Implications of Chaos Theory in Management Science

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Abstract. We live in a dynamic world that is most often described as being “chaotic” and unpredictable. From our human perspective, we do not see the greater framework of the system that we live in, and can only try to approximate its boundaries. However, with technological advances and continued adaptability, this does not limit our progression, because humans are complex creatures that seek to control chaos. It follows that we function in organizations that become complex systems, or systems that provide a balance between rigid order and random chaos. This realization defines a new paradigm for “emergent” leadership and management based on chaos theory, where emergent leaders become “strange attractors”; this means they are leaders that are flexible and have the skill set to accept unpredictability to enable the organization to adapt accordingly.

Keywords: Control, Non-linear systems, Uncertainty, Unpredictability, Attractors, Leadership, Emergent leader, Positive motivators.

1 Introduction

When we think of the word “chaos”, the prominent meanings that come to mind are confusion, disorder, and lack of control. However, these definitions represent the modern English meaning of the word. Chaos was first conceptualized and defined through mythology, which described the origins (or birth) of humankind. “Myth is as logical as philosophy and science, although the logic of myth is that of unconscious thought” (Caldwell[3]). The word itself is rooted in Greek origins, its authentic form being *Χάος* (Khaos). In Greek mythology, Chaos is “the embodiment of the primeval Void which existed before Order had been imposed on the universe” (Grimal and Kershaw[4]). In this definition it is evident how humankind had tried to contain a vastness that was (and is) difficult to comprehend in its natural form. Hesiod’s *Theogony* agrees with the undefinable origin concept, as “first of all, the Void came into being, next broad-bosomed Earth, the solid and eternal home of all… Out of Void came Darkness and black Night” (Brown [1]).

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The synonymy of “void” and “chaos”, and the birth of darkness from the “embodiment of the primeval void”, implies that chaos is an “impenetrable darkness and unmeasurable totality, of an immense opacity in which order is nonexistent or at least unperceived” (Caldwell[3]); that chaos describes the collection of everything that humankind cannot grasp and cannot control. Now, there is the duality of the controllable and uncontrollable; an unspoken demarcation of what the human mind is capable of elucidating.

However, the characterization of chaos is incomplete without the following line, as “Earth, the solid and eternal home of all” is formed, born as a separate entity and representative as “the primordial maternal symbol” (Caldwell[3]). The key to this is to note the synonymy of the “maternal symbol” and the “solid and eternal home of all”, the implication being that “solidity” and order is established so that the lineage of humankind can be traced back to something tangible. Again, there is a polarity of order and non-order, which can be seen in the chronologic succession of chaos, then the formation of earth. This sequence becomes significant in implying “that Chaos, a state prior to perception, represents the situation of the child in the symbiotic state” and “may be regarded as a representation of the symbiotic phase as un-differentiation and imperception, as a formless totality” (Caldwell[3]). Through this implication, chaos is better defined as being everything before perception, rather than confusion or disorder; chaos is what is unknown or intangible.

2 Control in Chaos

For decades we have repressed this unknown through systems and controls of mathematical equations and patterns. The reduction of chaos began with Sir Isaac Newton, in his attempt to mechanize reality through linearization of (what was later accepted as) a nonlinear, dynamic system. The theory had been based on the idea that with a linear reality, predictions could be made and phenomena could be controlled simply by deconstruction of the universe into its most “basic parts” and “logically” putting them back together (Burns[2]). In truth, this type of linearized reality has helped to advance humankind not only technologically, but also socially. “The social sciences have always attempted to model physical science paradigms” (Burns[2]). This is evident in the early formation of the field of Psychology, where Freud’s developmental stages build upon each other. It was assumed that if one stage is (or becomes) dysfunctional, problems in the human psyche occur. Other human systems, such as the development and function of political parties, economic systems, and the development of children’s concept learning strategies, are also built under the assumption of Newton’s linear reality (Burns[2]). With the help of quantum physicists, and theoretical meteorologist, Edward Lorenz, the way actual reality functions became easier to accept: the reality that the universe is chaotic and cannot be linearized and deconstructed into simpler mathematics. The realization that *social systems* could no longer be defined as linear.

Finally, the study of reality no longer models the constrained limitations of a linear way of thinking, and instead begins to model non-linear, dynamic chaos. By extension, because organizations exist in reality, it can be assumed that social systems develop within a chaotic system. Therefore, "organizations are nonlinear, dynamic systems" (Otten and Chen[10]) that make it imperative for leadership, and leadership practices, to be constructed through chaotic-system thinking. "In chaos theory leadership is not reduced to the 'leadership' behavior of a key position holder or team of 'top' people. Leadership is conducted throughout the organization, through all agents… Leadership is broadly conducted precisely because in chaotic systems, all agents have potential access to vital information from the environment" (Burns[2]).

The very definition of an organization is a body of people who share a purpose, vision, or mission. The primary functions of leadership within the organization are to: a) ensure that the agents of the organization keep the purpose and core values in mind, and b) ensure that the primary mission and values adapt (continuously) with environmental demands. By empowering all levels of the organization, the environment is monitored constantly and the overall mission is clarified because it is continuously evaluated and defined from different perspectives (Burns[2]). The acceptance of chaos in social systems is the basis that leaders must begin with. The assumption that outcomes are predictable is parallel to the assumption that chaos can be predicted. However, if chaos is defined as the unknown, the assumption that chaos is predictable is illogical. Therefore, it is the prerogative of leaders to influence the perspectives of the agents to accept unpredictability, so as to allow them to develop the capability to receive information and adapt accordingly. Leaders must have the skill set to shift thought processes in order to focus on the possibilities of outcomes and choose which ones are "desirable" to the organization, rather than fixate on a single possibility and try to control and direct chaos to produce this outcome.

3 Chaos Theory and Complex Systems Defined

Chaos theory states that the behavior of complex systems are highly sensitive to the slightest changes in conditions, which results in small changes to giving rise to more unpredictable, prominent effects on the system. With the introduction of quantum mechanics came a better understanding of how chaos theory applies to the real world. "Chaos theory, in essence, is an attempt to remove some of the darkness and mystery which permeates the classical concept of chaos by explaining, at least in some dynamic systems, how the system exhibits chaotic behavior" (Hite[7]). Chaos theory emphasizes that the conditions and state of change are no longer simple linear cause-and-effect relations; instead it assumes that both the cause and effect can originate and result from a multitude of variables that could come from various directions. This implies that a chaotic system is a flexible macro-structure that is vulnerable to the slightest disturbances on the micro level, although these changes are bounded by a pre-established framework.

Within the framework of an organization, chaos theory implies (but is not limited to) six critical points: 1) organizational life is predictable and unpredictable; 2) it is virtually impossible to define a single cause for any reactions; 3) diversity provides a more productive base; 4) self-organization will reduce concern for anarchy prevailing over chaos; 5) individual action in combination with a multiplier effect will focus responsibility on the individual; and 6) "scale-invariant properties and irreversibility are components of all chaotic organizations" (Grint[5]).

Organizational life may be predictable on the macro-level, as there will appear to be repetitive behaviors or patterns that appear aperiodically. On the other hand, at the micro-level of an organization, it will seem unpredictable because humans, as individuals, will appear to be random and to express unconnected, chaotic tendencies. One example found in nature is seen in the actions of ants; the activities of a single ant will appear random and disconnected, but the greater picture shows that it is a part of a larger social organization that has a single value. Because of this type of reasoning, the second critical point holds true: to define a single cause to explain an effect is impossible, as there could be many causes that occur simultaneously to produce an outcome. Every individual agent of the organization will establish multiple links, or connections, with other agents and various sources of information from the environment. Therefore, multiple reasons behind following directives or strategies will develop over time or simultaneously. Each unique link and motive must be taken into account when trying to align the goals of the organization with that of the individual. The strategies that are established should be aimed towards the acceptance of unpredictability and uncertainty, so as to give the impression "that they have control over something which is inherently uncontrollable" (Grint[5]).

The acceptance of uncertainty and unpredictably will help agents to recognize the value of dissenting voices and contrary cultures. The idea behind this is to shift the organization from a hierarchical top-down structure to a self-organizing structure, where the environment is defined by fundamental, interactive guidelines that allow for the flexibility in handling each situation uniquely. This idea is akin to giving an organization a set of standards and regulations that *suggest* how to handle general issues, instead of stating rigorous rules on how things should and should not be. It would be ideal to just hint at the overall culture and let each experimental, self-organizing group within the structure contribute to the definition of organizational life by facilitating their own resolutions (because it would be unique to each group) instead of following orders. The allowance of this kind of problem solving will enable the agents to voice their opinions and implement actions without reprimand, unlike positive- and negative-reinforcement managerial styles that may dissolve the organization into anarchy. Agents who do not feel constrained by rules and regulations feel that they are contributing to the overall system, and are less likely to cause destructive disorder. From this point, it is up to the leader or manager to be able to allow the loss of total control, and to allow for the birth and decay of motivational schemes in order to become effectively adaptive.

