The case of non-zero cannibalistic rates of adults on eggs and adults on pupae and no cannibalism of larvae on eggs. This assumption is necessary to make at least some calculations analytically. It is shown that the behavior can be continued to the generic model with non-zero cannibalistic rate of larvae on eggs. In the model exist both supercritical and subcritical strong 1:2 resonance. The bifurcation responsible for the change of topological type of the strong 1:2 resonance is study. This bifurcation is accompanied by the origination of the Chenciner bifurcation. The destabilization of the system is caused by two parametric bifurcation is study together with its biological consequences.

Keywords: Two parametric bifurcation; LPA model; Tribolium model; strong 1:2 resonance; Chenciner bifurcation.

1 Introduction

This article is based on original work of Robert F. Costantino, Ph.D., Jim Cushing, Ph.D., Brian Dennis, Ph.D., Robert A. Desharnais, Ph.D. and Shandelle Henson, Ph.D. about LPA model (Tribolium model). LPA model is a structured population model that describes flour beetle population dynamics of four stage populations: eggs, larvae, pupae, adults with cannibalism between these stages. Main results of the research were published from the year 1995 to nowadays. In published articles authors concentrate mainly on the chaotic behavior in the system. The nonlinear dynamics of the system associated with the LPA model is rich, there is a lot of studies that deal with basic analysis of equilibria and their stability (e.g. Cushing[6], Cushing[4] or Kuang and Cushing[10]), some works are devoted to one-parameter bifurcations (as Dennis et al.[7]) and its route to chaotic dynamics (e.g. Cushing[6], Constantino et al.[2], Cushing et al.[5], Cushing et al.[3]). To our best knowledge, there is not any
published work about two-parameter bifurcation analysis by far. The original analysis of Chenciner bifurcation and subcritical strong 1:2 resonance were done in our article which is under review in Journal of theoretical biology.

In this article we concentrate on the both supercritical a subcritical strong resonance 1:2 and the bifurcation responsible for the change of topological type of the strong 1:2 resonance, which is accompanied with Chenciner bifurcations. The mathematical background for these bifurcations, their normal forms and analysis can be found in Kuznetsov[11].

The structured population LPA model consists of three stages: larvae $L$, pupae $P$ and adults $A$, while the population of eggs as a function of the adult population is not included into the state space. We assume cannibalism between the stages. We have to point out that we concentrate on LPA model with non-zero cannibalistic rates of adults on eggs and adults on pupae and no cannibalism of larvae on eggs. Here this assumption of no cannibalism of larvae on eggs is used only to make the mathematical calculations more easy and clear (a lot of them may be done analytically in this case) and this case was also examined in e.g. Dennis et al.[7].

2 Model description and basic analysis

The dynamic of LPA model is (see e.g. Cushing[6] or Cushing[4]):

\[
\begin{align*}
L(t+1) &= bA(t) e^{-c_{EL}L(t)-c_{EA}A(t)} \\
P(t+1) &= (1 - \mu_L) L(t) \\
A(t+1) &= P(t) e^{-c_{PA}A(t)} + (1 - \mu_A) A(t),
\end{align*}
\]

(1)

where state variables $L, P, A$ represent number of larvae, pupae and adults in population. Parameter $b > 0$ represents natality. Parameters $\mu_L$ and $\mu_A$ represent mortality of larvae and adults. We assume natural inequalities $0 < \mu_L < 1$, $0 < \mu_A < 1$ to be satisfied. Parameters $c_{EL}, c_{EA}, c_{PA}$ denote cannibalistic rates. Namely, $c_{EL}$ is the cannibalistic rate of larvae on eggs, $c_{EA}$ is the cannibalistic rate of adults on eggs and $c_{PA}$ is the cannibalistic rate of adults on pupae. We assume $c_{EA} \geq 0, c_{PA} \geq 0$ and $c_{EL} \geq 0$, in this article we consider a special case $c_{EL} = 0$.

There can be two fixed points of the system (1). The trivial fixed point corresponds to extinction of the population, the non-trivial fixed point $[L^*, P^*, A^*]$ satisfies formulas

\[
\begin{align*}
L^* &= \frac{b \ln \left( \frac{b(1 - \mu_L)}{\mu_A} \right) e^{-c_{EA} \ln \left( \frac{b(1 - \mu_L)}{\mu_A} \right)}}{(c_{PA} + c_{EA})} \\
P^* &= \frac{b (1 - \mu_L) \ln \left( \frac{b(1 - \mu_L)}{\mu_A} \right) e^{-c_{EA} \ln \left( \frac{b(1 - \mu_L)}{\mu_A} \right)}}{(c_{PA} + c_{EA})} \\
A^* &= \frac{\ln \left( \frac{b(1 - \mu_L)}{\mu_A} \right)}{(c_{PA} + c_{EA})}.
\end{align*}
\]

(2)
We introduce the basic reproduction number $R_0 = \frac{b(1-\mu L)}{\mu A}$. The non-trivial fixed point exists for $R_0 > 0$, but for $R_0 \in (0, 1)$ it has no biological meaning. It can be easily shown that the trivial fixed point is stable for $R_0 < 1$, for $R_0 = 1$ $[L^*, P^*, A^*] = [0, 0, 0]$ and is unstable for $R_0 > 1$, while the fixed point $[L^*, P^*, A^*]$ is not stable for all values of parameters. In the words of biology, population will extinct for $R_0 \leq 1$ and can survive for $R_0 > 1$. In the words of the bifurcation theory, $R_0 = 1$ is a critical value of the transcritical bifurcation. The manifold of the transcritical bifurcation is included in $b = \frac{\mu A}{1-\mu L}$ of the parameter space. It’s good to mention that the transcritical bifurcation does not depend on cannibalistic rates.

The one-parameter bifurcations are already already described in Dennis et al.[7]. From the presented work it’s clear that the flip bifurcation curve (called there 2-cycles) and Neimark-Sacker bifurcation curve (called there loops) can intersect (see the figure 1 in Dennis et al.[7]). In next sections of this paper we will go on with deeper two-parameter bifurcation analysis. All our results are in agreement with the results presented in the paper Dennis et al.[7] as well as with sufficient conditions for stability of the non-trivial fixed point that can be found in Kuang and Cushing[10].

3 Routes to two-parameter bifurcations

There are two ways how we receive two-parameter local bifurcations of the fixed point. One of them is that the non-degeneracy conditions of the one-parameter bifurcation is violated. For example the Neimark-Sacker bifurcation non-degeneracy condition is violated in the Chenciner critical points. Qualitative changes in dynamics near the Chenciner bifurcation have globally destabilizing effect to the population and this will be discussed in the next separate section. The other way is that two eigenvalues reach the unit circle. Let’s consider this case now. Obviously, the two-parameter bifurcation manifold covers the intersection of one-parameter bifurcation manifolds. In our system there exists thee different one-parameter bifurcation manifolds: transcritical, flip and Neimark-Sacker bifurcation. There are two types of intersections of the flip and the Neimark-Sacker bifurcation manifolds:

(i) $b = \frac{\mu A e^{\frac{2}{5}}}{1-\mu L}, c_{EA} = \frac{(\mu A+1)c_{PA}}{1-\mu A}$ and $c_{EA} = \frac{(2\mu A-1)c_{PA}}{5-2\mu A}$.

(ii) $b = \frac{\mu A e^{\frac{2}{5}}}{1-\mu L}, c_{EA} = \frac{(2\mu A-1)c_{PA}}{5-2\mu A}$.

The manifold (i) exists for all allowed values of parameters. On the other hand the manifold (ii) exists for $\mu A > \frac{1}{2}$ only.

In this paper we will focus on manifold (ii). The manifold (ii) corresponds to the strong 1:2 resonance with associated eigenvalues are $-1, -1, \frac{1}{2}$.

For arbitrarily fixed parameters $\mu_L$, $\mu_A$, $c_{PA}$, the two-parameter bifurcation manifolds correspond to points of intersection of one-parameter bifurcation curves in the two-parameter space $c_{EA}$ versus $b$. The parameters $\mu_L$, $\mu_A$, $c_{PA}$ are fixed to common values (see e.g. Dennis et al.[8]).
4 Strong 1:2 resonance in LPA model

Strong 1:2 resonance is a two-parameter bifurcation that lies in the intersection of flip bifurcation manifold and the Neimark-Sacker bifurcation manifold. In our model two topological types of the strong 1:2 resonance exists: subcritical bifurcations of a node or a focus, supercritical bifurcation of a node or a focus. Then normal form for the supercritical bifurcation is similar to the subcritical, but the time variable has an opposite sign (see e.g. Kuznetsov[11]). Therefore the phase portraits of subcritical and supercritical bifurcations has an opposite stability.

![Subcritical strong 1:2 resonance diagram](image)

**Fig. 1.** Subcritical strong 1:2 resonance diagram in a two-parameter space. The N-S\(_+\) denotes the subcritical branch of the Neimark-Sacker curve, N-S\(_-\) denotes the neutral saddles, F\(_+\), F\(_-\) denote the flip bifurcation curves, LPC denotes the fold bifurcation of the invariant loop curve, P denotes the saddle separatrix loop curve. The phase portraits in each domain 1 - 6 are topologically generic. Similarly to the Chenciner bifurcation, a special heteroclinic structure of orbits appears in the neighbourhood of LPC and P. For more details see Kuznetsov[11].

Strong 1:2 resonance points lie in the intersection of Neimark-Sacker and flip manifolds, therefore we expect birth of the limit loop from a fixed point due to N-S bifurcation and split of the fixed point into a 2-cycle nearby the strong 1:2 resonance point. The figure 1 displays the generic transversal two-parameter space section of a canonical subcritical strong 1:2 resonance bifurcation manifold at zero with one-parameter N-S and flip manifolds at the horizontal and vertical axes.

As we move around the strong 1:2 resonance point, the topological structure of the state space change the way that is presented for the canonical form at the figure 1.
5 Chenciner bifurcation in LPA model

Transversal crossing of the Neimark-Sacker bifurcation manifold give rise to an invariant loop around a fixed point that changes its stability. There are two topological types of the Neimark-Sacker bifurcation: supercritical and subcritical. The supercritical type give rise to a stable invariant loop, reversely, the subcritical bring about an unstable loop. The Chenciner bifurcation is a critical change of these two types. There exists an accompanying bifurcation manifold of the Chenciner bifurcation. It is called the fold bifurcation of the invariant loop or the limit point bifurcation of the invariant loop and it gives a birth to the stable and unstable invariant loop around.

The Chenciner bifurcation is found strictly on the one branch of Neimark-Sacker bifurcation near the strong 1:2 resonance. We found even parameter values for which the Chenciner and strong 1:2 resonance bifurcations collide. This collision is responsible for change of topological type of strong 1:2 resonance.

6 Change of topological type of strong 1:2 resonance in LPA model

Both Chenciner bifurcation and subcritical strong 1:2 resonance occur for \( \mu_A \) sufficiently close to 1 in LPA model (remember that the necessary condition for the strong 1:2 resonance is \( \mu_A > \frac{1}{2} \)). For \( \mu_A \) sufficiently close to \( \frac{1}{2} \) there exists only supercritical strong 1:2 resonance. The critical change of subcritical and supercritical strong 1:2 resonance gives a birth to the Chenciner bifurcation. Here we present our original analysis of the phenomenon of changing topological type of the strong 1:2 resonance. We will describe the structure by equivalence classes of structurally stable domains with topologically equivalent state spaces for both topological types of the strong 1:2 resonance. The borders of these domains are the one-parameter bifurcation.

The transversal two-dimensional section \( b \) versus \( c_{EA} \) of supercritical strong 1:2 resonance is taken for fixed parameters \( \mu_L = 0.1613; \mu_A = 0.75; c_{PA} = 0.004348 \) (which is shown in the picture 2). The dynamic classes I. - VI. are displayed at the figures 3.

The topological structure of the parameter space near Chenciner bifurcation and subcritical strong 1:2 resonance give rise to a complicated state space dynamics with coexistence of different types of invariant sets. The transversal two-dimensional section \( b \) versus \( c_{EA} \) of both two-parameter manifolds (Chenciner and subcritical strong 1:2 resonance) is taken for fixed parameters \( \mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348 \). Striped and shadowed domains belong to the basins of attraction corresponding to weak and huge oscillations respectively. White domains belong to a stable fixed point basins of attraction. The two different branches of LPC (fold bifurcation of the invariant loop) collide in a typical V-shape in the cusp point, that is a two-parameter bifurcation point. The cusp point is typically connected with another phenomenon of hysteresis. The parameter space is divided into nine domains where the state
Fig. 2. Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters $c_{EA}$ and $b$ for fixed $\mu_L = 0.1613; \mu_A = 0.75; c_{PA} = 0.004348$.

Spaces stay topologically equivalent. All dynamic classes I. - IX. are displayed at the figures 6. We omit the stripe underneath the transcritical bifurcation curve, where the population is dying out. Here the only fixed point is the trivial equilibrium that is globally stable and so the population extinctions. For values of $b$ above the transcritical bifurcation curve, the trivial equilibrium is unstable and the orbits can tend to another attractors.

For parameter values near the change of topological type of strong 1:2 resonance the system is locally topologically equivalent to the system displayed in the picture 4. The global behavior is shown in the picture 5.
Fig. 3. Phase portraits near the strong resonance 1:2 and Chenciner bifurcation in LPA model with parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$ and free parameters $c_{EA}$ a $b$. In the left column, there are schematic phase portrait for each domain according to the figure 2. In the right column, there are computed stable sets at adults and pupae state variables.
**Fig. 4.** Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters $c_{EA}$ and $b$ for fixed $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$.

**Fig. 5.** Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters $c_{EA}$ and $b$ for fixed $\mu_L = 0.1613; \mu_A = 0.87; c_{PA} = 0.004348$. 
(a) I: stable invariant loop, e.g. $c_{EA} = 0.0013$, $b = 6$

(b) II: stable fixed point, e.g. $c_{EA} = 0.002$, $b = 8$

(c) III: stable fixed point, stable invariant loop, e.g. $c_{EA} = 0.0014$, $b = 6$

(d) IV: stable 2-cycle, stable invariant loop, e.g. $c_{EA} = 0.0015$, $b = 9.2$

(e) V: two stable symmetric coupled loops, stable invariant loop, e.g. $c_{EA} = 0.00154$, $b = 9.6$

(f) VI: two stable invariant loops, e.g. $c_{EA} = 0.00172$, $b = 10.05$

(g) VII: stable 2-cycle, e.g. $c_{EA} = 0.002$, $b = 10$

(h) VIII: stable symmetric coupled loops, e.g. $c_{EA} = 0.00185$, $b = 10.2$

(i) IX: stable invariant loop, e.g. $c_{EA} = 0.0016$, $b = 9.9$

Fig. 6. Phase portraits near the strong resonance 1:2 and Chenciner bifurcation in LPA model with parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$ and free parameters $c_{EA}$ a $b$. In the left column, there are schematic phase portrait for each domain according to the figure 4. In the right column, there are computed stable sets at adults and pupae state variables.
In our opinion, this complicated structure near the strong resonance 1:2 and the Chenciner bifurcation in LPA model has a very troublesome consequence, since in this quite a big area of parameters it's very hard to compare the simulated and real data. In real experiments, the natality $b$ and the cannibalistic rate $c_{EA}$ as parameters are not strict constants and they can vary during time due to temperature or attainability of other sources of food and other random events, there can be some measure errors also. The real data and simulations may become totally different even in the case of a proper model usage. Even the simulated data may be considered to be chaotic or random by mistake. Imagine parameters $b$ and $c_{EA}$ that vary slowly in their parameter domain near the described phenomenon. The simulated data look as chaotic or random, since they are very sensitive to the parameter changes, see the figure 7.

![Parameter changes](image1)

Fig. 7. Simulated time series with slowly varying parameters $b$ and $c_{EA}$ for parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$.