With the loss of control, it usually follows that there is a loss of responsibility placed solely on the leader of an organization. This happens because the agents create and form the culture, and therefore have the obligation to uphold the culture. The leader or manager, and even the individual agents, must also understand the irreversibility of individual actions; the multiple connections that form between various agents will contribute greatly to the multiplier effect, and propel smaller-scale decisions and strategies into larger arrangements. A component of chaotic structures that this is commonly compared to is called a fractal, where similar ordering properties can be seen at different levels of the organization, and be recognizable to all levels. And, like a fractal, these similar patterns will build upon each other to create a complex structure.

4 The “Strange” Attractor

The development of Lorenz’s mathematical model of a chaotic system emphasized the idea that dynamic, complex systems are highly dependent on initial conditions; his model of the system demonstrated that a slight change in the input values produced very different outputs. However, no matter what changes were made, the visual pattern that computers generated based on Lorenz’s model reflected that of butterfly wings. “The resulting figure displays a curve that weaves itself into a circular pattern, but never repeats itself exactly. Because it never returns to the initial state, though it may come arbitrarily close, the system is aperiodic” (Singh and Singh[12]). An embedded circular shape within the “wings” forms as the model continues; however, it is almost like a void space – the pathways never cross through this space. This void space is an “attractor” that will draw “point trajectories into its orbit, yet two arbitrarily close points may diverge away from each other and still remain within the attractor” (Burns[2]).

“Conventional theory asserts that the world is predictable and stable, and able to be explained by causal links that can be measured and monitored. Chaos theory implies that in the short term anything can happen, but that in the long term patterns, or ‘strange attractors’, are discernible” (Grint[5]). These strange attractors represent a key concept in this definition of chaos theory. “A system attractor, in essence, operates like a magnet in a system. It is the point or locus around which dynamical system activity coalesces… It is the attractor that provides the system with some sense of unity, if not uniformity. The attractor may be strong and definite, as with a fixed point, or it may be weak and indefinite, as with strange attractors" (Hite[7]). The strange attractor is not "weak" as in the classical sense of the definition. It is weak in the sense that it is flexible in its structure and has the ability to adapt infinitely. The strange attractor is better conceptualized as the pinpoint where the basis of the new or current dynamic system begins; this is similar to agents and how they interact within an organization. The difference between the agent and the attractor is that the attractor is an individual who possesses innate qualities that other agents may eventually gravitate towards.

In essence, the strange attractors of the organization are the values and vision that is shared, and "attractor" agents will exemplify these values and vision; but it is unlikely that individual agents will "orbit" the vision and values in the same way. This will result in the creation of multiple pathways to achieve the same overall mission of the organization. The "Butterfly Effect" theory was named after complexity science "where a butterfly flapping its wings in one location gives rise to a tornado or similar event occurring in another remote part of the world… the butterfly effect is nonlinear and amplifies the condition upon each iteration" (Osborn *et al.* [9]). And, as the butterfly effect explains, because these paths differ, these small changes in trajectories will result in larger changes to the overall system, though it will still be within the same framework. However, the timeframe of these changes, and to what extent the changes will have an effect, will be unknown; something small can begin a chain of events that will cause something relatively larger or smaller, in another part of the world or in close vicinity; but how quickly or slowly that happens will be unpredictable. At this point the difference between a complex system and chaotic system becomes difficult to define.

5 The Line between Chaotic and Complex Systems

"Where chaos theory addresses systems that appear to have high degrees of randomness and are sensitive to initial conditions, complexity theory has to do with systems that operate just at the line of separating coherence from chaos" (Hite[7]). Returning to the definition that chaos is everything unknown to humankind, it was also seen that the state of chaos thrives within the condition of symbiosis, by un-differentiating or non-delineating the self from the total. Now, instead of chaos being the unknown, as in uncertainty or ambiguity, it is transformed into being the unknown, as in the unawareness of individuality; there is no self or other, there is only totality; there is only interdependence in oneness (Singh [11]). Complex systems operate between order and chaos, where the state of symbiosis exists, but the conditions surrounding the symbiotic relationship are defined.

By extension of this thought, the theory of the "Butterfly Effect" is emphasized. The initial conditions put into the system are known, which is representative of imposing a type of order into the system. However, the outcome will always be unpredictable in the short-term. Nevertheless, in the long-term, there will be aperiodic behaviors that a complex system will adapt to. Thus, if new initial conditions based upon these behaviors are inputted into the system, no matter how unpredictable the outcomes, the system will iterate and adapt to try to return to a flexible state of equilibrium, even if the speed of this change is unknown. It must also be accepted that this state of equilibrium is fleeting, as there will be another change in the system occurring somewhere else at any given point in time, giving credence to the idea that complex systems are dynamic in nature. And, because the system will always be in flux and dynamic, it is logical to say that how leadership is defined and how management is applied also need to be continuously dynamic.

6 Leadership Actualized

There is no universal explanation for what leadership is, or how to define it -- only contextual examples of what leadership accomplishes. Through the understanding of chaos and complexity, it becomes easier to digest that a solid definition for leadership may never be found; the essence of leadership is continuously adapted and remolded to fit what the organization needs. There are a few reasons behind why leadership is so difficult to define. Like the Butterfly Effect, the extent, speed, and actual dimensions of the response(s) to leadership will never be clearly known, and so cannot be clearly defined. However, the connotations of leadership are known to be adaptable to the culture of the organization.

Therefore, defining the culture would mean determining the style of leadership that is needed. Because culture varies from organization to organization, what defines a leader will also differ, as they will need to adapt to specific and unique organizational needs. And, as a leader, it is important to note that leadership is not delivered by a single individual, but rather, is dependent on the interaction between an agent and its organization and is constructed from social recognition (Osborn *et al.* [9]). "The point of leadership is to initiate change and make it feel like progress… Leadership is what takes us and other people into a better world. Leadership insists that things must be done differently. Leadership rides the forces that are pulling individuals, groups, organizations, markets, economies, and societies in different directions, and lends a coherence that will enable us to benefit from the change around us. Leadership says, 'We cannot just carry on doing what we have done before. See all these forces of change around us; they are not just threats, they are also opportunities. But we must do this rather than that'" (Yudelowitz *et al.* [13]). Leadership seems to represent the "space between" what a leader does and how the organization responds; leadership manifests itself in the interaction, and what makes someone a leader is the leader's awareness of this fact and to what extent his or her influence can be recognized.

7 Organizations are Complex Adaptive Systems

In an adaptive organization, leaders monitor the overall well-being of the system, both internally and externally. Attractors influence the organization's culture and dynamics, while agents drive the system. A relatively new understanding of an organization is that it follows a "complex adaptive system" theory [CAST] -- a framework for explaining the emergence of system-level order that arises through the interactions of the system's interdependent components (agents)" (Lichtenstein and Plowman[8]). Because these interactions and influences can begin from anywhere within an organization, the model of an organization that seems to emerge is a decentralized structure that allows change to originate from anywhere, at any time. However, this does not mean that the unity and cohesiveness of the structure will become affected. What a complex adaptive system offers is a flexible structure that allows for the input of all the variables from the environment to influence the system, then adapts accordingly by beginning with individual agents. This is very reflective of the Butterfly Effect; "when an agent adjusts to new information, the agent expands his/her own behavioral repertoire, which, in effect, expands the behavioral repertoire of the system itself" (Lichtenstein and Plowman[8]).

In an empirical study, B.B. Lichtenstein and D.A. Plowmen found that there are four sequential conditions that form an element termed "emergence". Multiple cases were examined, where each case exemplified an organization undergoing the process of adaptation and how they "emerged" to be able to survive within the present environmental conditions. The four prevailing, sequential conditions found in each case are: dis-equilibrium, amplification of actions, recombination or self-organization, and stabilizing feedback.

Dis-equilibrium describes the system when it is in a state of dynamism and is usually initiated by the occurrence of an incongruity or change. This disruption can be caused by external or internal influences, such as, competition or new opportunities, and can be volatile enough to push the system beyond the existing perceptions of the norm. The study found that this state must be sustained for a long period of time in order to be considered a precursor to an emergent ordered system.

The second condition, amplifying actions, is when the dis-equilibrium caused by small actions and events begins to fluctuate and amplify throughout the system, seemingly to move toward a "new attractor", and grows until a threshold is reached. And, as learned from chaos theory, these actions and alterations will not follow a linear path throughout the organization; the change will easily "jump channels" (because all the agents are interconnected in some way) and can escalate in unpredictable, and unexpected ways (Lichtenstein and Plowman[8]).

The recombination, or self-organization, is the third (and most defining) condition that must be reached, as this is where a new order is established that increases the efficiency and capacity of the entire system. Once the organization has crossed the aforementioned threshold, it “emerges” as a “new entity with qualities that are not [yet] reflected in the interactions of each agent within the system” (Lichtenstein and Plowman[8]). The hope of this self-organization is that the system will recombine in such a way that new patterns of interaction between agents will improve the functions and capacity of the organization. In truth, this critical step will determine the survivability of the organization because, instead of restructuring progressively, the system could collapse or self-disorganize. This could be due to a) the lack of innovative ideas, b) poor assessment of the environment (because the reconstruction is dependent on reform), c) an inadequate “strategic fit” or core competency to handle the changes made, or d) a resistance to change (which is characteristic of a stable system) (Yukl and Lepsinger [14]).

The final condition of this emergent ordered system is the stabilizing feedback (“damping feedback”), or the anchors that keep the change in place and slow the amplification that produced the emergence in the initial stages. This anchoring is important, as it is reflective of how the interactions between agents sustain the change successfully and solidify legitimacy to the new paradigm. The new emergent order will dramatically increase “the capacity of the system to achieve its goals” (Lichtenstein and Plowman[8]). The study also surmised that leaders with certain characteristics will enable this emergence in an organization.

8 Characteristics of Leaders of Emergence

Leaders of emergence will generate or “enable” circumstances that will purposefully create the conditions needed to bring about the new emergent order. Lichtenstein and Plowmen noticed that certain characteristics were prominent and recognizable within each case used in the study. To achieve the dis-equilibrium condition, a leader will need to disrupt existing patterns and rally support for the uncertainty in the disturbance. Most importantly, a leader will need to acknowledge these conflicts and controversies with the intention that the farther the “ripple” spreads, the more perspective and diverse solutions will be generated. In this case, it is not the “people at the top” of the formal hierarchy that will brainstorm and decide what solution to take. Instead, the role of the leader becomes distributed through all branches of the organizations, where conflict and diversity are acknowledged, and can be accepted, equally. Next, it becomes the role of these emerging leaders to “amplify” the perspectives and conflict through the rest of the organization by encouraging innovative ideas and solutions, in order to instigate the second condition. By allowing experimental procedures, for example, to be enacted in a certain part of the organization, new ideas can be tested instead of just talked about; the belief or disbelief in the success of an experiment is only truly forged when the results are attainable. And, by encouraging the expression of innovation, “new attractors” may be birthed, and a type of “relational space” can be created, where “a certain high quality of interactions, reflecting a shared context of mutual respect, trust, and psychological safety in the relationship" is created (Lichtenstein and Plowman[8]). And, "as predicted by complexity theory (and managerial psychology), these rich interactions strengthened interpersonal networks, which helped to amplify the changes as they emerged" (Lichtenstein and Plowman[8]).

A leader who seeks the creation of a new emergent structure will assess the feasibility of the new structure that this attractor presents and not blindly following the new internal trend. Some points that a leader may ask about the proposed system are a) if it is attainable, b) if it will fit within the environment, and c) if it is progressive or retrogressive to the organization's values and vision. If the leader is fairly sure that the new regime is "better" for the organization, he or she will need to begin to rally other agents to support it, so that collective action can contribute to a solidified installment of the changes made.

The final condition of this complex adaptive system depends on the ability of the leaders to re-stabilize the structure. To do this, the leaders must remind the organization of the values and vision of the organization, and promote awareness of the cultural and environmental constraints that will affect the new emergent structure. It is the leader's job to keep the structure grounded in reality while allowing it to thrive at the increased capacity that was achieved. And, while it is true that these four sequential conditions and characteristics were founded upon a limited number of case studies, this model for understanding the functions and reactions of a complex adaptive system are relevant and supported by aspects grounded in chaos theory, presenting an "underlying order in chaos" (Otten and Chen[10]).

9 Possible Motivators for an Emergent System

Both models have only scratched the surface of the new order of leadership and management in an organization. They very clearly express that leaders are no longer the apex of the organization, but, instead, are more effective when they are "orbited" and "in-plane" with the agents. However, in order for the agents to begin to collect around a supported attractor, they must be motivated to do so. The leader will need to give purpose and meaning to the new attractor that will make sense to the emerging paradigm. The empirical study of the emergent system found that the creation of correlated language and symbols helped to initiate recombination or "self-organization". These symbols resonated the most when performed through symbolic actions that legitimize the change, while the language used helps to relate emotionally on a personal level with each agent. Another way to inspire meaning and connection to the new structure is to consolidate or recombine important resources, such as, capital, space, or skills, so as to give the impression that the system is expanding towards a "better" paradigm. The idea is that self-organization will be supported, and, thus, gain favor throughout the system. And, because there is not only a centralized leader within the structure of this complex system, the multiple leaders who emerge become symbols (Lichtenstein and Plowman[8]).

Hamel [6], discusses a management style called "Management 2.0" that humanizes the structure of an organization, acknowledges the autonomy of the individual, and sets a complex system motivated by humanistic, not materialistic, ideals: it redefines the language of the system, supporting ideals such as justice, community, and collaboration, as opposed to corruption, profit, and rivalry. The motivators behind this foundation are unique and requires a distinct leadership style to achieve it. One technique to increase motivation to uphold these ideals is to "reduce fear and increase trust". To reduce fear means to eliminate positive-negative reinforcement of actions, and encourage risk-taking innovations. With autonomy, now, comes an inherent trust between the leader and agents, where a leader trusts the agents of the organization to function within the values and boundaries established, and the agents trust the leader to provide stability and dynamism, without erasure of the individuality of the agent. And, democratization of information allows agents to act independently, thus preserving autonomy.

Summary and Conclusions

Empowering the agents allows them to have the capability to drive the system. However, without the presence of the attractors to influence the culture, the system may not emerge according to values of the organization. Ultimately, the obligation of the leader is to bridge the values with the vision and mission of the entity, and give purpose to the organization. Leaders will also need to monitor the internal and external influences to the system. The use of complex adaptive systems theory will enable the leaders to guide the adaptation of a system by creating an emergent structure that reconfigures the organization into new patterns that improve the function and capacity of the system, while still aligning with its core competency. Although it is fundamentally impossible to control chaos, it is possible to increase the survivability of an organization to adapt to the chaotic environment through complex adaptive systems theory.

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Lyapunov spectrum analysis of natural convection in a vertical, highly confined, differentially heated fluid layer

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Abstract. We use Lyapunov spectrum analysis to characterize the dynamics of a single convection roll between two differentially heated plates. 3D numerical simulation is carried out in a highly confined periodic domain. As the Rayleigh number increases, the intensity of the convection roll displays chaotic features while the roll remains stationary. For still higher values of the Rayleigh number, the roll intermittently moves between two positions separated by half a wavelength. We use Lyapunov spectrum analysis to help determine the characteristics of the flow in both regimes. We show that although the largest Lyapunov exponent is positive on average, the most probable value of the short-time Lyapunov exponent is negative. We compute the flow eigenvectors associated with the strongest variations in the exponent in the chaotic and the intermittent case and identify the corresponding hydrodynamic modes of instability.
Keywords: Natural convection, Period-doubling bifurcations, crisis-induced intermittency, Lyapunov spectrum.

1 Introduction

Natural convection between two vertical plates maintained at different temperatures is an important prototype to model heat transfer in industrial applications, such as plate heat exchangers or solar panels. The properties of heat transfer are deeply influenced by the nature of the flow, which is typically turbulent. It is therefore of interest to study the onset of chaotic dynamics in these flows. The development of instabilities in a differentially heated cavities with adiabatic walls has been studied numerically for a few decades [1,2]. Earlier studies are mostly limited to 2D geometries and relatively low Rayleigh numbers regimes (steady, periodis, quasi-periodic) with a focus on primary instabilities. Recent studies focus on the fully turbulent nature of the natural

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convection flow at high Rayleigh numbers [3], which remains a challenge owing to the double kinetic and thermal origin of the fluctuations.

Our studies attempt to bridge the gap between the relatively ordered flow observed at low Rayleigh numbers and the fully turbulent flow at high Rayleigh numbers. To this end, we carried out the three-dimensional direct numerical simulation (DNS) of a fluid layer between two vertical, infinite, differentially heated plates and determined the different stages leading to chaos [4]. The flow is characterized by co-rotating convection rolls which grow and shrink over time and interact with each other in a complex fashion. Similar rolls have also been observed in tall cavities of high aspect ratio [5]. A useful model of the problem can be obtained by limiting the dimensions of the plates in order to study the dynamics of a single convection roll. A cascade of period-doubling bifurcations and a crisis-induced intermittency have been observed in the vertically confined domain [6]. The goal of this paper is examine how Lyanunov exponent analysis can help characterize the chaotic dynamics of the flow in such a configuration.

2 Configuration

We consider the flow of air between two infinite vertical plates maintained at different temperatures. The configuration is represented in Figure 1. The distance between the two plates is D, and the periodic height and depth of the plates are L_z and L_y respectively. The temperature difference between the two plates is ΔT. The direction x is normal to the plates, the transverse direction is y, and the gravity g is opposite to the vertical direction z.