### 7 Conclusion

We presented a two-parameter bifurcation analysis of LPA model (for parameters $c_{EA}$, $b$ and $\mu_A$) with zero $c_{PA}$ cannibalistic rate to show complex dynamics in the model of the tribolium population. Here we mention that we did not concerned to the period doubling and chaos, since there is a lot of papers devoted to this topic, but we focused on another bifurcations that were overlooked so far and their destabilization effects were not mentioned yet.
We found strong 1:2 resonance of node or a focus in LPA model and we explained its topological structure. We explained the importance of the bifurcation type of the strong 1:2 resonance bifurcation, because both of the types (subcritical and supercritical) are present. The topological change of the strong 1:2 resonance gives a birth to the Chenciner bifurcation. We expressed the Chenciner bifurcation.

As the most important part of our paper we consider to be the finding of connection between the Chenciner bifurcation and strong 1:2 resonance and setting of the complete two-parameter bifurcation diagram of these manifolds (together with the nearby non-local bifurcation manifolds).

References

Stochastic Calculations for Fibre Raman Amplifiers with Randomly Varying Birefringence

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Abstract. For the first time for the model of real-world forward-pumped fibre Raman amplifier with the randomly varying birefringence, the stochastic calculations have been done numerically based on the Kloeden-Platen-Schurz algorithm. The results obtained for the averaged gain and gain fluctuations as a function of polarization mode dispersion (PMD) parameter agree quantitatively with the results of previously developed analytical model. Simultaneously, the direct numerical simulations demonstrate an increased stochasticization (maximum in averaged gain variation) within the region of the polarization mode dispersion parameter of 0.01÷0.1 ps/km¹/². The results give an insight into margins of applicability of a generic multi-scale technique widely used to derive coupled Manakov equations and allow generalizing analytic model with accounting for pump depletion, group-delay dispersion and Kerr-nonlinearity that is of great interest for development of the high-transmission-rates optical networks.

Keywords: Stochastic modeling, Raman amplifiers, Fibre optic communications.

1 Introduction

Rapid progress in overall capacity of optical networks based on the distributed fibre Raman amplifiers [1] (FRAs) faces the challenge of increased transmission impairments related to polarization depend gain (PDG) and gain fluctuations (GF), e.g. dependence of the Raman gain on the inputs state of polarization (SOP) and random nature of birefringence in optical fibres [2-5]. As was found previously [2, 3], the randomly varying birefringence contributes essentially into GF by de-correlating the signal and pump SOPs, e.g. preventing polarization pulling of signal SOP to the pump SOP. As a result, GF takes a maximum value as a function of PMD parameter [2, 3] that is similar to stochastic resonance (SR), dynamic...
localization phenomena and escape from a metastable state in an excitable system [4, 6-9].

2 Model of the Raman-scattering Induced Polarization Phenomena in Presence of the Stochastic Birefringence

To validate our previously developed analytical models [3, 4] and application of multiple-length-scale technique developed by Menyuk [10] to study PDG and GF in FRAs, for the first time to our knowledge we present the results of direct stochastic calculations of PDG and GF in FRAs. The starting point is the coupled Manakov-PMD equations describing co-propagation signal and pump waves in a fibre Raman amplifier [2,5,11]:

\[
i \frac{\partial |A_s\rangle}{\partial z} + \beta_s [\cos(\theta) \sigma_3 + \sin(\theta) \sigma_1] |A_s\rangle + i \frac{\alpha_s}{2} |A_s\rangle + \\
i \beta_s' [\cos(\theta) \sigma_3 + \sin(\theta) \sigma_1] \frac{\partial |A_s\rangle}{\partial t} - \frac{\beta_s''}{2} \frac{\partial^2 |A_s\rangle}{\partial t^2} + \frac{\gamma_s}{3} [2 \langle A_s | |s\rangle + |A_s^*\rangle \langle A_s^* | |s\rangle - \frac{i \gamma_R}{2} |A_p\rangle \langle A_p | |s\rangle = 0,
\]

\[
\frac{\partial |A_p\rangle}{\partial z} + \beta_p [\cos(\theta) \sigma_3 + \sin(\theta) \sigma_1] |A_p\rangle + i \frac{\alpha_p}{2} |A_p\rangle + \\
i \beta_p' [\cos(\theta) \sigma_3 + \sin(\theta) \sigma_1] \frac{\partial |A_p\rangle}{\partial t} - \frac{\beta_p''}{2} \frac{\partial^2 |A_p\rangle}{\partial t^2} \\
+ \frac{\gamma_p}{3} [2 \langle A_p | |p\rangle + |A_p^*\rangle \langle A_p^* | |p\rangle - \frac{i \gamma_R}{2} \omega_p |A_s\rangle \langle A_s | |p\rangle = 0.
\]

Here \( \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, |A_k\rangle = (A_{kx}, A_{ky})^T \) are the polarization vectors for pump \((k = p)\) and signal \((k = s)\), respectively. \( \beta_{sp} = 2\pi/L_{sp} \) is the corresponding birefringence strength \((L_{sp} \text{ are the beat lengths for pump and signal at the frequencies } \omega_p \text{ and } \omega_s, \text{ respectively})\), \( \beta_{sp}' \) are the group-delays, and \( \beta_{sp}'' \) are the group-delay dispersions. \( \gamma_{k,m} \) are the self-
and cross-phase modulation coefficients \((k = s, p; m = s, p)\). \(\alpha_{p,s}\) are the attenuation coefficients for pump and signal, respectively. At last, \(g_p\) is the Raman gain coefficient and \(\theta\) is the angle defining the birefringence axis [3].

For comparatively small propagation distances \((z < 20 \text{ km})\), long pulses (>2.5 ps), small pump and signal powers \((P_{in} < 1 \text{ W} \text{ and } s_0 < 10 \text{ mW}, \text{ respectively})\) one may neglect group-delay, its dispersion and nonlinear effects in a system. Then, the rotation \(|A_i⟩ = R|a_i⟩, R = \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}\) results in the equations for the pump and signal Jones vectors \(|a_p⟩\) and \(|a_s⟩\):

\[
i \frac{d|a_p⟩}{dz} + \Sigma_p|a_p⟩ + i \frac{\alpha_p}{2} |a_p⟩ = 0,
\]

\[
i \frac{d|a_s⟩}{dz} + \Sigma_s|a_s⟩ + i \frac{\alpha_s}{2} |a_s⟩ = 0,
\]

where \(\Sigma_i = \begin{bmatrix} \beta_i & -\theta_i \\ \theta_i & -\beta_i \end{bmatrix}\), \(\theta_z \equiv \frac{\partial \theta}{\partial z} = g_\theta\). \(i\)-index corresponds to pump \((p)\) or signal \((s)\), and \(g_\theta\) is a Wiener process defining the birefringence stochastic: \(\langle g_\theta(z) \rangle = 0, \langle g_\theta(z) g_\theta(z') \rangle = \sigma_\theta^2 \delta(z - z')\), and \(\sigma_\theta^2 = 1/L_c\), where \(L_c\) is the correlation length of the birefringence vector. Transition to the reference frame in the Stokes space where the local birefringence vectors \(\vec{W}_i\) have the components \((\beta_i, 0, 0)\) [3] results in the system of stochastic differential equations for the unit vectors \(\vec{s}\) and \(\vec{p}\) defining the signal and pump SOPs and their scalar product \(x = \vec{s} \cdot \vec{p}\):

\[
\frac{d(s_0 \vec{s})}{dz} = \frac{g_R}{2} P_0(z) s_0 \vec{p} + \begin{pmatrix} -s_2 \\ -s_1 \\ 0 \end{pmatrix} s_0 \theta + s_0 \beta_s \begin{pmatrix} 0 \\ -s_3 \\ s_2 \end{pmatrix},
\]

\[
\frac{ds_0}{dz} = \frac{g_R}{2} P_0(z) x(z) s_0(z),
\]

\[
\frac{d(s_0 x)}{dz} = \frac{g_R}{2} P_0(z) s_0 - (\beta_p - \beta_s)(p_3 s_2 - p_2 s_3) s_0,
\]

\[
\frac{d\vec{p}}{dz} = \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix} g_\theta + \beta_p \begin{pmatrix} 0 \\ -p_3 \\ p_2 \end{pmatrix}.
\]
Here $s_0$ is a length of signal Stokes vector normalized to \[ \exp \left( \int_0^L \frac{g_R P(z)}{2} \, dz - \alpha_s L \right), \] $L$ is the fibre length, and $P_0(z) = P_{in} \exp(-\alpha_P z)$. The average gain $\langle G \rangle$ and the PDG parameters are defined as $\langle G \rangle = 10 \log \left( \frac{\langle s_0(L) \rangle}{\langle s_0(0) \rangle} \right)$ and $PDG = 10 \log \left( \frac{\langle s_{0,\text{max}}(L) \rangle}{\langle s_{0,\text{min}}(L) \rangle} \right)$, respectively.

Equations (3) can be averaged over both regularly (the scales are defined by $L_{s,p}$) and randomly (the scale is defined by $L_c$) varying birefringence that results in [3]:

\[
\frac{d\langle s_0 \rangle}{dz'} = \frac{g_R P_{in} L}{2} \exp(-\alpha_s z') \langle x \rangle, \\
\frac{d\langle x \rangle}{dz'} = \frac{g_R P_{in} L}{2} \exp(-\alpha_s z') \langle s_0 \rangle - \beta_s \langle y \rangle, \\
\frac{d\langle y \rangle}{dz'} = \beta_s L (\langle x \rangle - p_1(0)s_1(0) \exp(-z' L/L_c)) - \langle y \rangle L/L_c,
\]

where $z' = z/L$. Further generalization of this analytical approach [4] allows characterizing a system in more complete way. In particular, the standard deviations for both $\langle G \rangle$ and $\langle x \rangle$ can be obtained.

Eqs. (3) have been solved with the Wolfram Mathematica 9.0 computer algebra system by the built-in Klöden-Platen-Schurz method with the propagation step $\Delta z = 10^{-4} \min(L_c, L_l)$. The averaging procedure was performed over an ensemble of 100 stochastic trajectories. We have used the following parameters: $g_R = 0.8 \, \text{W}^{-1}\text{km}^{-1}$, $s_0 = 10 \, \text{mW}$, $P_{in} = 1 \, \text{W}$, $L = 5 \, \text{km}$, $L_c = 100 \, \text{m}$. Stokes parameters for the pump and the input signal fields corresponding to the maximum and minimum gain were $\mathbf{p} = (1,0,0)$, $\mathbf{s} = (1,0,0)$ and $\mathbf{p} = (1,0,0)$, $\mathbf{s} = (-1,0,0)$, respectively.

### 3 Results and Discussion

In this work, for the first time to our knowledge we present the results of direct stochastic calculations of PDG and GF in FRAs. In terms of SR, correlation length $L_c$ and beat lengths $L_{s,p}$ play roles of inverse noise strength and frequency of external periodic modulation, respectively [4, 6]. The PMD parameter is defined as $\sqrt{2L_c \lambda_s / \pi c L_s}$ ($c$ is the light speed, $\lambda_s$ is the signal wavelength) and so according to the SR theory there is some
PMD value at which GFs, i.e. the relative gain standard deviation \( \sigma / \langle G \rangle \) is minimal. However, the gain fluctuations are resonantly enhanced in the vicinity of \( L_b \approx L_c / 4 \) (red curve in Fig. 1 (a)). Such a growth of GFs corresponds to enhancement of “wondering” of stochastic trajectories for \( s_0 \) and \( \langle x \rangle \), as it is demonstrated by insets in Fig. 1 (b). Thus, the phenomenon reported can be termed stochastic anti-resonance (SAR).

Since the PMD growth activates escape from polarization pulling SOP corresponding to \( \langle x \rangle \to 1 \) (black solid curve in Fig. 2), the maximum averaged gain \( \langle G \rangle \) decreases in the quantitative agreement with the analytical model of Eqs. (4) (black solid and dashed curves in Fig. 1 (a)). Simultaneously, a de-correlation of the pump-signal SOP (dashed red curve for the standard deviation \( \sigma_{\langle x \rangle} \) in Fig. 2) allows reducing the PDG with the PMD growth (black curve in Fig. 1 (b)).

An enhancement of stochasticity is illustrated by anti-resonance valley in the dependence of the Hurst parameter \( H \) [14] on the PMD (inset in Fig. 2), when an initially highly correlated SOP with \( H \approx 1 \) tends to become Brownian (\( H \to 0.5 \)). Such a switch between the statistical scenarios is typical for SAR, when the variance increases with growth of variability in a system [15].

Further growth of PMD suppresses the GFs (red curve in Fig. 1 (a)) and increases the Hurst parameter for \( \langle x \rangle \) (inset in Fig. 2) although the pump and signal SOPs remain de-correlated (red dashed curve for \( \sigma_{\langle x \rangle} \) in Fig. 2). Such a scenarios corresponds to the escape from a pulling state when the gain and signal SOPs evolve independently and substantially faster than the birefringence changes randomly (i.e. \( L_i < L_c \)). Such an intensification of polarization evolution affects the
Fig. 1. (a): the averaged maximum gain $\langle G \rangle$ vs. PMD from numerical simulations (solid black curve) and analytical model of Eqs. (4) (dashed black curve) as well as numerical (solid red curve) and analytic [4] (dashed red curve) relative standard deviation $\sigma/\langle G \rangle$. (b): the PDG parameter vs. PMD. The inserts correspond to red square, where the standard deviation is maximum, and show the stochastic trajectories of signal $s_0$ as well as the correlation of signal and pump SOPs $\langle x \rangle$ (dashed curves are the ensemble averages, filled regions are bounded by the corresponding standard deviation $\sigma$).
Fig. 2. Dependencies of $\langle x \rangle$ (black solid curve), its standard deviation $\sigma(\langle x \rangle)$ (red dashed curve), and the corresponding Hurst parameter (inset) on the PMD parameter for $\vec{p}=(1,0,0)$ and $\vec{s}=(1,0,0)$.

Fig. 3. Power density spectra of $\langle x \rangle$ for $\vec{p}=(1,0,0)$ and $\vec{s}=(1,0,0)$ and the signal beat lengths $L_s$ of 200 (a), 20 (b) and 10 m (c), respectively.
power spectral density of $\langle x \rangle$ which demonstrates a dramatic increase in a number of additional frequencies with the PMD growth (Fig. 3).

4 Conclusions

In conclusion, the Stratanovich-type equations describing Raman amplification in fibres in the presence of stochastic birefringence have been solved directly for the first time to our knowledge. The results obtained revealed a transition between different stochastic scenarios of mean gain and its fluctuations. It is shown that the PMD growth induces an escape from the regular metastable state corresponding to the pump and signal SOPs pulling to the state of de-correlated but comparatively regular SOPs. Such an escape can be characterized as the "stochastic anti-resonance" accompanied by substantial enhancement of GFs and quasi-Brownian evolution of the pump-signal coupling. A range of PMD parameters corresponding to such stochastization is typical for modern FRAs and, thereby, is of interest for high-capacity un-repeatered fibre networks. The demonstrated quantitative agreement of the numerical results with the analytical ones based on multi-scale averaging techniques promises developing a generalized approach to design and optimization of FRAs.

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References

Smoothing discontinuities in Predator Prey Models

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Abstract. Several modern predator prey like systems as well as a number of simple low dimensional chaotic systems involve discontinuous functions. This causes difficulties in simulating such systems in order to obtain characteristics such as stationary points, local and structural stability, Lyapunov Exponents or limit cycles. These parameters are very important in understanding the models. The distinction between differential equations and discretized maps is also of interest, particularly in two dimensional systems. In this work, we propose to replace discontinuous functions by continuous functions that approximate them in order to facilitate the analysis. For example, the step function can be replaced by an inverse tangent or hyperbolic tangent.

Keywords: Predator-Prey models, Discontinuous functions, Local and Structural Stability, Simulation of Chaotic Systems and Lyapunov Exponents.