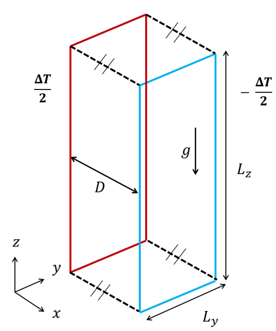


Fig. 1. (Color online) The simulation domain is constituted by two vertical plates, separated by a distance D and maintained at different temperatures. Periodic boundary conditions for the plates are enforced in both transverse and vertical directions (y and z). The aspect ratios of the periodic dimensions are $A_y = L_y/D = 1$ and $A_z = L_z/D = 2.5$. The temperature of the back plate at $x = 0$ (in red) is $\frac{\Delta T}{2}$, while that of the front plate at $x = 1$ (in blue) is $-\frac{\Delta T}{2}$. The distance between the plates is D.

The fluid properties of air, such as the kinetic viscosity ν, thermal diffusivity κ, and thermal expansion coefficient β, are supposed to be constant. Four nondimensional parameters characterizing the flow are chosen in the following way: the Prandtl number $\mathrm{Pr} = \dfrac{\nu}{\kappa}$, the Rayleigh number based on the width of the gap between the two plates $\mathrm{Ra} = \frac{g\beta \Delta T D^3}{\nu\kappa}$, and the transverse and vertical aspect ratio $A_y = L_y/D$ and $A_z = L_z/D$, respectively. Only the Rayleigh number is varied in the present study. The Prandtl number of air is taken equal to 0.71. The transverse aspect ratio is set to be $A_y = 1$, the vertical aspect ratio is set to $A_z = 2.5$, which corresponds to the critical wavelength $\lambda_{zc} = 2.513$ obtained by the stability analysis [4].

The flow is governed by the Navier-Stokes equations within the Boussinesq approximation. The nondimensional equations are:

$$\nabla \cdot \overrightarrow{u} = 0 \tag{1}$$

$$\frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u} \cdot \nabla \overrightarrow{u} = -\nabla p + \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ra}}} \Delta \overrightarrow{u} + \mathrm{Pr}\theta \overrightarrow{z} \tag{2}$$

$$\frac{\partial \theta}{\partial t} + \overrightarrow{u} \cdot \nabla \theta = \frac{1}{\sqrt{\mathrm{Ra}}} \Delta \theta \tag{3}$$

with Dirichlet boundary conditions at the plates

$$\overrightarrow{u}(0, y, z, t) = \overrightarrow{u}(1, y, z, t) = 0, \quad \theta(0, y, z, t) = 0.5, \quad \theta(1, y, z, t) = -0.5 \tag{4}$$

and periodic conditions in the y and z directions. The equations (1)-(4) admit an $O(2) \times O(2)$ symmetry. One $O(2)$ symmetry corresponds to the translation in the transverse direction y and the reflection $y \to -y$, while the other corresponds to the translations in the vertical direction z and a reflection that combines centrosymmetry and Boussinesq symmetry: $(x, z, \theta) \to (1 - x, -z, -\theta)$.

A spectral code [7] developed at LIMSI is used to carry out the simulations. The spatial domain is discretized by the Chebyshev-Fourier collocation method. The projection-correction method is used to enforce the incompressibility of the flow. The equations are integrated in time with a second-order mixed explicit-implicit scheme. A Chebyshev discretization with 40 modes is applied in the direction x, while the Fourier discretization is used in the transverse and vertical directions. 30 Fourier modes are used in the transverse direction y for $A_y = 1$, while 60 Fourier modes are used in the vertical direction z for $A_z = 2.5$.

2.1 Description

For low Rayleigh numbers, the flow solution is laminar. A cubic velocity and linear temperature profile, which depend only on the normal distance from the plates are observed. The flow presents similar features to those of a confined mixing layer [9,4]. As the Rayleigh number Ra is increased, steady two-dimensional convection rolls appear at $\mathrm{Ra} = 5708$, which then at $\mathrm{Ra} = 9980$ become steady three-dimensional convection rolls linked together through braids of vorticity (see Figure 2). For still higher Rayleigh numbers, the flow becomes periodic at $\mathrm{Ra} = 11500$. The convection roll essentially grows and shrinks with a characteristic period of $T = 28$ convective time units, which is in good agreement with the natural excitation frequency of a mixing layer [9].

As the Rayleigh number increases, a series of period-doubling bifurcations appears, as illustrated in Figure 3. More details can be found in [6]. The onset of chaos was predicted to occur at Ra $\sim$ 12320, in agreement with numerical observations. The variations of the roll size become more disorganized and intense, but the position of the roll remains quasi-stationary. When Ra $= 12546$, the variations in the intensity of the roll become so large that the roll breaks down and reforms at another location, separated by half a vertical wavelength from the original one. In terms of dynamics this corresponds to crisis-induced intermittency, which can be seen in Figure 3(b). The difference between the chaotic and the intermittent regimes in terms of phase portraits is illustrated in Figure 4 for two Rayleigh numbers taken in each regime.

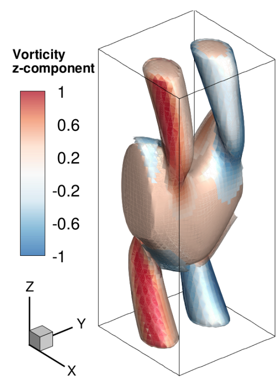
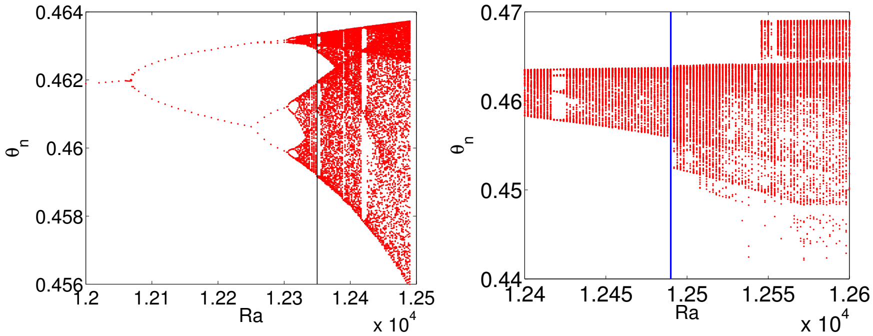


Fig. 2. (Color online) Q-criterion visualization of flow structures colored by the vertical vorticity Ω_z. Bi-periodic domain at Ra $= 12380$, $Q = 0.25$ in the present numerical configuration from Figure 1, i.e. with periodic boundary conditions in both y and z directions ($A_y = A_z = 1$);

3 Lyapunov spectrum

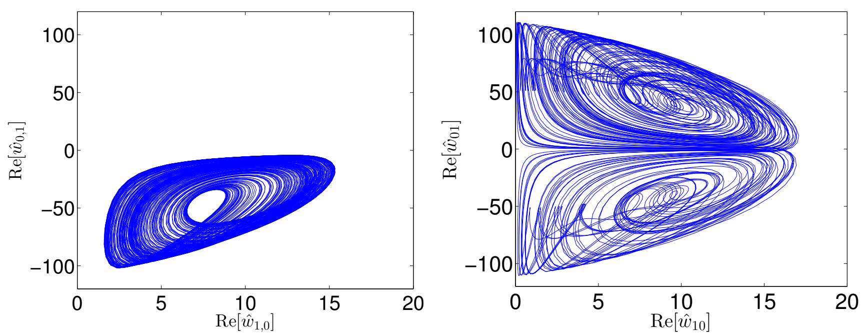
3.1 Definition

Several methods exist to distinguish between regular and chaotic dynamics in a deterministic system. The largest Lyapunov exponent, which measures the divergence rate of two nearby trajectories, is considered as a useful indicator to answer this question. Similarly, the n first Lyapunov exponents $\lambda_1 > \lambda_2 > \lambda_3 > ... > \lambda_n$ characterize the deformation rates of a n-sphere of nearby initial conditions. We applied the numerical technique proposed by Benettin *et al.* [8] to compute the Lyapunov spectrum of our fluid system, by parallelizing the DNS code described above with MPI library. On each processor, the flows are advanced independently in time. The flow on the processor-0 is the reference solution, which is obtained by numerical integration of the nonlinear equations. On the other processors, the randomly initiated perturbations $\delta\mathbf{X}$ are integrated in time by solving the linearized DNS code. The modified



(a) (b)

Fig. 3. (Color online) Bifurcation diagram obtained by using the local maxima θ_n of the temperature time series at the point (0.038 0.097 0.983).Note: the vertical line in the figure corresponds the largest Rayleigh number in Figure 3 (a) $12000 < Ra < 12500$ (b) $12400 < Ra < 12600$.



(a) Ra = 12380 (b) Ra = 12600

Fig. 4. (Color online) Phase portraits. Abscissa: real part of the the Fourier transform (in y and z) of the vertical velocity $\hat{w}_{01}(x)$ calculated on vertical plane $x = 0.0381$; ordinate: real part of the Fourier transform of the vertical velocity $\hat{w}_{10}$. (a) Ra = 12380, (b) Ra = 12600.