1 Introduction

Predator prey systems and their generalizations such as the HANDY(Human and Nature Dynamics) model (Motesharrei et al.[1]) or models used in ecological systems (Caia,Z. et al[2]) or systems that involve general functional responses such as the low dimensional chaotic systems proposed by Sprott and Linz.[3] involve discontinuous functions. In real world application, the biological dynamical systems are usually discontinuous. For example, in many models of renewable resource management, sliding mode feedback control or nonlinear circuits, stabilization favors the use of threshold policies involving discontinuous functions. As far as simulating the fiducial trajectory is concerned, discontinuities do not pose difficulties in integrating the system numerically. However, Lyapunov exponents are usually regarded as invariants that characterize chaotic behavior and their calculation requires varied trajectories about a fiducial
trajectory (Wolf, A et al. [4]) and discontinuities affect the varied trajectory calculation and produce stiffness or instabilities during the numerical integration. This has been pointed out in the case of the signum nonlinearity by Sun and Sprott [5].

There are ways with which some discontinuities can be approximated by continuous functions so that the derivative that is necessary for the varied trajectory can exist. An example is using $2/\pi \arctan(ax)$ or $\tanh(ax)$, $a>0$ for the step function. A list of similar discontinuous functions and their continuous equivalents are given below.

\[
sgn(x) = \frac{2}{\pi} \tan^{-1}(ax) \quad \text{or} \quad \tanh(ax)
\]

\[
|x| = x sgn(x)
\]

\[
\min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}
\]

\[
\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}
\]

A graph of the arc tangent and hyperbolic tangent functions is presented in Fig 1 where the parameter $a$ can be varied to increase the steepness as much as possible without affecting differentiability.

This is important since Sprott [5] correctly claims that the Wolf integration or even attractor reconstruction do not successfully analyze such systems.
In this work, the Sprott and HANDY models, the two examples mentioned above for discontinuities in the dynamical systems will be examined. Existence of chaotic behavior in the Sprott model and its nonexistence in the HANDY model for a range of values of the parameters as evidenced by the maximal Lyapunov Exponent will be presented.

2 HANDY (Human and Nature Dynamics) Model

Originally, the HANDY Model is derived from a predator-prey model as indicated in [1]. It is a socio economical model that aims to describe accumulated wealth and economic inequality in a predator–prey model of humans and nature. This model divides the society into two parts: commoners \( x_c \) and elites \( x_e \). Natural resources or nature \( y \) in which society lives and the wealth \( w \) that that society uses and spends is included in the model as additional variables. It is interesting that in this model the wealth is only generated by commoners \( x_c \), however the wealth is not equally distributed between commoners \( x_c \) and elites \( x_e \) such that elites get \( \kappa \) (which is the inequality factor) times more than commoners. The governing differential equations of the HANDY model are given below:

\[
\begin{align*}
\dot{x}_c &= \beta_c x_c - \alpha_c x_c \\
\dot{x}_e &= \beta_e x_e - \alpha_e x_e \\
\dot{y} &= \gamma y(\lambda - y) - \delta x_c y \\
\dot{w} &= \delta x_c y - C_c - C_e
\end{align*}
\]

Here \( \beta_c \) and \( \beta_e \) are the birth rates of commoners and elites, \( \gamma \) is the regeneration rate of nature, \( \lambda \) is the carrying capacity of nature, \( \delta \) is depletion (or production) factor. The problematic parts in this model come from \( C_c, C_e \) and \( \alpha_c, \alpha_e \) which are discontinuous functions of \( x_c, x_e \) and \( w \):

\[
\begin{align*}
C_c &= \min \left( 1, \frac{w}{W_{th}} \right) \cdot s \cdot x_c \\
C_e &= \min \left( 1, \frac{w}{W_{th}} \right) \cdot s \cdot \kappa \cdot x_e \\
\alpha_c &= \alpha_m + \max(0, 1 - \frac{C_c}{x_c})(\alpha_M - \alpha_m) \\
\alpha_e &= \alpha_m + \max(0, 1 - \frac{C_e}{x_e})(\alpha_M - \alpha_m)
\end{align*}
\]

Here \( \alpha_M \) and \( \alpha_m \) are maximum and minimum death rates, \( s \) is subsistence salary per capita. In addition to this, \( w_{th} \) is defined as \( w_{th} = \rho x_c + \kappa \rho x_e \), where is the threshold wealth per capita. It can be seen discontinuous function max () and min () are used for consumption functions \( C_c, C_e \) and death rate constants \( \alpha_c, \alpha_e \).
To eliminate the effect of the discontinuous functions we introduce the following modifications to $C_c, C_e, \alpha_c, \alpha_e$: Note that tanh can replace the $2/\pi \arctan$ function.

\[
C_c = \left(1 + \frac{w}{W_{th}}\right) - \frac{2}{\pi} \arctan \left(\frac{1 - \frac{w}{W_{th}}}{2}\right) s x_c
\]

\[
C_e = \left(1 + \frac{w}{W_{th}}\right) - \frac{2}{\pi} \arctan \left(\frac{1 - \frac{w}{W_{th}}}{2}\right) s x e
\]

\[
\alpha_e = \alpha_m + \left(1 - C_c s x_c\right) + \frac{2}{\pi} \arctan \left(\frac{1 - C_c s x_c}{2}\right) (\alpha_M - \alpha_m)
\]

\[
\alpha_c = \alpha_m + \left(1 - C_e s x_e\right) + \frac{2}{\pi} \arctan \left(\frac{1 - C_e s x_e}{2}\right) (\alpha_M - \alpha_m)
\]

In Reference [1], simulation results for different values of parameters introduced above are given. We also repeated the simulation of this model with the modification that we proposed for the case of the Egalitarian society in [1] and the result is given in Fig 2.

Fig 2a. Simulation results which compare commoners’ population result for Egalitarian Society where there is no elite (Soft Landing Solution).
Simulation results which compare wealth and nature result for Egalitarian Society where there is no elite (Soft Landing Solution).

As it can be seen from the graph, simulation results of the HANDY model are not too much affected and the modification and discontinuity in the model can thus be eliminated.

Attempting to calculate the Lyapunov exponents by the Wolf algorithm where the discontinuities are implemented in the code gives the following rather meaningless result, since the Lyapunov exponents should not depend on the initial conditions. Replacing the discontinuities by their continuous approximations involving the arctan function gives no positive Lyapunov exponents and corrects this defect. This implies the lack of chaotic behavior. The Poincare sections also confirm the lack of chaotic behavior.

<table>
<thead>
<tr>
<th>initial conditions</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
<th>$\Lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_c(0)=100, X_e(0)=0$ $Y(0)=100, W(0)=0$</td>
<td>0.1988668</td>
<td>0.0248482</td>
<td>-0.738032</td>
<td>-1.2567</td>
</tr>
<tr>
<td>$X_c(0)=10, X_e(0)=0$ $Y(0)=10, W(0)=0$</td>
<td>0.9456494</td>
<td>0.0288546</td>
<td>-0.07919</td>
<td>-1.0961804</td>
</tr>
<tr>
<td>$X_c(0)=100, X_e(0)=10$ $Y(0)=100, W(0)=0$</td>
<td>0.0119613</td>
<td>0.0114135</td>
<td>-0.0114135</td>
<td>-1.0534976</td>
</tr>
<tr>
<td>$X_c(0)=100, X_e(0)=10$ $Y(0)=100, W(0)=100$</td>
<td>0.1245292</td>
<td>0.0215223</td>
<td>-0.0719403</td>
<td>-1.2414130</td>
</tr>
<tr>
<td>$X_c(0)=100, X_e(0)=20$ $Y(0)=200, W(0)=100$</td>
<td>0.1260577</td>
<td>0.019006</td>
<td>-0.0872434</td>
<td>-1.2701920</td>
</tr>
</tbody>
</table>

Table 1: Lyapunov Exponents of HANDY System for different initial conditions
3 Sprott Model

The other nonlinear model that Sprott proposed is one of the several that have been proposed by this author. It belongs to a restricted class of three dimensional dynamical systems, referred to as jerky dynamics. These are ordinary differential equations in one scalar real dynamical variable $x(t)$ which are of third order, explicit and autonomous. Their functional form reads $\dddot{x} = f(x, \dot{x}, \ddot{x})$ where $\dddot{x}$, is mechanically the rate of change of acceleration or the jerk function. Under certain restrictions, jerky dynamics can be interpreted as the direct extension of a one-dimensional Newtonian dynamics to spatially or temporally non-local forces, such as radiation reaction.

The nonlinear Sprott system of interest in this work is given below:

$$\dddot{x} + \dot{x} + x + f(\dot{x}) = 0$$

where $f(\dot{x}) = \text{sign}(1 + 4\dot{x})$

When we introduce the following change of variables, we get the following set of equation, suitable for simulations:

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = -y - x - f(1 + 4y)$$

A simulation of the given system yields the graph shown in Fig 3.

![Graph showing simulation results for Sprott System](image_url)

Fig3. Simulation results for Sprott System $x(t)$ vs. $y(t)$ with the initial conditions $x(0) = -0.5, y(0) = z(0) = 0$
We work on this system to try to find the possible chaotic behavior using TISEAN package. In Fig 4 it can be seen that delay time for the Sprott system is close to 5. In Fig 5. The result of False Nearest neighbors analysis is plotted and from there it can be claimed that the embedding dimension is bigger than 5.

We have calculated the largest Lyapunov Exponent for the Sprott system and find its value is 1.34933 which shows that this is system presents chaotic behavior. (Fig 6).
The Lyapunov spectrum obtained by direct use of the Wolf simulation gives the following values displayed in Table 2, thus confirming the presence of chaotic behavior.

<table>
<thead>
<tr>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4360020</td>
<td>0.4235678</td>
<td>-0.08662078</td>
</tr>
</tbody>
</table>

Table 2. Lyapunov Exponents for the given Sprott System

We also calculated the Lyapunov exponents of the Sprott system by changing the discontinuous function with both $\tanh(a*x)$ and $\frac{2}{\pi \arctan(a*x)}$ functions and we got the following results in Table 3 and Table 4.

<table>
<thead>
<tr>
<th>a</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.4190020</td>
<td>0.4075673</td>
<td>-0.07841198</td>
</tr>
<tr>
<td>8</td>
<td>0.4112314</td>
<td>0.4000936</td>
<td>-0.071223512</td>
</tr>
<tr>
<td>6</td>
<td>0.3898712</td>
<td>0.39634</td>
<td>-0.42424063</td>
</tr>
<tr>
<td>4</td>
<td>0.3877567</td>
<td>0.3745321</td>
<td>-0.21213898</td>
</tr>
<tr>
<td>2</td>
<td>0.3764682</td>
<td>0.3675292</td>
<td>-0.00162423</td>
</tr>
<tr>
<td>1</td>
<td>0.3701262</td>
<td>0.35325125</td>
<td>0.01261420</td>
</tr>
</tbody>
</table>

Table 3. Lyapunov Exponent for Modified Sprott System (by $\frac{2}{\pi \arctan(a*x)}$)
As it can be seen from Table 3 and 4, changing discontinuous function in Sprott System with \( \tanh(a*x) \) and \( \arctan(a*x) \) functions gives similar results and both choices give satisfactory results since both lead to results that are close to the Lyapunov Exponents of the original system.

<table>
<thead>
<tr>
<th>a</th>
<th>( \Lambda_1 )</th>
<th>( \Lambda_2 )</th>
<th>( \Lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.4260020</td>
<td>0.4135678</td>
<td>-0.07662078</td>
</tr>
<tr>
<td>8</td>
<td>0.4120014</td>
<td>0.4012876</td>
<td>-0.075662708</td>
</tr>
<tr>
<td>6</td>
<td>0.3998712</td>
<td>0.3987625</td>
<td>-0.45376685</td>
</tr>
<tr>
<td>4</td>
<td>0.3877567</td>
<td>0.3766513</td>
<td>-0.24567602</td>
</tr>
<tr>
<td>2</td>
<td>0.3775645</td>
<td>0.3688565</td>
<td>-0.00156670</td>
</tr>
<tr>
<td>1</td>
<td>0.3847020</td>
<td>0.3635767</td>
<td>0.01423355</td>
</tr>
</tbody>
</table>

Table 4. Lyapunov Exponent for Modified Sprott System (by \( \tanh(a*x) \))

**Conclusions**

It has been demonstrated that in a number of models on discontinuous dynamical models, difficulties have been encountered in calculating Lyapunov exponents. Sprott acknowledges these difficulties and has an approach parallel to that used in this work, while Handy and similar ecological models do not mention chaotic behavior or Lyapunov exponents, since in most cases only a stable limit cycle (or otherwise) is sufficient. Nevertheless, an attempt has been made to replace discontinuities by functions that nearly approximate it and bypass these difficulties. In the original form of the discontinuous Handy model, the Wolf algorithm gives meaningless results while replacing the discontinuities by the arc tangent function corrects this defect, while not affecting the trajectory seriously. It is clear that the system is not chaotic; this fact can also be confirmed by looking at the trajectories in their paper. The Sprott system can also be handled by replacing the discontinuities by the arc tangent function. One can adjust the steepness of the step function by changing the free parameter. The change also helps alleviate possible stiffness problems in the numerical integration of trajectories coming from discontinuous derivatives. Thus, a solution to the differentiability problem in discontinuous models has been proposed. This should aid other dynamical systems calculations with such models such as central manifolds, normal forms or bifurcation analysis which relies on the existence of derivatives.
References


On the design of proportional integral observer for a rotary drilling system

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Abstract. Torsional drill-string vibrations, also known as "stick-slip" oscillations appearing in oil-well drill-strings are a source of failures which reduce penetration rates and increase drilling operation costs. Some strategies based on drillers recommendations are evaluated in order to reduce stick-slip oscillations, the use of the angular velocity at the drill-string upper part, and the weight on the bit is shown to have a key effect in the reduction of drill-string torsional vibrations. In this paper we design a Proportional Integral Observer (PIO) to estimate the down hole speed and detect stick-slip vibrations.

Keywords: component; stick-slip oscillations, drilling system, dry friction, Proportional Input observer, Unknown Input observer

1 Introduction

The drilling technique used mostly in the oil industry, called rotary drilling, is the creation of a bore hole by mean of a tool, called a bit. This technique relies on a mechanical system for energy transport from surface to the bit and hydraulic system for material transport from the bit to surface. The mechanical part is composed of a rotating bit to generate the bore-hole, a drill-string to rotate the bit, a rotary drive at surface to rotate the drill-string. The hydraulic part consists of mud (drilling fluid), pumps, and a transport channel[1]. When drilling, drill-string is exposed to vibrations, they are classified depending on direction they appear, we distinguish three main types of them: torsional, axial, and lateral (see[1][2]). These vibrations can exist separately or can be present together. Rotary drilling systems using drag bits, which consist of fixed blades or cutters mounted on the surface of a bit body, are known to exhibit, mainly, torsional vibrations, which may lead to torsional stick-slip, characterized by sticking phases with the bit stopping completely and slipping phases with the angular velocity of the tool increasing up to times of the angular velocity at the surface. These stick-slip oscillations decrease drilling efficiency, accelerate the wear of drag bits and may cause drill-string failure because of fatigue. In this paper we focus on modeling the behavior of the drill-string torsional vibration and designing an observer for the Bottom Hole Assembly (BHA) dynamic in...
order to detect stick-slip vibrations. Most of the models describing stick-slip motion in drill-string consider this last as a torsional pendulum with different degrees of freedom [3][4]. A simple and reliable model can be obtained with two degrees of freedom as in [3][5][6][7]. An unknown input proportional integral observer is designed in order to estimate the unmeasured states and the unknown input.

2 Torsional model of a drill-string

A simple model of torsional drill-string vibrations is obtained by assuming that the drill-string behaves as a torsional pendulum, i.e. the drill pipes are represented as torsional spring, the drill collars behave as a rigid body and the top drive rotates at constant speed, (see Figure 2). In this study, It is supposed that no lateral or axial motions of the bit are present, and the only part of the drill-string interacting with the bore hole is the bit. This interaction is usually modeled by frictional forces (dry friction model) [3][1][7]. In this paper, we consider the interaction between the bit (drag bit) and the rock as a combination of two processes: cutting of the rock and frictional contact[8][9][5]. The corresponding equation of motion are written.