Gram-Schmidt procedure is applied every 20 time-steps of dt to renormalize the perturbations. At each renormalisation step, the instantaneous Lyapunov exponents were computed as

$$\lambda_i^{inst} = \frac{1}{\Delta t} \ln \frac{\|\delta \mathbf{X}(j\Delta t)\|_i}{\|\delta \mathbf{X}(0)\|_i} \tag{5}$$

Their asymptotic mean values form the long-time Lyapunov spectrum:

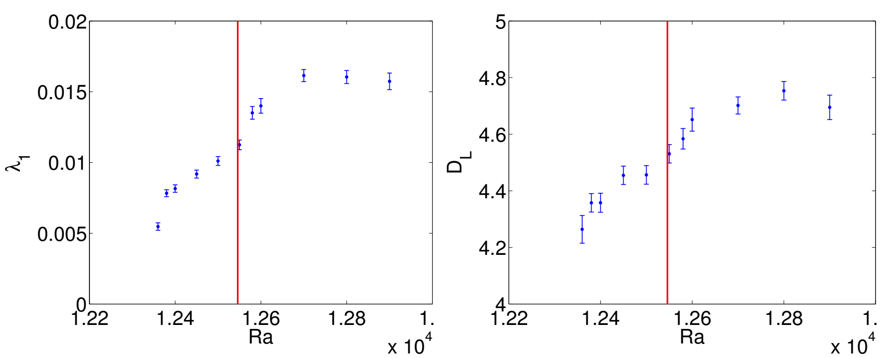
$$\lambda_i = \lim_{N\to+\infty} \frac{1}{N\Delta t} \sum_{j\in N} \ln \frac{\|\delta \mathbf{X}(j\Delta t)\|_i}{\|\delta \mathbf{X}(0)\|_i} \tag{6}$$

where λ_i is the i-th Lyapunov exponent and the norm mesuring the distance between two nearby trajectories is chosen as $\|\delta \mathbf{X}(t)\| = \sqrt{\int_V [\delta \overrightarrow{u}(t)^2 + \delta\theta(t)^2] dV}$.

3.2 Long-time Lyapunov exponents

The computation of Lyapunov spectrum for our fluid system was carried out at different Rayleigh numbers between Ra = 12360 and Ra = 12900. Errorbars for the Lyapunov exponent are estimated from the standard error of the mean assuming a Gaussian distribution and a 95% confidence interval. We note that the error on the exponent may be somewhat underestimated, as we do not take into account other sources of error, such as the distance to the attractor.

In all that follows, we focus on two Rayleigh numbers: one corresponds to the chaotic, non-intermittent system Ra = 12380. The other Ra = 12600 corresponds to a chaotic, intermittent case. Convergence tests were run for these two Rayleigh numbers Ra = 12380 and Ra = 12600 and two different time-discretizations $dt = 1 \times 10^{-3}$ and $dt = 1 \times 10^{-2}$. The 15 leading Lyapunov exponents are computed, among which the first 8 ones are listed in Table 1.



(a) (b)

Fig. 5. (Color online) (a) The largest Lyapunov exponent λ_1 for different Rayleigh numbers; Error bars are 1.96 times the standard error. (b) Fractal dimension obtained by application of the Kaplan-Yorke formula as a function of the Rayleigh number. The position of the solid line spanning each figure represents the value of the Rayleigh number at the onset of the crisis.

As shown in Figure 3.2, the largest asymptotic Lyapunov exponent is positive for Ra $\geq$ 12360, and increases quasi-linearly for 12400 < Ra < 12546. This suggests that temporal chaos has been reached. For all Rayleigh numbers considered, only one single positive Lyapunov exponent is found and is on the order of 0.01. The test $0 - 1$ for chaos proposed by Gottwald and Melbourne [12,13] was applied to an appropriately sampled temperature time series, and returned a value close to 1, which confirms that our flow is chaotic. The Lyapunov exponent is considerably larger for the intermittent case Ra = 12600 than for the chaotic case Ra = 12380.

We find that the asymptotic value of exponents 2 to 4 is close to zero. We observe that the temporal oscillations of the short-time exponents 2 to 4 decrease with the time step, as can be expected. As shown by Sirovich and Deane [10] for Rayleigh-Bénard convection, three exponents should be zero:

Table 1. First 8 Lyapunov exponents at two different Rayleigh numbers for two different time steps.

	Ra = 12380		Ra = 12600	
λ_i	$dt = 1 \times 10^{-3}$	$dt = 1 \times 10^{-2}$	$dt = 1 \times 10^{-3}$	$dt = 1 \times 10^{-2}$
1	0.0094 ± 0.0004	0.0078 ± 0.0002	0.0199 ± 0.0005	0.0140 ± 0.0005
2	-0.00047 ± 0.00067	-0.00027 ± 0.00043	-0.0001 ± 0.0008	-0.0002 ± 0.0005
3	0.00075 ± 0.00048	0.00031 ± 0.00026	0.0036 ± 0.0006	0.0009 ± 0.0006
4	0.00010 ± 0.00053	-0.00090 ± 0.00031	0.00011 ± 0.00062	0.00047 ± 0.00066
5	-0.0579 ± 0.00020	-0.0220 ± 0.0001	-0.0594 ± 0.00017	-0.0230 ± 0.0001
6	-0.0726 ± 0.0006	-0.0485 ± 0.0004	-0.0696 ± 0.0006	-0.0464 ± 0.0006
7	-0.0709 ± 0.0006	-0.0318 ± 0.0004	-0.0732 ± 0.0006	-0.0328 ± 0.0006
8	-0.0843 ± 0.0006	-0.0571 ± 0.0004	-0.0919 ± 0.0006	-0.0594 ± 0.0006

one comes from the fact that the time derivative $\dfrac{\partial \mathbf{X}}{\partial t}$ of the reference solution $\mathbf{X}$ satisfies the linearized equation, since the system is autonomous. The other two zero exponents reflect the fact that $\dfrac{\partial \mathbf{X}}{\partial y}$, $\dfrac{\partial \mathbf{X}}{\partial z}$ also satisfy the linearized equation on account of the homogeneous boundary conditions.

All exponents of order $n \geq 5$ were found to be negative. Convergence was more difficult to reach for these higher-order exponents. However even if some uncertainty is present, this does not affect significantly the value of the fractal dimension.

The Lyapunov dimension was estimated using the Kaplan-Yorke formula [11]:

$$D_L = K + \frac{S_K}{|\lambda_{K+1}|} \tag{7}$$

where K is the largest n for which $S_n = \sum_{i=1}^{n} \lambda_i > 0$. It was found to be between 4.2 and 4.6, as can be seen in Figure 3.2 (b). An inflection point, corresponding to a sharp increase in the largest exponent, is observed at the onset of intermittency for both the largest exponent and the Lyapunov dimension.

4 Short-time Lyapunov exponent

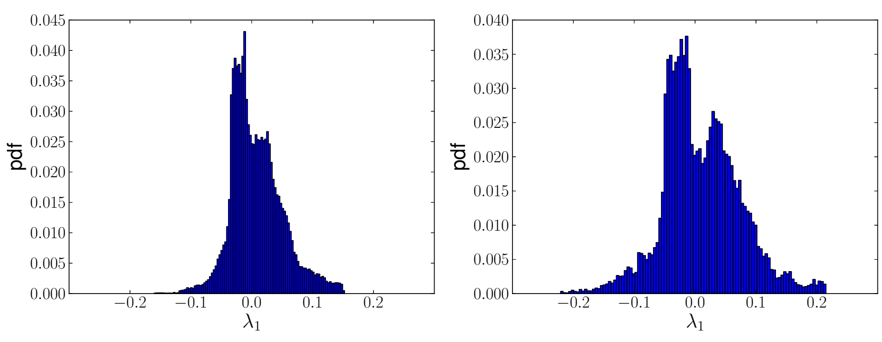
As pointed out by Vastano and Moser [15], examination of the short-time Lyapunov exponent provides additional information about the flow. Figure 6 and 7 shows the distribution of the first Lyapunov exponent for the two Rayleigh numbers and the two time resolutions. We can see that the distributions are very similar for both time intervals, which shows the convergence of the computations. Corresponding time series of the largest Lyapunov exponent and their Fourier spectrum are represented in Figure 8. The fundamental excitation frequency $f = 0.22$ is dominant in the chaotic case. Lower frequencies become important in the chaotic case.

A striking fact is that for both Rayleigh numbers, although the mean value of the exponent is positive, the maximum value of probability distribution function (p.d.f.) is actually negative. This is markedly different from the results reported by Kapitaniak [14] for quasi-periodically forced systems, where the mean value of the exponent appeared to correspond to the maximum of the distribution. We note that no external forcing is imposed in our configuration, which is characterized by self-sustained oscillations. The distributions at Ra = 12380 and Ra = 12600 present many similarities. The main difference is that in the intermittent case the local maximum of the distribution for small positive values in Figure 6 disappears, while a band of significantly higher positive values (larger than 0.2) is created in Figure 7.