A. The Bottom-hole-assembly (BHA)

The bottom-hole-assembly dynamic is governed by the following equation:

\[ J_b \ddot{\phi} = k (\phi_t - \phi_b) - C_b \dot{\phi}_b - Tob (\dot{\phi}_b) \]  

Let us set:

\[ \phi = \phi_t - \phi_b , \text{ And, } \dot{\phi}_b = \Omega_b \]

Then:

\[ J_b \dot{\Omega}_b = k \phi - C_b \Omega_b - Tob (\Omega_b) \]  

Where: (\phi_b) (\Omega_b) (J_b) (C_b) are respectively, the angular displacement, angular velocity, equivalent of mass moment of inertia, equivalent viscous damping coefficient at the bottom of the drill-string, and (\phi) is the angular displacement at the top of the drill-string, (k) is the torsional stiffness coefficient, and Tob is a nonlinear function which will be referred to be the torque-on-bit.

B. The Drive system

The mechanical behavior of the drill sting at the surface is dominated by two components: a gearbox with combined gear ratio of (n:1), and an electric motor (here considered as a separately excited DC motor). The equations of this system are given as follow.

- Mechanical Equation:

\[ J_t \dot{\Omega}_t = T - C_t \Omega_t - k \phi \]

300
Where \( \Omega_t \), \( J_t \), and \( C_t \) are respectively, the angular velocity, equivalent of mass moment of inertia, equivalent viscous damping coefficient at the top of the drill-string. \( T \) is the torque delivered by the motor to the system multiplied by the gearbox ratio \( n \).

\[
T = n T_m
\]

Figure 1. Mechanical model describing the torsional behavior of a generic drill-string.

- Electrical Equation:

\[
v = l \frac{di}{dt} + r i + v_{cem}
\]

(4)

Where \( l \), \( r \), \( i \) and \( v \) are respectively defined as motor current, motor resistance, motor inductance and motor input voltage, \( v_{cem} \) is the counter-electromotive force (back-emf), \( K \) is the motor constant multiplied by the gearbox ratio such as \( K = n K_m \).

The counter-electromotive force, and the motor torque, are linearly related to the motor speed and the motor current, respectively.

\[
v_{cem} = K \Omega_t, \quad \text{And,} \quad T = K i
\]

Finally

\[
J_t \dot{\Omega}_t = K i - C_t \Omega_t - k \phi
\]

(5)

\[
l \frac{di}{dt} = v - r i - K \Omega_t
\]

(6)
C. Bit-Rock interaction (Torque-On-Bit)

As we say before, the Torque-On-Bit is the result of, not only a frictional contact, but also a cutting process [10].

\[ \text{T}_{ob} = \text{T}_c + \text{T}_f \]  \hspace{1cm} (7)

According to [10], the cutting torque resulting from the cutting process is given by

\[ \text{T}_c = \frac{1}{2} R_{bit}^2 \varepsilon d \]  \hspace{1cm} (8)

Where \((\varepsilon)\) is the intrinsic specific energy (the amount of energy required to cut a unit volume of rock), \((d)\) the depth of cut, and \((R_{bit})\) is the bit radius. Unlike [8] [5] where the frictional contact is modeled as static continuous model, here the frictional contact is modeled as a dynamic discontinuous dry friction contact. One of the difficulties with modeling friction is the complexity of the phenomenon at low velocity and, in particular, of the stick-slip process. In the slip phase, the macroscopic relative motion is null and friction appears as constraint maintaining the zero-velocity condition between the rubbing surfaces. One extensively-used model is the classical Coulomb model shown in Figure 3, this model exhibits numerical problems in the vicinity of zero-velocity, therefore KARNOOP [11] introduce a zero velocity band in his model where a condition for switching from the stick to the slip motion is established, outside this band a standard Coulomb model is used. The Karnopp model is given by:

\[ T_f = \begin{cases} \text{T}_c \text{ si } |\Omega_b| < \text{T}_s \\
\text{T}_c \text{ sign}(\dot{\phi}_b) \text{ si } |\Omega_b| \geq \text{T}_s \\
\text{T}_s \text{ sign}(\dot{\phi}_b) \text{ si } |\Omega_b| \geq \text{T}_s 
\end{cases} \]  \hspace{1cm} (9)

Where \((\text{T}_c)\) is the applied external torque that must overcomes the static friction torques \((\text{T}_s)\), \((\text{T}_d)\) is the dynamic friction torque, and \((\text{T}_b)\) is the zero-velocity band. With:

\[ T_{s(d)} = \frac{1}{2} Wob g \mu_{s(d)} R_{bit} \]  \hspace{1cm} (10)

Where \((\mu_{s(d)})\) is the static (dynamic) dry frictional coefficient, \((Wob)\) is the Weight-On-Bit. Finally, the torsional model of the whole system is writing as:

\[
\begin{align*}
\dot{\phi} &= \Omega_i - \Omega_b \\
\dot{\Omega}_i &= -\frac{k}{J_i} \phi - \frac{C_i}{J_i} \Omega_b + \frac{nK}{J_i} i \\
\dot{\Omega}_b &= -\frac{k}{J_b} \phi - \frac{C_b}{J_b} \Omega_b - \frac{1}{J_b} \text{T}_{ob}(\Omega_b, Wob) \\
\frac{di}{dt} &= -\frac{nK}{I} \Omega_i - \frac{1}{I} v + \frac{1}{I} \phi
\end{align*}
\]  \hspace{1cm} (11)
Figure 2. Models for the friction between the bit and the rock given by expression (9)

The overall system architecture was implemented as shown in Figure 4

Figure 3. Block diagram of a torsional model of the drill-string

4 Observer design of the rotary drilling system

The proportional integral observer (PIO) is an observer in which an additional term, which is proportional to the integral of the output estimation error is added in order to achieve some desired robustness performance. Here we design a Proportional Integral Observer (PIO) for unknown input (Tob) which estimates both the state and the bounded nonlinear unknown input as shown in figure (5), (See [12][13]).

Figure 4. Block diagram of a PI Observer for unknown input.
A continuous-time linear state space model of the drill-string torsional behavior can be derived from (11) as:

\[
\begin{align*}
\dot{X} &= AX + Bu + Ed \\
Y &= CX
\end{align*}
\] (12)

Where \( X = (\Phi \Omega \Omega i \Omega j) \) is the state vector, \( Y = (\Omega i \Omega j) \) is the output vector, \((u = v)\) is the known input (voltage), \((d = Tob)\) is an unknown input (Torque-On-Bit). \( A, B, C, \) and \( E \) are known matrices with appropriate dimensions.

\[
A = \begin{pmatrix}
0 & 1 & -1 & 0 \\
-k & -C & 0 & K \\
J & 0 & J & 0 \\
0 & J & 0 & -r \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}
\]

\[
E = \begin{pmatrix}
0 \\
0 \\
-1 \\
J \\
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

The PI observer (14) is synthesized on the basis of the following augmented system

\[
\begin{align*}
\dot{Z} &= A_z Z + B_z u \\
\dot{\hat{Y}} &= C_z \hat{X}
\end{align*}
\] (13)

Where:

\[
Z = \begin{pmatrix} X \\ d \end{pmatrix}, \quad A_z = \begin{pmatrix} A & E \\ 0 & 0 \end{pmatrix}, \quad B_z = \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad C_z = \begin{pmatrix} C & 0 \end{pmatrix}
\]

Since the pair \((A_z, C_z)\) is observable, a PI observer for the system (12) can be defined as follows:

\[
\begin{align*}
\dot{\hat{X}} &= A \hat{X} + Bu + E \hat{d} + K_p (Y - C \hat{X}) \\
\dot{\hat{d}} &= K_i (Y - C \hat{X})
\end{align*}
\] (14)

Where \( (K_p) \) and \( (K_i) \) are proportional and integral gains respectively, (14) is written in the following augmented form:

\[
\begin{align*}
\dot{\hat{Z}} &= A_z \hat{Z} + B_z u + K_z (Y - C_z \hat{Z}) \\
\dot{\hat{Y}} &= C_z \hat{\hat{X}}
\end{align*}
\] (15)

Where: \( \hat{Z} = [\hat{X} \ \hat{d}]' \); \( K_z = [K_p \ K_i]' \)

The state estimation error of the augmented system is

\[
e = Z - \hat{Z}
\]

So:

\[
\dot{e} = Z - \dot{\hat{Z}} = (A_z - K_z C_z) e
\] (16)
Estimation errors converge asymptotically to zero if the matrix \((A_o - K C)\) is Hurwitz. Since the pair \((A_z, C_z)\) is observable, the gain \((K_z)\) can be calculated by pole placement.

5 Simulation results

Simulations have been performed to investigate the efficiency level of the proposed model and observer for different scenarios of wight-on-bit and motor voltage, these simulations are performed with Lab-View. The numerical values used in these simulations are listed in table 1 and table 2 (see Appendix).

**TABLE 1. NUMERICAL VALUES OF THE DRILLING SYSTEM**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(J_t)</td>
<td>Equivalent of mass moment of inertia at the top of the drill-string</td>
<td>999.35</td>
<td>kgm²</td>
</tr>
<tr>
<td>(J_b)</td>
<td>Equivalent of mass moment of inertia at the bottom of the drill-string</td>
<td>127.27</td>
<td>kgm²</td>
</tr>
<tr>
<td>(C_t)</td>
<td>Equivalent viscous damping coefficient at the top of the drill-string</td>
<td>51.38</td>
<td>Nms/rad</td>
</tr>
<tr>
<td>(C_b)</td>
<td>Equivalent viscous damping coefficient at the bottom of the drill-string</td>
<td>39.79</td>
<td>Nms/rad</td>
</tr>
<tr>
<td>(k)</td>
<td>Drill-string stiffness</td>
<td>481.29</td>
<td>Nm/rad</td>
</tr>
<tr>
<td>(r)</td>
<td>Motor resistance</td>
<td>0.01</td>
<td>Ω</td>
</tr>
<tr>
<td>(l)</td>
<td>Motor inductance</td>
<td>0.005</td>
<td>H</td>
</tr>
<tr>
<td>(K_m)</td>
<td>Motor constant</td>
<td>6</td>
<td>Nm/A</td>
</tr>
<tr>
<td>(n)</td>
<td>Gearbox ratio</td>
<td>7.20</td>
<td>-</td>
</tr>
</tbody>
</table>

**TABLE 2. NUMERICAL VALUES OF THE BIT-ROCK INTERACTION**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e)</td>
<td>Intrinsic specific energy of the rock</td>
<td>130</td>
<td>MJ/m³</td>
</tr>
<tr>
<td>(d)</td>
<td>Depth of cut</td>
<td>4</td>
<td>mm/rev</td>
</tr>
<tr>
<td>(\mu_s)</td>
<td>Static dry frictional coefficient</td>
<td>0.6</td>
<td>-</td>
</tr>
<tr>
<td>(\mu_d)</td>
<td>Dynamic dry frictional coefficient</td>
<td>0.4</td>
<td>-</td>
</tr>
<tr>
<td>(R_{dp})</td>
<td>Bit radius</td>
<td>0.10</td>
<td>m</td>
</tr>
</tbody>
</table>

First, to approve the model (11), we test it on the basis of drillers observations and recommendations. In practice, to avoid stick-slip vibrations on the field, the driller operator typically controls the surface-controlled drilling parameters,
such as the weight on the bit, the speed at the surface and the viscosity of the drilling fluid to limit or eliminate stick-slip vibrations.

- Manipulation of the weight on the bit and the speed at the surface

From field data experience, it is concluded that the reduction of the weight on the bit and/or the augmentation of the speed at the surface can limit the severity of stick-slip vibrations, we can see these facts by simulating model (11) as shown in figures (6,7).

- Manipulation of the viscosity of the drilling fluid

Another strategy for reducing stick-slip oscillations at the BHA is by increasing the damping at the down end of the drill-string. This can be done by modifying the drilling fluid characteristics, we can also see that by simulating model (11) with increasing equivalent damping coefficients $C_b$ and $C_t$ as shown in figure(8).

As the above simulations reflect experimental results, the model (11) is validated.

Second, to estimate the speed of BHA and the torque on bit, eigenvalues of the observer are located at (-10, -50, -70, -100, -130), therefore
the gain is calculated by “poleplace” commande of LabView MathScript node. Figures bellow (10,11) show the performance of the PIO (19) for the wight on bit and motor voltage inputs shown in figure (9).

Figure 7. Angular velocity of the BHA ($\Omega_b$) and the speed at the surface ($\Omega$) under (Wob=15T) and (v=125V). Top: with (Ct=51.38 Nms/rad, Cb=39.79 Nms/rad), Bottom: with (Ct=63.54 Nms/rad, Cb= 56.85 Nms/rad)

Figure 8. weight-on-bit and Motor voltage

Figure 9. BHA speed and BHA speed estimation (on the left), estimation error of the speed at the surface (on the right)
Because of the weight-on-bit is usually perturbed due to axial vibrations, PIO (19) is tested under a perturbed Wob which causes a perturbation on torque-on-bit (unknown input) as shown in figure (12). To enhance our results, eigenvalues of the observer are placed at (-10 -100 -110 -120 -130).

Figure 10. Torque-on-bit and Torque-on-bit estimation (on the left). Estimation error of Tob (on the right)

Figure 11. weight-on-bit

Figure 12. BHA speed and BHA speed estimation (on the left). Estimation error of the speed at the surface, under Wob shown in figure 12 and motor voltage =125V (on the right)

Figure 13. Torque-on-bit and Torque-on-bit estimation (on the left). Estimation error of Tob (on the right)
6 Conclusion

This paper has studied stick-slip vibrations in oil-well drill-string. A models for describing the torsional drill-string behavior have been given. We presented also a model for the rock-bit interaction, the modeling of exact friction characteristic is not an easy problem, because the friction characteristic can be changed easily due to the environment’s changes. A PI observer has been developed to estimate the speed of the BHA, in which the unknown disturbance (Torque-on-bit) has been also estimated, this estimation can be used to compensate the torque-on-bit in vibration suppression of a drill-string.

Appendix

Numerical values listed in table 1 are calculated by the following equations:

- Mass moment of inertia

  \[ J_t = J_m + \frac{1}{3} J_p \quad \text{and} \quad J_b = J_c + \frac{1}{3} J_p \]

  \((J_m, J_c, J_p) : \) mass moment of inertia of the motor, the drill-collar, the drill-pipe respectively. \((J_{b(t)}) : \) equivalent mass moment of inertia at the top of the drill-string (at the bottom of the drill-string).

  Where:

  \[ J_p(L_p) = \frac{1}{2} \rho \pi (R_p^4 - r_p^4) L_p \quad \text{and} \quad J_c(L_c) = \frac{1}{2} \rho \pi (R_c^4 - r_c^4) L_c \]

  \(R_p(c)\) is the outer radius of the drill-pipe (drill-collar), \(r_p(c)\) is the inner radius of the drill-pipe (drill-collar), \(L_p(c)\) is the length of the drill-pipe (drill-collar), \(\rho\) is the density of steel.

- The torsional stiffness coefficient

  \[ k(L_p) = \frac{G I}{L_p} \]

  Where:

  \[ I = \frac{\pi}{2} (R_p^4 - r_p^4) \]

  \((k) : \) the torsional stiffness coefficient, \((G) : \) shear modulus of steel, \((I) : \) is the polar moment of inertia.

- The viscous damping coefficients

  \[ C_t = C_m + \frac{1}{2} C_p \quad \text{and} \quad C_b = C_c + \frac{1}{2} C_p \]

  \(C_{p(b)} \) viscous damping coefficient of the drill-pipe (drill-collar), \((C_m) \) viscous damping coefficient of the motor.
Where:

\[ C_{p(c)} = 120 \eta \frac{R_h^2 R_{p(c)}^2}{R_h^2 - R_{p(c)}^2} L_{p(c)} \]

\((R_h)\) is the radius of the borehole, \((\eta)\) is the dynamic viscosity

References


Experimenting Chaos with Chaotic Training Boards

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Abstract. Chaos Training Boards, consist of four sets, have been designed and implemented in a systematic way for experimenting chaos and chaotic dynamics. After a long design period and primitive prototypes, we completed final versions of the training boards. In this study, it is aimed to introduce these training boards, their design methodology and experimental studies with the boards. Thanks to these boards, the mainly chaotic oscillator systems including Chua’s oscillators (Chua’s, MLC and Mixed-Mode Chaotic circuits), Lorenz-Family Systems, Chaotic Systems based on Jerk Equations, Rössler oscillator, Chaotic Wien Bridge oscillator, Chaotic Colpitts oscillator, RLD oscillator and Transistor based nonautonomous chaotic oscillator can be investigated experimentally, and these laboratory tools provide new educational insights for practicing chaotic dynamics in a systematic way in science and engineering programs.