We computed the vector associated with local extrema of the short-time Lyapunov exponent which were identified in the time series. This gives us insight into the perturbations most likely to disorganize the flow. We checked that observations made at a particular time held for other times.

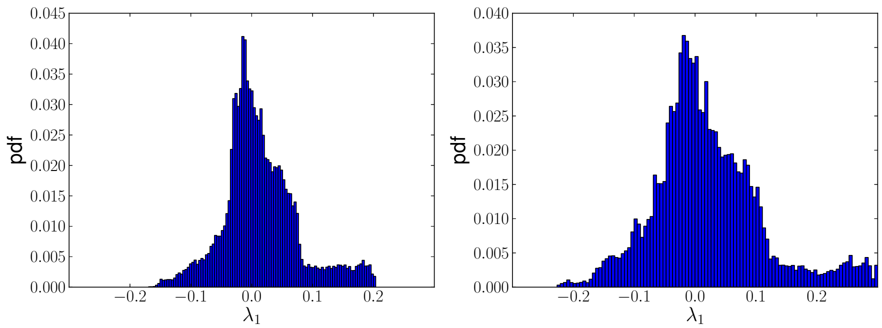
Results are presented in Figure 9 for the chaotic case. For the chaotic case, we have identified two types of relative extrema: (i) relatively small excursions, associated with the local maximum and the local minimum in the histogram from Figure 6 corresponding to positions marked with filled circles in Figure 8 (a). We find that the perturbation associated with a local maximum consists of almost 2D rolls (Figure 9 (a)), while the minimum corresponds to a strongly 3D flow and a relatively weaker convection roll (Figure 9 (b)). (ii) stronger excursions, where both extrema are associated with an essentially 2D flow (positions marked with filled squares in Figure 9 (c)(d)). 2D convection rolls correspond to the most unstable linear modes. However the convection rolls associated with maxima seem to be stronger than those associated with minima.

In the intermittent case, we focus exclusively on largest extrema. Figure 10 (a) shows that the maxima in time corresponds to a flow which is in fact almost 1-D (note the much lower value for the criterion $Q = 0.05$), while the minima in time corresponds to a 2D flow (see Figure 10 (b)). These two states can be associated with the break-up and formation of the roll.



(a) $dt = 1 \times 10^{-2}$ (b) $dt = 1 \times 10^{-3}$

Fig. 6. (Color online) Probability distribution function (p.d.f.) of instantaneous 1st Lyapunov exponent λ_1^{inst} at Ra = 12380.



(a) $dt = 1 \times 10^{-2}$ (b) $dt = 1 \times 10^{-3}$

Fig. 7. (Color online) Probability distribution function (p.d.f.) of instantaneous 1st Lyapunov exponent λ_1^{inst} at Ra = 12600.

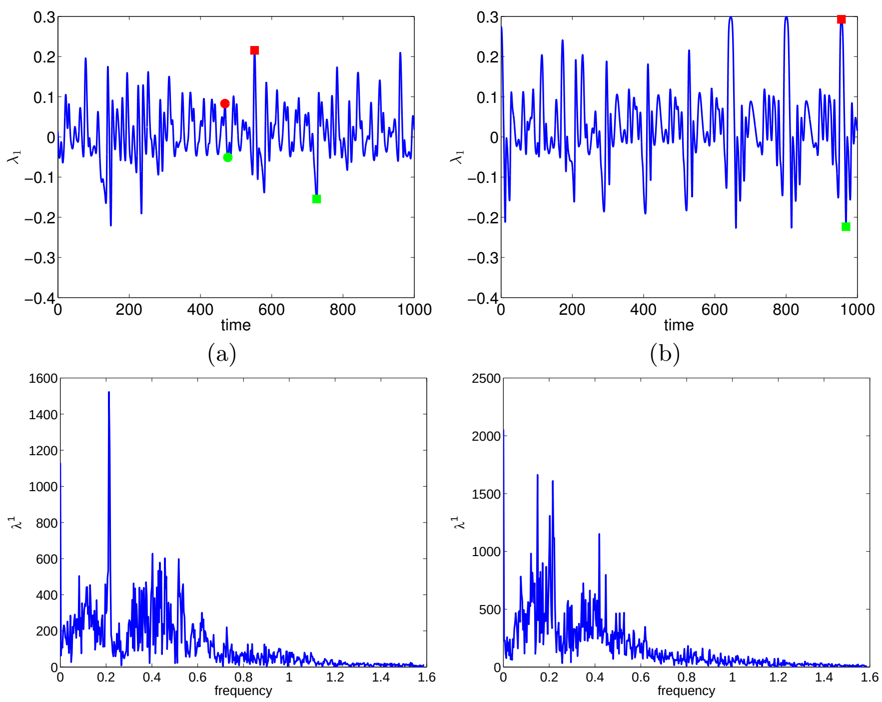


Fig. 8. (Color online) (a) (b) Evolution of the largest short-time Lyanunov exponent λ_1^{inst} at (a) Ra = 12380 (b) Ra = 12600; (c) (d) Temporal Fourier spectrum of the largest short-time exponent λ_1^{inst} at (c) Ra = 12380 (d) Ra = 12600.

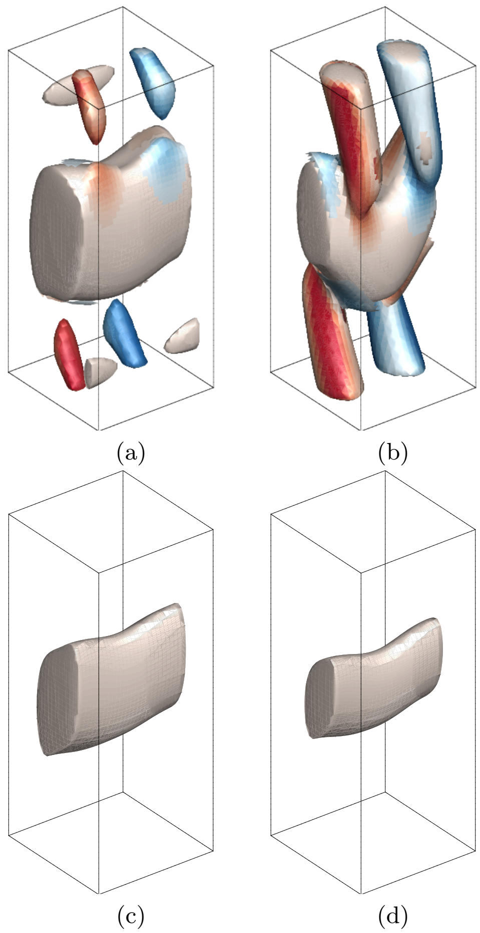
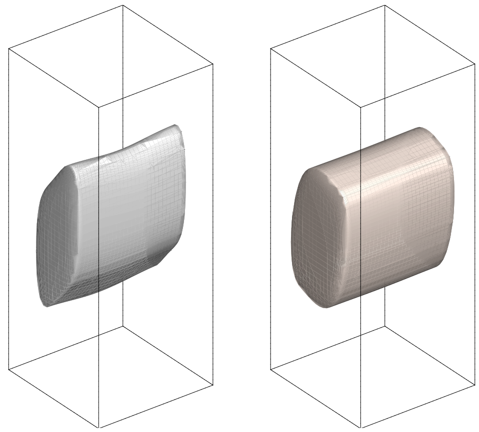


Fig. 9. (Color online) Eigenvector associated with a local extremum of the short-time exponent at Ra = 12380 at the positions indicated in Figure 8 (a). Value of the Q isosurface Q = 0.3 (a) t=469 (maximum) (b) t=479 (minimum) (c) t=552 (maximum) (d) t=726 (minimum)

5 Conclusion

We have considered the numerical simulation of a convection roll between two differentially heated plates of small periodic dimensions. As the Rayleigh number increases, the convection roll shrinks and grows in a periodic, then quasi-periodic, then chaotic. For still higher values, the convection roll breaks down and reforms intermittently at another location. Lyapunov spectrum analysis was used to characterize the dynamical features of the flow. Two cases in the purely chaotic and intermittent regime were examined in detail. We found that although the asymptotic value of the largest exponent is positive, its most probable value is negative. We showed that intermittency corresponds to the occurence of higher positive values in the Lyapunov exponent corresponding



(a) (b)

Fig. 10. (Color online) Eigenvector associated with a local extremum of the short-time exponent at Ra = 12600 at the positions indicated in Figure 8 (b). Value of the Q isosurface (a) t=954 (maximum) $Q = 0.05$ (b) t=968 (minimum) $Q = 0.3$

to the break-up and reformation of the convection roll. The perturbations associated with the extremal values of the short-time largest exponent were identified. In the chaotic case, the perturbations associated with the largest extrema are 2D convection rolls. Maxima are associated with larger rolls, while minima are associated with less intense rolls. In the intermittent case, maxima were associated with a quasi 1-D flow, which corresponds to the break-up of the roll, while minima corresponded to 2D convection rolls and therefore the roll formation stage. These results confirm that the analysis of short-time Lyapunov exponents provides insight into the physics of the flow and suggests that it could be useful for low-order modelling of its complex dynamics.

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From Nonlinear Oscillations to Chaos Theory

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Abstract

In this work we propose to reconstruct the historical road leading from nonlinear oscillations to chaos theory by analyzing the research performed on the following three devices: the *series-dynamo machine*, the *singing arc* and the *triode*, over a period ranging from the end of the XIXth century till the end of the Second World War.

Thus, it will be shown that the *series-dynamo machine*, *i.e.* an electromechanical device designed in 1880 for experiments, enabled to highlight the existence of *sustained oscillations* caused by the presence in the circuit of a component analogous to a "negative resistance".

The *singing arc*, *i.e.* a spark-gap transmitter used in *Wireless Telegraphy* to produce oscillations and so to send messages, allowed to prove that, contrary to what has been stated by the historiography till recently, Poincaré made application of his mathematical concept of *limit cycle* in order to state the existence of *sustained oscillations* representing a stable regime of sustained waves necessary for radio communication.

During the First World War, the *singing arc* was progressively replaced by the *triode* and in 1919, an analogy between *series-dynamo machine*, *singing arc* and *triode* was highlighted. Then, in the following decade, many scientists such as André Blondel, Jean-Baptiste Pomey, Élie and Henri Cartan, Balthasar Van der Pol and Alfred Liénard provided fundamental results concerning these three devices. However, the study of these research has shown that if they made use of Poincaré's methods, they did not make any connection with his works.

In the beginning of the twenties, Van der Pol started to study the oscillations of two coupled *triodes* and then, the forced oscillations of a *triode*. This led him to highlight some oscillatory phenomena which have never been observed previously. It will be then recalled that this new kind of behavior considered as "bizarre" at the end of the Second World War by Mary Cartwright and John Littlewood was later identified as "chaotic".

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1 Introduction

The aim of this work is to trace the history of the foundations of Chaos theory through the analysis of the works performed on the following three devices: the *series-dynamo machine*, the *singing arc* and the *triode*, over a period ranging from the end of the XIXth century till the end of the Second World War.

In 1880, by sending the current produced by a dynamoelectrical into a magnetoelectrical machine forming thus a *series-dynamo machine*, the French engineer Jean-Marie Gérard Anatole Lescuyer highlighted a *nonlinear phenomenon* that will be later considered by Paul Janet as *sustained oscillations* and by Balthasar Van der Pol as *relaxation oscillations*[1]. If the cause of this phenomenon was rapidly identified as being the presence in the circuit of a component analogous to a "negative resistance", its mathematical modeling was out of reach at that time.

A quarter of a century later, at the time of the emergence of *Wireless Telegraphy*, it became of tremendous need to find the condition for which the oscillations produced by a spark-gap transmitter called *singing arc* were sustained. Actually, this condition representing a stable regime of sustained waves necessary for radio communication was established by Henri Poincaré in 1908 during a series of "forgotten lectures" he gave at the École Supérieure des Postes et Télécommunications (today Telecom ParisTech). Contrary to what was stated by the historiography till recently, Poincaré made thus the first correspondence between the existence of *sustained oscillations* and the concept of *limit cycle* that he had introduced in his second memoir "On the curves defined by differential equations". In other words, he proved that the *periodic solution* of the nonlinear ordinary differential equation characterizing the oscillations of the *singing arc* corresponds in the phase plane to an attractive *closed curve*, i.e. a *stable limit cycle.*

During the First World War, the *singing arc* was progressively replaced by the *triode* which was also able to sustain oscillations but even more importantly to amplify the electric signal.

In 1919, the French engineer Paul Janet established an analogy between the *series-dynamo machine*, the *singing arc* and the *triode* and stated thus that their *sustained oscillations* belong to the same *nonlinear phenomenon.* Then, in the following decade, many scientists such as André Blondel, Jean-

[1] A brief history of *relaxation oscillations* can be found in Ginoux and Letellier [15]. However, let's notice that this article has been entirely republished by M. Letellier in the chapter 2 of his last book while omitting to make correct reference to this work. For a detailed history of *relaxation oscillations*, see Ginoux [13, 18, 19].

Baptiste Pomey, Élie and Henri Cartan, Balthasar Van der Pol and Alfred Liénard provided fundamental results concerning these three devices. However, it appears that if they made use of Poincaré's methods, they did not make any connection with his works.

In the beginning of the twenties, Van der Pol started to study the oscillations of two coupled *triodes* and then, the forced oscillations of a *triode*. This led him to highlight new oscillatory phenomena that he called *oscillation hysteresis*, *automatic synchronization* and *frequency demultiplication*. Nevertheless, in this case, if the oscillations are still sustained, the solution is no more periodic but exhibits a new kind of behavior that will be called "bizarre" at the end of the Second World War by Mary Cartwright and John Littlewood and that will be later identified as "chaotic".

2 The series-dynamo machine: *the expression of nonlinearity*

At the end of the nineteenth century, *magneto-* or *dynamo-electric machines* were used in order to turn mechanical work into electrical work and *vice versa*. With the former type of machine, the magnetic field is induced by a permanent magnet, whereas the latter uses an electromagnet. These machines produced either *alternating* or *direct current* indifferently. Thus, in 1880, a French engineer named Jean-Marie-Anatole Gérard-Lescuyer made an experiment by associating a *dynamo-electric machine* used as a generator with a *magneto-electric machine*, which in this case can be considered as the motor (Fig. 1).

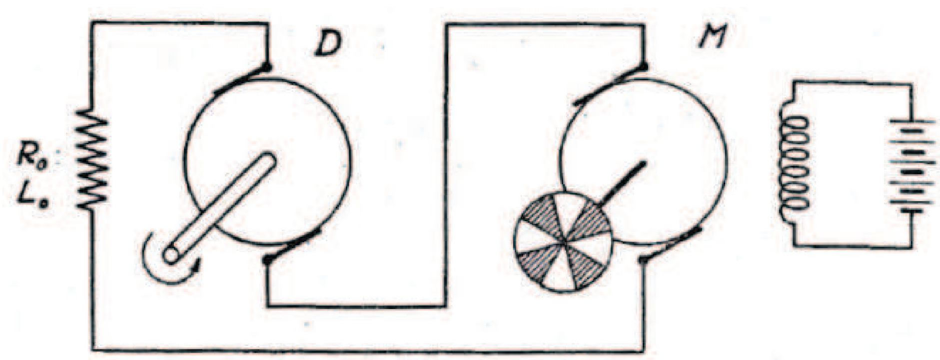

Figure 1: The Gérard-Lescuyer's paradoxical experiment [25].

Gérard-Lescuyer [20, 21] reports on the found effects in a note published in the *Comptes rendus de l'Académie des Sciences de Paris* and in the *Philosophical Magazine* in the following way:

> "As soon as the circuit is closed the magnetoelectrical machine begins to move; it tends to take a regulated velocity in accordance with the intensity of the current by which it is excited; but suddenly it slackens its speed, stops, and start again in the opposite direction, to stop again and rotate in the same direction as before. In a word, it receives a regular reciprocating motion which lasts as long as the current that produces it."

While observing the periodical reversal of the magneto-electric machine's circular motion, despite the direct current, he wondered about the causes of this oscillatory phenomenon that he was unfortunately unable to isolate. Gérard-Lescuyer [20, 21] wrote in his conclusion:

> "What are we to conclude from this? Nothing, except that we are confronted by a scientific paradox, the explanation of which will come, but which does not cease to be interesting."

It was actually proven by the count Théodose du Moncel [28] a few weeks later, then by Aimé Witz [50, 51], and by Paul Janet [23], that the gap situated between the brushes of the dynamo is the source of an electromotive force (e.m.f.), i.e. a potential difference at its terminals symbolized by a nonlinear function of the intensity that flows through there. However, the mathematical modeling of this e.m.f. was out of reach at that time. Therefore the essence of Gérard-Lescuyer's paradox is the presence of an e.m.f, which has a *nonlinear current-voltage characteristic* acting as a a *negative resistance* and leading to *sustained oscillations*.

Half a century later, the famous Dutch physicist Balthasar Van der Pol [46] noted:

> "Relaxation oscillations produced by a motor powered by a D.C. series-dynamo. The fact that such a system is able to produce relaxation oscillations was already briefly discussed. In an article written by Mr. Janet (we find a reference to Gérard Lescuyer (CR 91, 226, 1880) where this phenomenon had already been described."