Keywords: Chaos, Chaotic Circuits and Systems, Training Boards, Laboratory Tools.

1 Introduction

In addition to the theoretical and practical studies on chaotic circuits and systems, the laboratory tools designed in a systematic way are required for studying and introducing chaotic circuits and systems in graduate and undergraduate research and education programs. To meet this requirement, we had designed and implemented laboratory tools, namely “Chaos Training Boards” and we had introduced these laboratory apparatus in different scientific platforms [1, 2].

Chaos Training Board-I, shown in Fig.1a, consists of eight pre-constructed circuit blocks. The core of training board-I is Mixed-Mode Chaotic Circuit (MMCC) [3] which offers an excellent educational circuit model for studying and practical experimenting chaos and chaotic dynamics. MMCC operates either in the chaotic regime determined by autonomous circuit part, or in the chaotic regime determined by nonautonomous circuit part depending on static switching, and it is capable to operate in mixed-mode which includes both autonomous and nonautonomous regimes depending on dynamic switching. As a result of being very flexible and versatile design methodology of training board, the user can configure the main chaotic circuit (MMCC) with alternative nonlinear resistor structures and inductorless forms [4].
Chaos Training Board-II is shown in Fig. 1b. It presents a collection of five chaotic circuits including chaotic Colpitts oscillator, RLD circuit, Rossler circuit, Wien-bridge based circuit and transistor based nonautonomous circuit. These circuits are selected for illustrating a variety of ways in which chaos can arise in simple analog oscillator structures containing active elements, specifically BJT and Op-amp [5, 6].

Chaos Training Board-III, shown in Fig.1c, is based on a common chaotic system using Jerk equations. It can be configured with three optional nonlinear function blocks on the board. Periodic and chaotic dynamics can be easily observed via this training board. And also, this dynamical system has a wide operating frequency and its frequency changes can be configured via Chaos Training Board-III [7, 8].

Chaos Training Board-IV in Fig.1d consists of a comprehensive combination of three chaotic systems based on Lorenz family [9-11]. In the literature, there are several chaotic systems similar to Lorenz system, but their chaotic dynamics are different. Thus, they are referred as Lorenz system family. Chaotic Training Board-IV has been designed by utilizing these similarities.

Fig.1. Chaos Training Boards; (a)Board-I, (b)Board-II, (c) Board-III, (d)Board-IV.
In this study, it is aimed to introduce these Chaotic Training Boards with sample experiments. In section 2, the structures of the boards and design methodologies are given. Experimental studies with the proposed boards are summarized in section 3. Concluding remarks are discussed in the last section.

2 Chaos Training Boards: Design Methodology and the Hardware Structures

In this section, design methodology and the hardware structures of Chaos Training Boards are presented in detail.

A. Chaos Training Board-I

Chaos training board-I in Fig.1a consists of eight pre-constructed circuit blocks. The numbered blocks on training board are labeled as follow:

1. Mixed-Mode Chaotic Circuit (MMCC) Part
2. Switching Signal Unit-A Square wave generator
3. Switching &Control Unit
4. Wien-Bridge Oscillator
5. Current Feedback Operational Amplifier (CFOA)-Based Floating Inductance Simulator
6. Current Feedback Operational Amplifier (CFOA)-Based Grounded Inductance Simulator
7. Voltage mode Operational Amplifier (VOA)-Based Nonlinear Resistor
8. CFOA-Based Nonlinear Resistor

The core of training board-I is Mixed-Mode Chaotic Circuit (MMCC) [3] block shown in Fig.2a. MMCC Circuit operates either in the chaotic regime determined by autonomous circuit part, or in the chaotic regime determined by nonautonomous circuit part depending on static switching, and it is capable to operate in mixed-mode which includes both autonomous and nonautonomous regimes depending on dynamic switching. The autonomous chaotic oscillator is Chua’s Circuit, and it is defined by Eq.1 [12]. The nonautonomous chaotic oscillator is MLC Circuit and it is defined by Eq.2 [13]. Because of having these

![Fig. 2. a) MMCC circuit, b) i-v characteristic of nonlinear resistor in MMCC circuit.](attachment:fig2.png)
versatile features, MMCC circuit offers an excellent educational circuit model for studying and practical experimenting chaos and chaotic dynamics.

\[
C_1 \frac{dv_{C1}}{dt} = \frac{(v_{C2} - v_{C1})}{R} - f(v_{C1})
\]

\[
C_2 \frac{dv_{C2}}{dt} = \frac{(v_{C1} - v_{C2})}{R} + i_L
\]

\[
L \frac{di_L}{dt} = -v_{C2} - i_L R_s
\]

\[
C \frac{dv_z}{dt} = i_L - f(v_R)
\]

\[
L \frac{di_L}{dt} = -R_i i_L - i_L R_s - v_R + A \sin(\omega t)
\]

As a result of being very flexible and versatile design methodology of training board, the user can configure the main chaotic circuit block in two forms: First, the user can configure the circuit as conventional way by placing discrete inductor elements to related sockets in the training board. And as an alternative way, the user can configure the chaotic circuit in inductorless form by using CFOA-based grounded and floating inductance simulators and Wien-Bridge Oscillator block located to the left side of the board. In this inductorless configuration, the user can also use Wien-Bridge oscillator block instead of LC resonator part on training board.

B. Chaos Training Board-II

Chaos Training Board-II consists of five pre-constructed chaotic circuit blocks. The numbered blocks on training board are labeled as follow: Chaotic Rössler Circuit, Chaotic Wien Bridge Oscillator, Chaotic RLD Circuit, Transistor-based Nonautonomous Chaotic oscillator, Chaotic Colpitts Oscillator.

Chaotic Rössler Circuit is introduced by Otto Rössler, in 1976 [14]. This system is defined by Eq.3. Different electronic circuit implementations of the Rössler system are available in the literature [15, 16]. Circuit configuration used in Chaos Training Board-II of Rössler system is shown in Fig.3a.

\[
\begin{align*}
\dot{x} &= -\alpha(\Gamma x + \beta y + \lambda z) \\
\dot{y} &= -\alpha(-x - y + 0.002z) \\
\dot{z} &= -\alpha(-g(x) + z)
\end{align*}
\]

\[
g(x) = \begin{cases} 0, & x \leq 3 \\ \mu(x - 3), & x > 3 \end{cases}
\]
Fig. 3. Chaotic circuits in Chaos Training Board-II; a) Chaotic Rössler Circuit, b) Chaotic Wien Bridge Oscillator, c) Chaotic RLD Circuit, d) Transistor-based Nonautonomous Chaotic oscillator, e) Chaotic Colpitts Oscillator.

The second example on the board is Chaotic Wien-Bridge oscillator [17] shown in Fig.3b. This chaos generator is formed by two circuit blocks using VOAs as active elements as shown in Fig. 3b. The first block is a Wien bridge oscillator with the gain $K_1 = 1 + \frac{R_3}{R_4}$ and the second block plays the role of a Negative Impedance Converter (NIC). In this design, by including a diode in the positive feedback loop of A2, the NIC is only activated that case $\frac{V_{C2}(K_2 - 1)}{V_D}$ where $K_2 = \left(1 + \frac{R_5}{R_6}\right)$ is the gain of A2 and $V_D$ is forward voltage drop of the diode.
The third example on the board is a nonautonomous chaotic circuit referred to as RLD chaotic circuit [18]. As shown in Fig. 3c, RLD circuit consists of an AC voltage source, a linear resistor, a linear inductor and a diode which provides the nonlinearity in the circuit. One can observe in this circuit that the current can be chaotic while the input AC-voltage is a linear oscillator.

The fourth example on the training board-II, shown in Fig. 3d, is the transistor-based nonautonomous chaotic circuit [19] coupling of an oscillator and a transistor. This is a very useful example showing that a coupling mechanism, which consists of two electronic circuits not in harmony, can exhibit chaotic dynamics. The mechanism behind this circuit is based on a capacitor charging process by means of “forward-reverse fighting” of the transistor.

The last chaotic circuit example on the training board-II, shown in Fig. 3e, is the chaotic Colpitts oscillator [5]. Undergraduate students are familiar with this oscillator circuit and this circuit is introduced as a sinusoidal oscillator circuit in analog electronic courses, but yet it can be driven to chaos. In a classical Colpitts oscillator circuit, transistor plays the role of both the active amplifying device and nonlinear element. So, the sinusoidal oscillator circuit in Fig. 3e is able to exhibit nonlinear chaotic dynamics according to the certain parameter setting in the circuit structure.

C. Chaos Training Board-III

Chaos Training Board-III is based on the “Jerk equations”. Jerk equation is defined by a third-order ordinary differential equation as follow:

\[ \dddot{v} + A \dddot{v} + \dot{v} = F(v) \]  \hspace{1cm} (4)

where A is a constant parameter and F(v) is a nonlinear function. This nonlinear function affects the behavior of the system and it has different mathematical descriptions as in Table 1 [7].

Table 1. Nonlinear functions of Jerk Equation.

<table>
<thead>
<tr>
<th>Nonlinear Functions F(v)</th>
<th>Parameters of F(v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(v) = -Bv + C \text{sgn}(v) )</td>
<td>( B=1.2, C=2 )</td>
</tr>
<tr>
<td>( f(v) = -B \text{max}(v,0) + C )</td>
<td>( B=6, C=0.5 )</td>
</tr>
<tr>
<td>( f(v) =</td>
<td>v</td>
</tr>
<tr>
<td>( f(v) = B(v^2/C + C) )</td>
<td>( B=0.58, C=1 )</td>
</tr>
<tr>
<td>( f(v) = Bv^2/C + C )</td>
<td>( B=1.6, C=5 )</td>
</tr>
<tr>
<td>( f(v) = -Bv^2/C - C )</td>
<td>( B=0.9, C=0 )</td>
</tr>
<tr>
<td>( f(v) = -B[v - 2 \tanh(Cv)/C] )</td>
<td>( B=2.15, C=1 )</td>
</tr>
</tbody>
</table>
These nonlinear functions have been implemented by using discrete circuit elements. A main circuit block was constructed for Jerk equation and then each of the nonlinear function circuits was coupled with this main circuit block respectively. After the studies of Sprott [7], it has been improved these chaotic circuits. For example, FTFN (Four Terminal Floating Nullor)-based circuit topologies are constructed. By using different capacitor values in FTFN-based circuit, the high frequency performance of Sprott’s chaotic circuits have been verified experimentally in [8]. Chaos Training Board-III is built on Jerk equation; it consists of five basic blocks about Sprott’s circuit as seen in Fig.1c. These blocks are listed as follow: Main Chaotic Circuit Block, Nonlinear Circuit Blocks, DC characteristic of nonlinear circuit blocks, Capacitor sets and Training area.

The main circuit block defined by Eq.4 is common to all Sprott’s circuits. This block is positioned in the middle of the board as seen in Fig.1c. On the board, there are three different nonlinear circuit blocks on the left side of the main circuit. While the main circuit block is presented in Fig.4a, the circuit schemes of three nonlinear functions in Table 1, namely nonlinear function blocks, are illustrated in Fig.4b.

To observe the dc characteristic of the nonlinear function, a simple block has been added to the bottom of the main circuit. Three capacitor sets are located the

![Figure 4](image-url)
around the main circuit block as seen in Fig. 1c to obtain chaotic circuits having different frequency ranges. On the right side of the board, a training area is located for configuring different nonlinear circuit blocks by the user.

**D. Chaos Training Board-IV**

Circuit implementations of three chaotic systems, which belong to the General Lorenz System Family, are objected in the Chaos Training Board-IV as shown in Fig. 1d. The General Lorenz System Family is defined by following equations [11]:

\[
\begin{align*}
\dot{x} &= a_x x + a_y y + a_{13} x z + a_{23} y z \\
\dot{y} &= b_1 x + b_y y + b_{13} x z + b_{23} y z + d_2 \\
\dot{z} &= c_1 z + c_{11} x y + c_{11} x^2 z + c_{22} y^2 + c_{33} z^2 + d_1
\end{align*}
\]

After some simplifications and parameter adjustments, this general system is called with different systems. Table 2 summarize these systems and their parameter values. Three different systems in Table 2 can be implemented by using Chaos Training Board-IV. First of them is Lorenz System, because it is

<table>
<thead>
<tr>
<th>General Lorenz System Family</th>
<th>Equations</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorenz System</td>
<td>$\dot{x} = a(y - x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{y} = cx - xz - y$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{z} = xy - bz$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a=10$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b=8/3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c=28$</td>
<td></td>
</tr>
<tr>
<td>Chen System</td>
<td>$\dot{x} = a(y - x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{y} = (c - a)x - xz - cy$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{z} = xy - bz$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a=35$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b=3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c=28$</td>
<td></td>
</tr>
<tr>
<td>Lü System</td>
<td>$\dot{x} = a(y - x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{y} = -xz + cy$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{z} = xy - bz$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a=36$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b=3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c=20$</td>
<td></td>
</tr>
<tr>
<td>Lorenz-like system</td>
<td>$\dot{x} = -(ab) / ((a + b)) x - yz + c$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{y} = ay + xz$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{z} = bz + xy$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a=10$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b=4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c=8$</td>
<td></td>
</tr>
<tr>
<td>Modified Lorenz System</td>
<td>$\dot{x} = y - x$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{y} = ay - xz$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{z} = xy - b$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a=0.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b=0.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c=0$</td>
<td></td>
</tr>
</tbody>
</table>
the basic structure of General Lorenz System Family. To have a simple mathematical definition, Modified Lorenz System is selected as second circuit. On the other hand, Lorenz - Like system has the most multiplication operations, namely it has the most complex definition and last circuit is Lorenz-Like system on this training board.

Although chaotic systems in Table 2 have different behaviors, their mathematical equations are very similar to each other. Chaos Training Board-IV is designed by utilizing these similarities and it has some common blocks for mentioned three general Lorenz-based systems. These system blocks consist of Inverting amplifier block, Differential amplifier block, Multipliers, Adder blocks, Integrator blocks and System identification blocks as seen in Fig.5.

The operations of the blocks on the training board-IV can be summarized as follow:

- To implement \((y - x)\) operations, a differential amplifier,
- To implement \(-x\) or \(-y\) operations, an inverting amplifier,
- To implement \(xz\), \(xy\), \(yz\) or their negative, three multiplier circuits,
- To implement addition or subtraction operations, several resistors in various values,
- To implement \(\dot{x}\), \(\dot{y}\) or \(\dot{z}\) operations, three integrator blocks,

have been used on the Chaos Training Board-IV. By using system identification blocks about chosen three systems, the connections between operation blocks can be made easily. Finally, three different systems in Table 2 can be implemented with this board by using essential blocks.

3 Experimental Studies with the Training Boards

Chaos Training Boards are introduced in previous sections. Some of the observed experimental results by using these training boards are presented in the following section.
A. Experimental Results of Chaos Training Board-I

By applying the jumper adjustments, the users can easily realize laboratory experiments on relevant chaotic circuits on Chaos Training Board-I. After the autonomous Chua’s circuit \[12\] configuration is obtained, the user can investigate the autonomous chaotic dynamics. The experimental observations for autonomous mode of the board have been illustrated in Fig.6a. These results illustrate chaotic time series of voltage across of C1 and C2 in MMCC circuit and chaotic attractor measured in VC1-VC2 plane.

The user can investigate chaotic dynamics of MLC circuit by making the necessary arrangements on the training board-I. VAC sinusoidal signal required for this nonautonomous system is taken from sine-wave output of an external function generator. Its amplitude and frequency are determined as Vrms=100mV and f=8890Hz, respectively. By adjusting amplitude of the AC signal source and/or R1 potentiometer located in nonautonomous part of MMCC circuit, the user can easily observe the complex dynamics of bifurcation and chaos phenomenon. Some experimental observations for this nonautonomous mode of the board have been illustrated in Fig.6b.

Mixed-mode chaotic phenomenon which includes both autonomous and nonautonomous chaotic dynamics can be observed via training board-I. In this mode switching time which determines the durations of autonomous and nonautonomous chaotic oscillations can be easily adjusted via R3 potentiometer located in square-wave generator block on the board. By adjustments of R1 and R2 potentiometers in MMCC circuit, a variety of mixed-mode chaotic dynamics are observed. Some experimental observations for this mixed-mode of the board have been illustrated in Fig.6c.

![Fig 6. Experimental measurements of MMCC circuit on the board-I; a) with the autonomous mode configuration, b) with the nonautonomous mode configuration, c) with the mixed-mode configuration.](image-url)
B. Experimental Results of Chaos Training Board-II

Since all chaotic circuit models on the training board-II are mounted to board as pre-constructed circuit blocks, laboratory experiments related to these circuits are easily made by using virtual measurement system. Fig. 7a shows measurements on chaotic Rössler circuit. This figure includes time domain illustration and chaotic attractor for \( V_{C2} \) and \( V_{C3} \).

In Wien bridge-based chaotic circuit model, there are two oscillation modes for investigations. By configuring J1 jumper adjustment on the board-II, a classical linear Wien-bridge oscillator configuration is obtained. By configuring J2 jumper adjustment on the board-II, a nonlinear circuit block is coupled to classical linear Wien-bridge oscillator configuration and this nonlinear oscillator oscillates chaotically. Fig.7b shows measurement results illustrating the chaotic phenomena.

A VAC sinusoidal signal source must be used to observe the results of RLD circuit. Its amplitude and frequency are determined as \( V_p=2V \) and \( f=100kHz \), respectively. The recorded experimental measurements on the board-II for chaotic RLD circuit are given in Fig.7c.

By varying the amplitude of the oscillator, some interesting complex dynamic series from periodic behaviors to chaotic behavior can be observed via transistor-based chaotic circuit model. Fig.7d shows measurement results illustrating its chaotic phenomena.

![Fig 7. Experimental measurements of circuits on the board-II.](image-url)
Fig. 7e shows measurements on Colpitts oscillator. These results illustrate the time series of voltage across of C1 and C2 in Colpitts oscillator circuit, and chaotic attractor measured in VC1-VC2 plane. The parameters of Colpitts oscillator have been determined that the circuit oscillates in chaos mode.

C. Experimental Results of Chaos Training Board-III

In these experiments, nonlinear circuit blocks in Fig.4b are connected to the main circuit block respectively. While the phase portrait illustrations of Sprott’s circuits based these systems are seen in Fig. 8a, dc characteristics of nonlinear circuit blocks are presented in Fig. 8b.

As mentioned in previous section, there are three capacitor set blocks on Chaos Training Boards-III and these blocks provide chaotic signals in variable frequencies. For third nonlinear circuit block in Fig. 4b, time domain illustrations in different frequency are given in Fig. 9. In addition to these properties, there are several potentiometers on the nonlinear circuit blocks as seen in Fig. 1c. By changing their values, the user can easily observe the complex dynamics of bifurcation and chaos phenomenon clearly.

![Fig. 8. Experimental results for the Chaos Training Board-III: (a) The phase portrait illustrations of Sprott’s chaotic circuits, (b) DC characteristics of the nonlinear circuit blocks.](image)

![Fig. 9. Time domain responses of third nonlinear circuit block in different frequency, x: 2 V/div, y: 0.2 V/div a) time: 2.5 ms/div, b) time: 250 µs/div, c) time: 25 µs/div.](image)
D. Experimental Results of Chaos Training Board-IV

In Chaos Training Boards-IV, three different chaotic systems, which are the members of general Lorenz System family, can be constructed by using common blocks via system identification blocks. The experimental results of these systems are given in Fig.10 respectively.

![Fig.10](image)

Fig.10. Experimental results of; a) Lorenz, b) Modified Lorenz, c) Lorenz-Like chaotic systems on Chaos Training Board-IV.

Conclusions

Chaos training boards have been introduced in this study with sample experimental studies. These experiments have been selected for demonstration of functionality and versality of the boards. Chaos Training Board-I is based on the MMCC and thanks to its switching mechanism, it is capable to operate autonomous, nonautonomous and mixed-mode mode chaotic dynamics. The circuits on the Chaos Training Board-II are selected for illustrating a variety of ways in which chaos can arise in simple analog oscillator structures containing active elements, specifically BJT and Op-amp. While the user has the possibility of examination for a Jerk equation based chaotic system with different nonlinear functions on Training Board-III, Lorenz family systems are easily configured and constructed in the Training Board-IV. A laboratory program arranged with the proposed chaos training boards can be easily accompanied with nonlinear courses in science and engineering programs. We hope this study will provide a practical guide for the readers and the boards presented here will be very useful laboratory apparatus for nonlinear studies in science and engineering research and education programs.

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Control of Chaotic Finance System using Artificial Neural Networks

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Abstract. In this paper, the control of a chaotic finance system is applied by using Artificial Neural Networks (ANNs). Economic systems become more complicated and have undesired nonlinear factors. It is difficult to control when a chaotic behaviour occurs. So that the ANNs have the ability of learning functional relations, they can achieve the control of chaotic systems more effectively. On-line neural training algorithms are used for regulating the chaotic finance system to its equilibrium points in the state space. For faster training in back-propagation, Levenberg-Marquardt algorithm is preferred. Numerical simulations are performed to demonstrate the effectiveness of the proposed control technique.

Keywords: Chaotic Finance System, Chaos Control, Neural Control, Artificial Neural Networks, Neural Networks.

1 Introduction

Nowadays, the financial systems are being more complicated and the markets are rising rapidly in an asymmetrical economic growth. The economic progresses are resulting in nonlinearity which leads difficult to control. Upon having some unpredictable nonlinear factors in interest rate, investment demand, price, per investment cost and stock, the financial systems can reveal chaotic behaviour. Nonlinearity and chaos in a financial system are undesired characteristics and traditional econometric approaches, strict assumptions, statistics based methods may be inadequate for a stable economic growth and control. Furthermore, the control of chaos in a financial system has significant importance from the management point of view of avoiding undesirable situations such as economical crises.
A nonlinear system under chaotic behaviour might have undesired trajectories; therefore it is required to control for eliminating chaos. Since the successful study of Ott, Grebogi, and Yorke named OGY control [1], various methods for the control and synchronization of chaotic systems have been presented. These approaches mainly include linear feedback [2], nonlinear feedback [3], time-delayed feedback [4], adaptive [5], sliding mode [6], impulsive [7], passive [8] control methods. In the most of these control methods, it is assumed that the dynamical model of the chaotic system is known. However, some of the chaotic system models do not exactly represent the real situation, the parameters may be unknown and many chaotic systems do not have any mathematical equations. Intelligent control techniques generally attempt to control the chaos by using the output values of states, so they can be more comprehensive solution. Recently, the control of Lorenz [9, 10], Rossler [11], Chen [12], Lü [13], unified [14], and unknown [15] chaotic systems have been implemented with Artificial Neural Networks (ANNs). Fuzzy logic, the other popular intelligent technique, is used in the control of Lorenz [16, 17], Chua [17, 18], Rossler [18], Chen [18, 19], unified [20], Mathieu–van der Pol [21] and some other chaotic systems. With the Adaptive Neuro-Fuzzy Inference Systems (ANFIS), which is a combination of ANN and fuzzy logic systems, there are only a few papers for the control of chaotic systems [22–25].

The first chaotic finance system has been introduced in 2001 [26, 27]. Then, a new chaotic finance attractor has been built in 2007 [28]. Afterwards, two different hyperchaotic finance systems have been presented respectively in 2009 and 2012 [29, 30]. Some papers have been published concerning the dynamic behaviours of these chaotic finance systems [30–33]. The synchronization of the chaotic finance systems have been implemented with active [34, 35], nonlinear feedback [36, 37], and adaptive [38] control methods. For the control of the chaotic finance systems, several control methods have been proposed [29, 30, 39–44]. Yang and Cai have achieved the control of chaotic finance via linear feedback, speed feedback, selection of gain matrix, and revision of gain matrix controllers in 2001 [39]. Chen has employed the time-delayed feedbacks to provide the control of this system in 2008 [40]. Emiroglu et al. have used a passive controller for controlling this nonlinear system in 2012 [41]. Cai et al. have constructed the control of modified chaotic finance system by means of linear feedback, speed feedback and adaptive control methods in 2011 [42]. The control of the former hyperchaotic finance system has been performed with the effective speed feedback control method by Ding et al. in 2009 [29], with the linear feedback control method by Uyaroglu et al. in 2012 [43], and with the time-delayed feedback control methods by Gelberi et al. in 2012 [44]. Yu et. al have realized the control of the latter hyperchaotic finance with the linear feedback and effective speed feedback control methods in 2012 [30].

Motivated by the previous intelligent chaos control papers, in this study, further investigations on the control of chaotic finance systems are explored. Although there are several studies in financial chaos control, as to the knowledge of the
authors there is no intelligent approach for controlling the chaotic finance systems. In this study, neural controllers are employed for achieving the control of a chaotic finance system. The effectiveness of the proposed ANN control technique has been presented visually by using simulation results.

2 Materials and Methods

2.1 Chaotic Finance System

The chaotic finance system is described by a set of three first-order differential equations as

\[
\begin{align*}
\dot{x} &= z + (y - a)x, \\
\dot{y} &= 1 - by - x^2, \\
\dot{z} &= -x - cz, 
\end{align*}
\]

where \(x, y, z\) are state variables and \(a, b, c\) are positive constant parameters, they represent the interest rate, investment demand, price exponent, saving amount, per investment cost, and elasticity of demands of commercials, respectively [26, 27]. It exhibits chaotic behaviour when the parameter values are chosen as \(a = 3, b = 0.1,\) and \(c = 1\) [40]. The time series, 2D phase portraits and 3D phase plane of the chaotic finance system under these parameter values and the initial conditions \(x(0) = 1.5, y(0) = 4.5,\) and \(z(0) = -0.5\) are illustrated in Fig. 1, Fig. 2, and Fig. 3, respectively.

The equilibria of chaotic finance system (1) can be found by solving the following equation:

\[
\begin{align*}
z + (y - a)x &= 0, \\
1 - by - x^2 &= 0, \\
-x - cz &= 0. 
\end{align*}
\]

Then, the chaotic finance system possesses three equilibrium points;

\[
\begin{align*}
E_1(0, 1/b, 0), \\
E_2(\sqrt{-c(b - c + abc)}/c, (1 + ac)/c, \sqrt{-c(b - c + abc)}/c^2), \\
E_3(\sqrt{-c(b - c + abc)}/c, (1 + ac)/c, -\sqrt{-c(b - c + abc)}/c^2). 
\end{align*}
\]

When the parameter values are taken as \(a = 3, b = 0.1,\) and \(c = -0.5,\) the equilibrium points become \(E_1(0, 10, 0), E_2(-0.7746, 4, 0.7746),\) and \(E_3(0.7746, 4, -0.7746).\)
Fig. 1. Time series of the chaotic finance system for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.
Fig. 2. 2D phase portraits of the chaotic finance system in (a) $x$–$y$ phase plot, (b) $x$–$z$ phase plot, and (c) $y$–$z$ phase plot.

Fig. 3. 3D phase plane of the chaotic finance system.
2.2 Artificial Neural Networks (ANNs)

ANNs, which are inspired from biological neural networks, are basically a parallel computing technique. An ANN consists of processing elements called neurons and connections between them with coefficients called weights. Each processing element makes its computation based upon a weighted sum of its inputs and an activation function is also used for determining the output value. ANNs adapt themselves to the given inputs and desired outputs with a learning algorithm, and then they respond to the unknown situations rationally. If using only input layer and output layer is not sufficient, increasing the number of layers called hidden layers can solve the learning problem. There are different kinds of ANNs, the most commonly preferred one is the three-layered Feed-Forward Neural Network (FFNN). As shown in Fig. 4, elementary FFNNs have three layers of neurons: input layer, hidden layer and output layer.

![Fig. 4. Basic architecture of feed-forward neural networks](image)

In Fig. 4, \(X(i)\) and \(Y(k)\) are the input-output data pairs, \(\beta_1\) and \(\beta_2\) are the bias values, \(w\) is the interconnection weight, and \(f\) is the activation function. \(i, j, \text{ and } k\) represent the number of inputs, neurons in the hidden layer, and outputs, respectively. The values of each neuron in the hidden layer of FFNN can be calculated by

\[
H(j) = f\left(\sum_{i=1}^{n} X(i)w(i,j) + \beta_1(j)\right),
\]

and the output layer of FFNN can be found as

\[
Y(k) = f\left(\sum_{j=1}^{l} H(j)w(j,k) + \beta_2(k)\right).
\]
Sigmoid and tangent sigmoid functions are the commonly used activation functions in FFNNs. While the sigmoid function produces only positive numbers between 0 and 1, the tangent sigmoid function produces numbers between -1 and 1. The formula of sigmoid function is given by

\[ f(x) = \frac{1}{1 + e^{-x}}, \]  

and the tangent sigmoid function can be denoted as

\[ f(x) = \frac{\sinh x}{\cosh x} = \frac{2}{e^{2x} + 1} \approx 1, \]

where \( e \) is the base of natural logarithm.

FFNNs must be trained for adapting themselves to the given inputs and desired outputs. Back-propagation can be used as a teaching method for FFNNs. It is also known as gradient descent training algorithm. But, it is often too slow. Therefore, several high-performance training algorithms such as scaled conjugate gradient; Fletcher-Reeves conjugate gradient; Powell-Beale restarts conjugate gradient; resilient back-propagation; Broyden, Fletcher, Goldfarb, Shanno quasi-Newton; one step secant and Levenberg-Marquardt algorithms are preferred in recent years.

### 3 Controlling Chaotic Finance System with ANNs

In this study, a direct ANN control technique is proposed to control the chaotic finance system. The direct adaptive control technique is widely used for controller design with intelligent methodologies. Its goal is to get a control system with backpropagating the errors. The parameters of controllers are adjusted to minimize the error between the plant’s output and desired output. If the system includes nonlinearity, then linear controllers may not produce performance satisfactorily. In such cases, artificial intelligence techniques such as ANNs can be used in a direct adaptive control system. Its fast response time, general approximation, and learning abilities make the ANN an attractive method for nonlinear control. Fig. 5 shows the diagram of proposed direct ANN control technique for the control of chaotic finance system.
In Fig. 5, $Y_d$ is the desired process output (an equilibrium point of the chaotic system), $Y(t+1)$ is the actual process output, $e(t)$ equals to $Y(t) - Y_d$ is the input of the ANN, and $u(t)$ is the output of the ANN. The network's output error $e(t+1)$ is defined as $Y_d - Y(t+1)$. The goal of the control is to determine the bounded input $u(t)$ as $\lim_{t \to \infty} e(t+1) \leq 0$. The ANN controllers are trained to control the chaotic system by backpropagating the errors so that the differences between desired output and actual output are minimized. This training path is shown as a dashed line in Fig. 5. For each of the $x$, $y$, and $z$ states, there exists a different ANN controller in the proposed control model. The weights of ANNs are adjusted online without a specific pre-training stage. If the error between desired output and actual output is too small, the control is achieved at that moment and no need to train the ANN controllers for this situation. Instead of using back-propagation (gradient descent) as a training method, the Levenberg–Marquardt algorithm is preferred because of learning relatively very fast. The inputs and outputs of ANNs are normalized to values between -1 and 1. Tangent sigmoid function is taken as the activation function because it produces numbers between -1 and 1. The training parameters of ANN are considered as $epochs = 3$, $goal = 10^{-10}$ and $min\_grad = 10^{-10}$. Default values are used for all the other parameters. In order to control successfully, the ANN controllers are employed to train again for the new $Y(t+1)$ situations by adjusting the weights of ANNs simultaneously.

4 Numerical Simulations

This section of the paper demonstrates the control results of chaotic finance system to verify the effectiveness of proposed ANN control technique. The simulation results are performed using the Matlab software. The numerical analyses are carried out using fourth-order Runge–Kutta method with variable time step. The same parameter values and initial conditions of finance system
described in Section 2 are taken to ensure the chaotic behaviour. When the ANN controllers are activated at $t = 50$, the simulation results for the control of chaotic finance system to $E_1(0, 10, 0)$, $E_2(-0.7746, 4, 0.7746)$, and $E_3(0.7746, 4, -0.7746)$ equilibrium points are shown in Fig. 6, Fig. 7, and Fig. 8, respectively.

Fig. 6. Time responses of controlled chaotic finance system to $E_1(0, 10, 0)$ with the ANN controllers are activated at $t = 50$ for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals
Fig. 7. Time responses of controlled chaotic finance system to $E(0.7746, 4, 0.7746)$ with the ANN controllers are activated at $t = 50$ for (a) $x$ signals, (b) $y$ signals, and (c) $z$ signals.
Fig. 8. Time responses of controlled chaotic finance system to $E(0.7746, 4, -0.7746)$ with the ANN controllers are activated at $t = 50$ for (a) $x$ signals, (b) $y$ signals, (c) $z$ signals.

As expected, the Figs. 6–8 show that the proposed ANN controllers have stabilized the chaotic motion of the finance system towards its equilibrium points. When the controllers are activated at $t = 50$, the control is observed at $t \geq 52$ for all equilibrium points. The errors between desired output and actual output signals converge to zero with an appropriate time period. Hence, the simulation results verify the effectiveness of proposed online direct ANN control technique.
5 Conclusions

Although several papers have concerned on the control of chaotic finance system, in this study, it is the first time its control is investigated with an artificial intelligence methodology. A direct adaptive ANN control technique is proposed to achieve the control. The weights of ANN controllers are adjusted online without a specific pre-training stage. Levenberg-Marquardt algorithm is preferred for faster training. The simulation results in Figs. 6–8 have shown that the chaotic finance system is stabilized towards its equilibrium points effectively owing to the ANN controllers. The proposed method differs from the previous finance chaos control techniques in that it is feasible even if the equations of the finance system is unknown. As a future work, the other intelligent techniques such as fuzzy logic, neuro-fuzzy and genetic algorithm can be applied for the control of chaotic finance system.

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Anti-Synchronization of Chaotic Systems with Adaptive Neuro-Fuzzy Inference System

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Abstract. In this study, Adaptive Neuro-Fuzzy Inference System (ANFIS), which is a combination of fuzzy inference system and artificial neural network learning algorithms, is proposed for the anti-synchronization of chaotic systems. Based on an adaptive model reference control technique, two identical chaotic systems that have different initial conditions are trained by backpropagating the anti-synchronization errors. In the simulations, well-known Lorenz chaotic system is used. Simulation results show that the proposed approach is very effective for the anti-synchronization of chaos.

Keywords: Anti-synchronization, Chaos, Adaptive Neuro-Fuzzy Inference System, Neuro-Fuzzy, ANFIS.

1 Introduction

The aim of synchronization is to use a master system’s output to induce a slave system so that the slave system’s output could follow the master system’s output asymptotically. Anti-synchronization means that the synchronized slave system's output has the same absolute values but opposite signs. Since the synchronization of chaotic systems was first proposed by Pecora and Carroll in 1990 [1], chaos synchronization has become one of the most interesting research subjects and many control techniques have been proposed for the synchronization and anti-synchronization of the chaotic systems. Active control method was used for the synchronization of chaotic Lorenz [2], Rössler [3], Chen [3], Chua [4], between Lorenz and Rössler [5], and many other identical and non-identical systems. The synchronization and anti-synchronization were applied with active control for chaotic Colpitts [6], extended Bonhöffer–van der Pol [7], and hyperchaotic Chen [8] systems. Active controllers were also constructed for chaos anti-synchronization between chaotic Lü and Rössler [9], and between hyperchaotic Lorenz and Liu [10] systems. Anti-synchronization between hyperchaotic Lorenz and Liu [10], hyperchaotic Lorenz and Chen [11], between two different hyperchaotic four-scroll [12], and a modified three-
The synchronization and anti-synchronization of chaotic systems were achieved Lorenz [17], Tigan [17], between Tigan and Lorenz [17], between Genesio and Rössler [18], hyperchaotic Chen [19], hyperchaotic Lü [19], and between hyperchaotic Chen and Lü [19] systems on the basis of nonlinear control scheme. Sliding mode control was applied to Rikitake [20], hyperchaotic Lorenz [21], hyperchaotic Lü [22], and hyperchaotic Qi [23] systems. Anti-synchronization of chaotic systems were also presented with passive control [24], $H\infty$ control [25], and backstepping design [26] techniques.

Furthermore, the synchronization and anti-synchronization of chaotic systems implemented with artificial intelligence approaches. Artificial Neural Networks (ANNs) were used for the synchronization of chaotic Lorenz [27], Rössler [27, 28], unified [29], Genesio-Tesi [30], Duffing-Holmes [31] systems. The synchronization of chaotic Lorenz [32, 33], Rössler [32], Chen [32], Duffing-Holmes [33], Chua [34], and Rikitake [35], between Chen and Lü [36], between Chen and hyperchaotic Lorenz [36] systems were applied with fuzzy logic. Anti-synchronization between hyperchaotic Wu and hyperchaotic Lorenz systems [36], chaotic Lorenz [37], and hyperchaotic Lorenz [37] were achieved owing to the fuzzy logic controllers in recent years. ANFIS was used for the chaos synchronization only in a few papers [38, 39].

According to the literature review, the anti-synchronization of chaos has not been investigated with ANFIS based controllers. In this paper, the anti-synchronization of two identical Lorenz chaotic systems is applied by using an adaptive model reference ANFIS control technique.

The rest of this paper is organized as follows: In Section 2, the Lorenz chaotic system, and ANFIS are described briefly. Then, the proposed ANFIS model is constructed for chaos anti-synchronization in Section 3. Afterwards, ANFIS controllers assigned to Lorenz chaotic system and the simulation results are presented graphically to verify the anti-synchronization in Section 4. Finally, the paper is concluded in Section 5.

2 Materials and Methods

2.1 Lorenz Chaotic System

The Lorenz model is used for fluid conviction that describes some feature of the atmospheric dynamic. The differential equations of the Lorenz chaotic system is described by

\[
\begin{align*}
\frac{dx}{dt} &= \sigma (y - x), \\
\frac{dy}{dt} &= x (r - z) - y, \\
\frac{dz}{dt} &= xy - bz,
\end{align*}
\]

where $\sigma$, $r$, and $b$ are parameters.
\[ \begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= -xz + rx - y, \\
\dot{z} &= xy - \beta z,
\end{align*} \tag{1} \]

where \( x, y, z \) are state variables that represent measures of fluid velocity, horizontal and vertical temperature variations, and \( \sigma, r, \beta \) are positive real constant parameters that represent the Prandtl number, Rayleigh number and geometric factor, respectively \[40\]. The Lorenz system is a chaotic attractor according to the parameters \( \sigma = 10, r = 28, \) and \( \beta = 8 / 3 \)[40]. The time series of the Lorenz chaotic system with the initial conditions \((x(0), y(0), z(0)) = (9, 15, 17)\) are shown in Fig. 1, the 2D phase portraits are shown in Fig. 2, and the 3D phase plane is shown in Fig. 3.

![Fig. 1. Time series of Lorenz chaotic system for (a) \( x \), (b) \( y \), and (c) \( z \) signals.](image)

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Fig. 2. Phase portraits of Lorenz chaotic system in (a) $x$–$y$, (b) $x$–$z$, and (c) $y$–$z$ phase plot.

Fig. 3. $x$–$y$–$z$ phase plane of Lorenz chaotic system.

2.2 Adaptive Neuro-Fuzzy Inference System (ANFIS)

Introduced by Jang in 1992 [41], ANFIS is a Sugeno fuzzy model where the final Fuzzy Inference System (FIS) is optimized with an ANN training. It is a universal intelligent computing methodology and it is capable of approximating any real continuous function on a compact set to any degree of accuracy [42].
ANNs, which are inspired from biological neural networks, have the ability of learning functional relations with limited amounts of training data. There are mainly two approaches for FISs, namely Mamdani [43] and Sugeno [44]. The differences between them arise from the outcome part where fuzzy membership functions are used in Mamdani’s approach, while linear or constant functions are used in Sugeno’s approach. Since ANFIS is based on the Sugeno type fuzzy model, it should be always interpretable in terms of function based fuzzy If-Then rules. Then, its parameter values are determined by a learning algorithm of ANN. Either a backpropagation method or a hybrid method which is a combination of least squares estimation with backpropagation can be utilized.

For better understanding of ANFIS, an example with two inputs \( x \), \( y \) and one output \( f \) is given briefly. In an ANFIS model, the output of each rule is a linear combination of input variables by adding a constant term. For a first-order Sugeno fuzzy model, when it is assumed that each input has two membership functions, the fuzzy If-Then rules can be written as

Rule 1: If \( x \) is \( A_1 \) and \( y \) is \( B_1 \), then \( f_1 = p_1x + q_1y + r_1 \),

Rule 2: If \( x \) is \( A_1 \) and \( y \) is \( B_2 \), then \( f_2 = p_2x + q_2y + r_2 \),

Rule 3: If \( x \) is \( A_2 \) and \( y \) is \( B_1 \), then \( f_3 = p_3x + q_3y + r_3 \),

Rule 4: If \( x \) is \( A_2 \) and \( y \) is \( B_2 \), then \( f_4 = p_4x + q_4y + r_4 \),

where \( A_i \) and \( B_i \) are the membership functions for inputs \( x \) and \( y \), respectively, and \( p_i, q_i, r_i \) are the parameters of output function with \( i = 1, 2, 3, \ldots, n \) corresponding to Rule 1, Rule 2, Rule 3, …, Rule \( n \). In ANFIS, the final output \( f \) is computed by the weighted average of each rule output as:

\[
f = \sum_i \bar{w}_i (p_i x + q_i y + r_i)
\]

where

\[
\bar{w}_i = \frac{w_i}{\sum_i w_i}
\]

Fig. 4 shows the composed layers of an ANFIS structure: input fuzzification, fuzzy rules, normalization, defuzzification, and total output. In layer 1, the fuzzy membership functions are represented. Layer 3 calculates the firing strength of the signals received from layer 2 and forwards it to layer 4, which calculates an adaptive output for giving them as input to the layer 5, which computes the overall output [45]. More detailed information about ANFIS technique can be found in [41, 42, 45].
3 Anti-Synchronization with ANFIS

In this paper, the model reference adaptive control technique with ANFIS is proposed for the anti-synchronization of chaotic systems. The goal of model reference adaptive control technique is to get a control system behaving like the reference model, which specifies the desired response of the system. The parameters of controllers are adjusted to minimize the error between the outputs of the model and the actual system. If the system has nonlinearity, intelligent algorithms such as ANFIS would rather be used in the model reference adaptive control due to getting better performance.

In the reference model, the master Lorenz system is described as

\[
\begin{align*}
\dot{x}_1 &= \sigma(y_1 - x_1), \\
\dot{y}_1 &= -x_1 z_1 + r x_1 - y_1, \\
\dot{z}_1 &= x_1 y_1 - \beta z_1, 
\end{align*}
\]

(5)

and the slave Lorenz system is

\[
\begin{align*}
\dot{x}_2 &= \sigma(y_2 - x_2) + u_1, \\
\dot{y}_2 &= -x_2 z_2 + r x_2 - y_2 + u_2, \\
\dot{z}_2 &= x_2 y_2 - \beta z_2 + u_3,
\end{align*}
\]

(6)

where \(u_1\), \(u_2\), and \(u_3\) are the nonlinear controllers. Sundarapandian obtained the controllers for anti-synchronization as
where error dynamics are \( e_1 = x_2 + x_1 \), and \( e_2 = y_2 + y_1 \) [17].

The diagram of adaptive model reference ANFIS control technique for anti-synchronization of chaotic systems is shown in Fig. 5. The adaptive model reference ANFIS controllers are trained to drive the slave system so that the differences between the anti-synchronization errors of ANFIS and the outputs of a reference model are minimized. If the ANFIS output error \( e_c(t+1) \) is defined as \( e_c(t+1) = y_d(t+1) + y_m(t+1) \), then the goal of the anti-synchronization is to determine the bounded input \( u(t) \) as \( \lim_{t \to \infty} e_c(t+1) = 0 \). The parameters of ANFIS controllers are adjusted by backpropagating the differences between anti-synchronization errors of ANFIS and nonlinear control reference model if the distance of error \( e_c(t+1) \) is greater than \( e_r(t+1) \). This training path is shown as a dashed line in Fig. 5. Once the ANFIS controllers are trained successfully, they are ready to use for anti-synchronization and there is no need to the reference model anymore.

The inputs of ANFIS include the state values of master and slave Lorenz chaotic systems. The output is the control signal. MATLAB is used for training the ANFIS controllers. This process is conducted with the command ‘genfis1’. Triangular (trimf) type membership functions with the number of 3 are taken for all inputs. Therefore, the ANFIS controllers have 729 Sugeno type rules. The membership function of output variable is selected as linear type. The training process proceeded with the command ‘anfis’. It identifies the parameters of Sugeno-type FISs. In the training stage, the hybrid learning rule, which is a combination of least-squared error and backpropagation gradient descent methods, with 5 epochs and zero error tolerance are preferred. Default values

\[
\begin{align*}
    u_1 &= \sigma e_2, \\
    u_2 &= -r e_1 + x_2 z_2 + x_1 z_3, \\
    u_3 &= -x_2 y_2 - x_1 y_1.
\end{align*}
\]
are used for all the other parameters. The training process finishes when the maximum epoch number is reached. In order to anti-synchronize successfully, the trained ANFIS controllers are employed to train again with a loop. After iterating the loop 10 times, the outputs of ANFIS controllers have 0.00025, 0.0007, and 0.00031 mean squared error as to $x$, $y$, and $z$ states, respectively.

4 Simulation Results

In this section, numerical simulations are performed to show the anti-synchronization of two identical Lorenz chaotic systems having different initial conditions with adaptive model reference ANFIS control technique. The fourth-order Runge–Kutta method with variable time step is used in the numerical simulations. The above-mentioned parameter values of Lorenz system are considered to ensure the chaotic behaviour. The initial conditions are taken as $(x_1(0), y_1(0), z_1(0)) = (9, 15, 17)$ for the master system and $(x_2(0), y_2(0), z_2(0)) = (13, 8, 38)$ for the slave system. When the ANFIS controllers are activated at $t = 10$, the simulation results of anti-synchronization and error signals are demonstrated in Fig. 6 and Fig. 7, respectively.

![Fig. 6. Time responses of anti-synchronization of Lorenz chaotic systems when the ANFIS controllers are activated at $t = 10$ for (a) $x$, (b) $y$, and (c) $z$ signals.](image)
Fig. 7. The anti-synchronization error signals of Lorenz chaotic systems when the ANFIS controllers are activated at $t = 10$.

When the ANFIS controllers are activated at $t = 20$, the simulation results of anti-synchronization and error signals are demonstrated in Fig. 8 and Fig. 9, respectively.

Fig. 8. Time responses of anti-synchronization of Lorenz chaotic systems when the ANFIS controllers are activated at $t = 20$ for (a) $x$, (b) $y$, and (c) $z$ signals.
As expected, the anti-synchronization of two identical Lorenz chaotic systems starting from different initial conditions is achieved with the ANFIS controllers in Fig. 6 and Fig. 8. The anti-synchronization error signals that are shown in Fig. 7 and Fig. 9 converge asymptotically to zero. When the ANFIS controllers are activated at \( t = 10 \), the anti-synchronization is provided at \( t \geq 12.5 \). Also, the anti-synchronization is observed at \( t \geq 21.5 \), when the controllers are activated at \( t = 20 \). Hence, the computer simulations validate the effectiveness of proposed adaptive model reference ANFIS control technique.

5 Conclusions

In this paper, a novel approach to the anti-synchronization of a chaotic system is applied with an ANFIS technique. The ANFIS controllers are trained on the bases of adaptive model reference control technique. Famous Lorenz chaotic system is preferred for simulations and a nonlinear control method is considered as the reference system. Numerical simulations show that ANFIS controllers achieve the anti-synchronization of two identical Lorenz chaotic systems in an proper time period. As a future work, non-identical chaos anti-synchronization may be investigated with ANFIS.

References


Information Security of the Chaotic Communication Systems

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Abstract. Nowadays are being held intensive researches involving the use of deterministic chaos in communication systems. Indeed chaotic oscillations in such systems serve as a carrier of information signals, and means of encryption, both in hardware and in software. This work is devoted to the complex analysis of the practical implementation of chaotic secure communication system. The generalized synchronization as synchronization response was selected. The circuit implementation of a modified Colpitts oscillator as a source of chaos was proposed. We also discussed questions of modeling chaotic oscillations and, in this context, analyzing the differences between chaotic, pseudo-chaotic and pseudo-random oscillations.

1 Introduction

The discovery of deterministic chaos Lorenz [1] (50 years ago) led to the rapid growth of both fundamental and applied scientific research related to the description of behavior of this nonlinear phenomenon. When two important properties of chaotic oscillations had been discovered – namely synchronization and sensitive dependence on initial conditions Pecora and Carroll [2], it became possible to investigate the use of deterministic chaos in information systems. At present, all works on using chaos in communication systems can be divided into the following four groups:

- **generation and general properties of chaotic oscillations** – these are works devoted to both chaos generators of different dimensions (including hyperchaos) and basic researches of new properties of deterministic (or dynamic) chaos;
- **synchronization and control of chaos** – these include studies that analyze the possibility of synchronous chaotic reviews without which it is impossible to realize chaotic communication system. Nowadays the studies related to the release of unstable periodic orbits, which are also used in chaotic communication are developing intensively;
- **chaotic cryptography** – this is another important direction of the use of chaos in secure communication systems. First chaotic encryption was performed on logistic map based on the fact that a certain range of variation of chaotic amplitude corresponds to a particular character Baptista [3]. Modern methods of chaotic cryptography use two, three
or more chaotic mappings and securely encrypt texts, images and videos in both on-line and off-line modes;

- **information systems using deterministic chaos** – here we should talk about several subgroups. First, all chaotic communication systems can be attributed to either analog or digital. In addition, the solutions can be represented both in hardware and software. Some groups include systems in integrated performance and chaotic communication based on symbolic dynamics. It is clearly that all these subgroups can overlap.

The analysis of works on chaotic information systems in recent years makes it possible to make several important conclusions. First, the predominant number of works devoted to the information security through the chaos include studies of chaotic cryptography implemented in software. Second, schematic implementations of both chaotic communication systems in general and of their receiving and transmitting units in the integrated performance are studied intensively. And third, according to many researchers, direct chaotic communication systems are especially perspective, where the change of actually chaotic system parameter transmits the information.

And another important point, which concerns the nature of chaotic oscillations. The real chaotic oscillations can be realized only if the exact values of their chaotic time series are fully taken into account. Since all computing devices have limited accuracy, then when studying the chaotic systems we always deal with pseudo-chaotic fluctuations. We clearly understand their fundamental difference from the pseudo-random oscillation, which is a sensitive dependence on initial conditions of chaotic mappings. Preliminary calculations show that in most cases, if there is the accuracy of 16 decimal places in computer simulations than we can talk precisely about chaotic oscillations Kushnir et al. [4]. Repetition period in this case is large enough and allows to carry out safely all information transfer and receiving operations.

This work is a continuation of a series of works carried out at the Department of Radio Engineering and Information Security of Chernivtsi National University and devoted to the study of chaotic secure communication systems. The work has the following structure. The second part provides the results of the study of generalized synchronization for hidden information transfer. The third part deals with the modification of Colpitts oscillator for covert communication system. And in the fourth part we proposed the generation of stable chaotic oscillations that can be used in the transmitting-receiving parts of the communication systems.

### 2 Chaos synchronization in communication systems

Synchronization is a universal phenomenon that occurs between two or more connected nonlinear oscillators. The ability of nonlinear oscillators to synchronize with each other is fundamental to understanding many processes in nature, because synchronization plays a significant role in science Boccaletti et al. [5].
An interesting type of chaotic synchronization for hidden transmission of information is generalized synchronization (GS) Lorenz [1]. GS occurs if after the end of transient processes, the functional relationship $Y = F(X)$ exists between the states of two chaotic systems, where $X$ and $Y$ – state vectors of driving and driven systems. The functional dependence of $F$ can be complex or fractal, which complicates the detection of the information from the carrier signal. High noise resistance distinguishes GS among the other types of chaotic synchronization Moskalenko et al., Abarbanel et al. [6, 7].

The other advantage of using GS in chaotic communication systems is also the ability to synchronize nonidentical or parametrically different systems and systems with different dimension of phase space.

The method for secure data transmission proposed in Moskalenko et al. [6] is based on the properties of stability of boundaries of GS to noise and switching chaotic modes using small change parameter of driving system. This method consists in the following. One or several control parameters of the driving generator of transmitter are modulated by a binary information signal. The obtained signal is transmitted through the communication channel. The receiver consists of two identical generators which can be in the mode of GS with transmitter. Depending on the transmitted binary bit 0 or 1, the parameters of modulation of the control parameter of the transmitter should be chosen so as to lead to the presence or absence of generalized synchronization between the transmitter and receiver. Thus between connected systems we can detect an occurrence of generalized synchronization or desynchronization, that is used to recover the information signal. To detect the GS one should use the method of auxiliary system Abarbanel et al. [7].

To use the generator of chaos in this system it is necessary that the minimum value of the coupling strength between systems (the boundary of synchronization) changed quickly at small change of the control parameter of generator. A sharp change in the GS border may be caused by two different ways of the occurrence of GS. In the first case GS arises due to suppression regime of natural oscillations of the driven system. Otherwise GS is established, when the driven generator captures the fundamental frequency of driving signal.

Our research has shown that these properties are not inherent in the most of known chaotic systems. Therefore the question appeared of the possibility of constructing artificial systems, which are characterized by the mentioned properties of GS.

The most suitable for solving this problem in terms of circuit implementation can be various modifications of the chaos ring oscillator, which consists of a filter and nonlinear element serially connected through buffer and closed by feedback. Simplest chaos ring oscillator consists of a first-order low-pass filter (LPF1), second-order low pass filter (LPF2) and nonlinear element (Figure 2.1).
For the occurrence of capture frequency between connected systems signals of chaotic systems should have distinct frequency component in the signal spectrum. Therefore, in terms of the generator should include narrowband filter. Replace the first-order low-pass filter (Figure 2.1) with the second-order narrowband one. Scheme of the modified generator is shown in Figure 2.2.

To investigate synchronization let’s consider two unidirectionally coupled generators, which are described by the following system of differential equations:

\[
\begin{align*}
\dot{x}_1 &= w_{11}^1 (v_1 - y_1) \\
\dot{y}_1 &= x_1 - by_1 \\
\dot{z}_1 &= -v_1 \\
\dot{v}_1 &= w_{12}^1 (e(nl(y_1) - v_1) + z_1) \\
\dot{x}_{2,3} &= w_{2,13}^2 (v_{2,3} - y_{2,3}) \\
\dot{y}_{2,3} &= x_{2,3} - by_{2,3} \\
\dot{z}_{2,3} &= -v_{2,3} + e(v_1 - v_{2,3}) \\
\dot{v}_{2,3} &= w_{2,23}^2 (e(nl(y_{2,3}) - v_{2,3}) + z_{2,3})
\end{align*}
\]  

(2.1)

where \(x, y, z, w\) – the state variables, the indices \(i = 1, 2, 3\) correspond to the signals of driving, driven and auxiliary systems respectively, \(e\) – coupling coefficient, \(nl(*)\) – nonlinear function, \(w_{11} = 5, w_{21} = 6.28, b = 1.38, c = 10\) – parameters of the system.

If \(w_1 > 4.83\) system (2.1) is in chaotic regime. Changing of the parameter \(w_{11}\) corresponds to the change of the fundamental frequency oscillations of driven system. Through numerical simulations we investigate the dependence of synchronization error of GS on \(w_{11}\) for \(w_{21} = 5\). If the synchronization error is close to zero, we can say that synchronization is established.

We calculated the dependence of synchronization error GS on coupling strength \(e\) and parameter \(w_{11}\). As shown in figure 2.3, there exists a range of parameter...
values for which synchronization error is small. This means that generators are synchronized if their parameters are selected from this range.

Fig. 2.3. The dependence of synchronization error $r_{GS}$ on coupling strength $e$ and parameter $w_{ij}$.

We studied the resistance of GS to noise in the channel. We found that GS is established if SNR $> 1.5$ dB (figure 2.4).

Therefore we proposed a method of creating chaos generator circuits for communication using chaos and we investigated the noise immunity of their generalized synchronization.

3 The properties of Colpitts oscillator signals for hidden communication systems

The development of HF hidden communication systems based on the use of nonlinear dynamic, requires the creation and study of chaotic generators the signals of which occupy the bandwidth of several hundred MHz. There are many scientific works on the research of analog and digital communication
systems using generators of chaotic oscillations Vovchuk et al., Vovchuk et al., Yang, Ivanyuk et al. [8-11], however, the information transfer speed is very slow, because the simplest chaotic generators (Chua’s circuit, Rossler, Lorentz systems etc. Fortuna et al., Chen et al., Alsafasfeh and Al-Armi [12-14]) are used, which generate the signals whose spectrums are at the low frequencies range. Thus Colpitts oscillator is more perspective, which has simple implementation and generates signals at the wide frequency range (figure 3.1a) Efremova, Dmitriev et al. [15, 16].

![Colpitts Oscillators Diagram]

**Fig. 3.1.** Colpitts oscillators: a) classic scheme with common emitter; b) modified scheme with common base and inductor in base subcircuit; c) modified scheme with common base and inductor in emitter subcircuit.

At the values of system parameters $R_L = 450 \, \Omega$, $R_E = 2400 \, \Omega$, $L = 1 \, \mu\text{H}$, $V_C = V_E = 20 \, \text{V}$, $C_1 = 150 \, \text{pF}$ and $C_2 = 70 \, \text{pF}$, spectrums of signals $U_{C1}$, $U_{C2}$ and $U_L$ occupy about 100 MHz ranges (figure 3.2).
Fig. 3.2. Spectral characteristics of signals generated by Colpitts oscillator:
  a) $U_{C1}$; b) $U_{C2}$; c) $U_L$.

For information secrecy it is necessary that the chaotic signals were similar to white noise and it takes that the statistical characteristics were similar to normal distribution ones. However the Colpitts oscillator signals are not characterized by properties similar to random variables, because the signals’ distributions are not symmetry (figure 3.3), proving the values of the mathematical expectation $\mu$ and standard deviation $\sigma$ and the values of kurtosis and skewness $|\kappa|$ and $|\alpha|$ are more than 0.25.

To solve this issue, we proposed to modify the scheme a bit. One more inductor was placed into one of the generator subcircuits. Several options of modification were considered, but only the signals $U_{L2}$ of the scheme that is shown in figure 3.1b, and $U_{L2}$ of the scheme that is shown in figure 3.1c, have statistical characteristics approximate to noise. The distributions of those signals and their spectral characteristics are shown in figure 3.4.

The statistical characteristics of the signals are shown in table 3.1. One can see that the Colpitts oscillator signals differ significantly from normal distribution. The signal generated by the modified scheme, shown in fig. 3.1c., is the most similar to noise.
Fig. 3.3. Signals’ distribution generated by Colpitts oscillator: a) $U_{C1}$; b) $U_{C2}$; c) $U_L$.

Fig. 3.4. The characteristics of signal $U_{L2}$ of the scheme that is shown in figure 1b: a) distribution; c) spectrum; and the scheme that is shown in figure 1c: b) distribution; d) spectrum.
Table 3.1. Statistical characteristics of studied chaotic generators

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Signal</th>
<th>μ</th>
<th>σ</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Figure 5a</td>
<td>$U_{C1}$</td>
<td>19.43</td>
<td>0.86</td>
<td>3</td>
<td>0.75</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$U_{C2}$</td>
<td>0.07</td>
<td>0.64</td>
<td>0.31</td>
<td>0.63</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>$U_L$</td>
<td>20.1</td>
<td>0.62</td>
<td>3</td>
<td>0.21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Figure 5b</td>
<td>$U_{L2}$</td>
<td>0</td>
<td>1.05</td>
<td>0.82</td>
<td>0.48</td>
<td></td>
<td></td>
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<tr>
<td>Figure 5c</td>
<td>$U_{L2}$</td>
<td>0</td>
<td>0.99</td>
<td>0.35</td>
<td>0.1</td>
<td></td>
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</tbody>
</table>

4 Method of synthesis of chaotic signals generators

An important task in the design of chaotic signals generators is a complex study of mathematical models that describe them. In particular, it is necessary to determine the exact boundaries of limits of system parameters and when they are changed various types of oscillatory modes can be formed, in particular – chaotic, hyperchaotic, periodic or quasi-periodic. This allows to calculate values of the Lyapunov exponent spectrum that characterizes the dependence of the dynamic system on initial conditions and determines the rate of divergence of trajectories in phase space.

Equally important tasks that help to determine the structure of the generated signals are mathematical modeling and numerical analysis of oscillatory modes of the chaotic systems under study, described by difference equations or systems of nonlinear differential equations of different orders.

Modeling of the general-circuit solution of chaotic signals generators is made possible by using the designed electrical circuit and modern approaches to circuit simulation using specialized software, such as, Micro-Cap or Multisim.

In the practical implementation of the electrical circuit of chaotic signals generator it is necessary to attribute the dynamic system parameters, its dynamic variables and mathematical operations over them in the equations of the system to signals and their processing circuits and perform the following steps:

1. Develop a block diagram of chaotic signals generator based on the systems of nonlinear differential equations.
2. Develop an electrical circuit of chaotic signals generator based on the block diagram.
3. Calculate the value of the electric circuit components on the basis of systems of differential equations of electrical circuit and the system of nonlinear differential equations.
4. Choose the base components according to the calculated values, considering the precision of circuit elements up to two decimal places.
5. Implement the electrical circuit of chaotic signals generator.
6. Make adjustment of modes of chaotic signals generator according to the types of oscillatory modes that can be generated by changing their values of system parameters.

Liu chaotic system Wang [17] was chosen as the basis for the realization of chaotic signals:
\[
\begin{align*}
\dot{x} &= a(y - x) \\
y &= bx - hxz + \lambda w \\
\dot{z} &= cx^2 - dz \\
\dot{w} &= -ny.
\end{align*}
\] (4.1)

Calculation of Lyapunov exponent spectrum, the results of mathematical modeling as time diagrams of dynamic variables and phase portraits, development of block diagram and electrical circuits, calculation of values of the electrical circuit components are given in Ivanyuk et al., Ivanyuk [11, 18]. The results of mathematical modeling and modeling of general-circuit solutions of chaotic signals generator were confirmed by experimental results.

The final stage of designing is to adjust the prototype of chaotic signals generator and experimental studies of oscillatory modes which may arise therein. Figure 4.1 shows the prototypes of communication system based on chaotic signals generators for experimental studies of secure data transfer process.

![Figure 4.1. Experimental studies of communication system based on chaotic signals generators: (1) – transmitter; (2) – receiver; (3) – power supply; (4) – scheme of synchronization; (5) – scheme of subtraction, addition of signals; (6) – audio speaker; (7) – audio amplifier](image)

Figure 4.2 shows the prototype of communication system based on chaotic signals generators. Figure 4.3 shows the phase portraits of the chaotic signals generator for chaotic and hyperchaotic modes of oscillations obtained by experimental studies.
Conclusions

This paper presents practical results on the implementation of chaotic secure communication system. Particular attention is paid to the issue of generalized chaotic synchronization, formation the reliable generators of chaos and their hardware implementation. Current prototype of the chaotic communication system confirms the proposed algorithms of transfer and data protection using deterministic chaos.

References