3 The singing arc: *Poincaré's forgotten lectures*

At the end of the nineteenth century a forerunner to the incandescent light bulb called *electric arc* was used for lighthouses and street lights. Regardless of its weak glow it had a major drawback: the noise generated by the electrical discharge which inconvenienced the population. In London, physicist William Du Bois Duddell (1872-1917) was commissioned in 1899 by the British authorities to solve this problem. He thought up the association of an oscillating circuit made with an inductor L and a capacitor C (F on Fig. 2) with the *electrical arc* to stop the noise (see Fig. 2). Duddell [10, 11] created a device that he named *singing arc*.

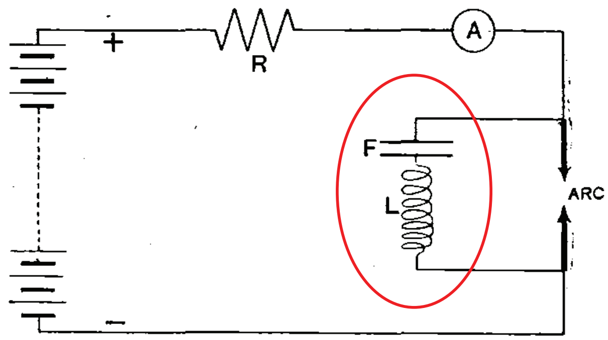

Figure 2: Diagram of the singing arc's circuit, from Duddell [10, 11].

Duddell had actually created an oscillating circuit capable of producing not only sounds (hence its name) but especially electromagnetic waves. This device would therefore be used as an emitter for *wireless telegraphy* until the triode replaced it. The *singing arc* or *Duddell's arc* was indeed a "spark gap" device meaning that it produced sparks which generated the propagation of electromagnetic waves shown by Hertz's experiments as pointed out by Poincaré [30, p. 79]:

> "If an electric arc is powered by direct current and if we put a self-inductor and a capacitor in a parallel circuit, the result is comparable to Hertz's oscillator... These oscillations are *sustained* exactly like those of the pendulum of a clock. We have genuinely an electrical escapement."

On July 4^{th} 1902, Henri Poincaré became Professor of Theoretical Electricity at the École Supérieure des Postes et Télégraphes (Telecom ParisTech) in Paris where he taught until 1910. The director of this school, Édouard Éstaunié (1862-1942), then asked him to give a series of conferences every two years in May-June from 1904 to 1912. He told about Poincaré's first lecture of 1904:

> "From the first words it became apparent that we were going to attend the research work of this extraordinary and awesome mathematician... Each obstacle encountered, a short break marked embarrassment, then a blow of shoulder, Poincaré seemed to defy the annoying function."

In 1908, Poincaré chose as the subject: Wireless Telegraphy. The text of his lectures was first published weekly in the journal *La Lumière Électrique* [31] before being edited as a book the year after [32]. In the fifth and last part of these lectures entitled: Télégraphie dirigée : oscillations entretenues (Directive telegraphy: sustained oscillations) Poincaré stated a necessary condition for the establishment of a stable regime of sustained oscillations in the *singing arc*. More precisely, he demonstrated the existence, in the phase plane, of a *stable limit cycle*.

To this aim Poincaré [31] studied Duddell's circuit that he represented by the following diagram (Fig. 3) consisting of an electromotive force (e.m.f.) of direct current E, a resistance R and a self-induction, and in parallel, a *singing arc* and another self-induction L and a capacitor.

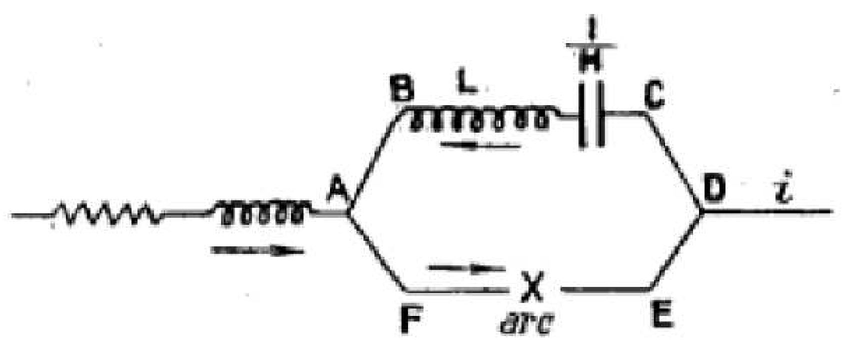

Figure 3: Circuit diagram of the singing arc, from Poincaré [31, p. 390].

Then, he called x the capacitor charge, x' the current intensity in the branch including the capacitor, $\rho x'$ the term corresponding to the internal resistance of the self and various damping and $\theta(x')$ the term representing the e.m.f. of the arc the mathematical modeling of which was also out of reach for Poincaré at that time. Nevertheless, Poincaré was able to establish the *singing arc equation*, i.e. the second order nonlinear differential equation (1) for the sustained oscillations in the *singing arc*:

$$Lx'' + \rho x' + \theta\left(x'\right) + Hx = 0 \tag{1}$$

Then, by using the *qualitative theory of differential equations* that he developed in his famous memoirs [33, 37], he stated that:

> "One can construct curves satisfying this differential equation, provided that function θ is known. Sustained oscillations correspond to closed curves, if there exist any. But every closed curve is not appropriate, it must fulfill certain conditions of stability that we will investigate."

Thus, he plotted a representation of the solution of equation (1):

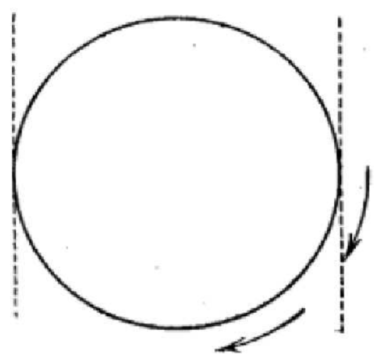

Figure 4: Closed curve solution of the *sing arc equation*, from Poincaré [31, p. 390].

Let's notice that this closed curve is only a *metaphor* of the solution since Poincaré does not use any graphical integration method such as *isoclines*. This representation led him to state the following *stability condition*:

> "*Stability condition.* – Let's consider another non-closed curve satisfying the differential equation, it will be a kind of spiral curve approaching indefinitely near the closed curve. If the closed curve represents a stable regime, by following the spiral in the

direction of the arrow one should be brought back to the closed curve, and provided that this condition is fulfilled the closed curve will represent a stable regime of sustained waves and will give rise to a solution of this problem."

Then, it clearly appears that the *closed curve* which represents a stable regime of sustained oscillations is nothing else but a *limit cycle* as Poincaré [34, p. 261] has introduced it in his own famous memoir "On the curves defined by differential equations" and as Poincaré [35, p. 25] has later defined it in the notice on his own scientific works [35]. But this, first *giant step* is not sufficient to prove the stability of the oscillating regime. Poincaré had to demonstrate now that the periodic solution of equation (1) (the *closed curve*) corresponds to a *stable limit cycle*. So, in the next part of his lectures, Poincaré gave what he calls a "condition de possibilité du problème". In fact, he established a stability condition of the periodic solution of equation (1), i.e. a stability condition of the limit cycle under the form of the following inequality.

$$\int \theta \left(x' \right) x' dt < 0 \tag{2}$$

It has been proved by Ginoux [12, 13, 16, 18, 19] that this stability condition (2) flows from a fundamental result introduced by Poincaré in the chapter titled "Exposants caractristiques" ("Characteristics exponents") of his "New Methods of Celestial Mechanics" [38, Vol. I, p. 180].

Until recently the historiography considered that Poincaré did not make any connection between sustained oscillations and the concept of limit cycle he had introduced and credited the Russian mathematician Aleksandr' Andronov [1, 2] for having been the "first" to establish this correspondence between periodic solution and limit cycle.

Concerning the *singing arc*, Van der Pol [49] also noted in the beginning of the thirties:

> "In the electric field we have some very nice examples of relaxation oscillations, some are very old, such as spark discharge of a plate machine, the oscillation of the electric arc studied by Mr. Blondel in a famous memoir (1) or the experience of Mr. JANET, and other more recent…"
>
> (1) BLONDEL, *Eclair. Elec.*, 44, 41, 81, 1905. See also *J. de Phys.*, 8, 153, 1919.

4 The triode: *from periodic solution to limit cycle*

In 1907, the American electrical engineer Lee de Forest (1873-1961) invented the *audion*. It was actually the first *triode* developed as a radio receiver detector. Curiously, it found little use until its amplifying ability was recognized around 1912 by several researchers. Then, it progressively replaced the *singing arc* in the *wireless telegraphy* devices and underwent a considerable development during the First World War. Thus, in October 1914, a few months after the beginning of the conflict, the French General Gustave Ferrié (1868-1932), director of the *Radiotélégraphie Militaire* department, gathered a team of specialists whose mission was to develop a French *audion*, which should be sturdy, have regular characteristics, and be easy to produce industrially. Ferrié asked to the French physicist Henri Abraham (1868-1943) to recreate Lee de Forests' audions. However, their fragile structure and lack of stability made them unsuitable for military use. After several unsuccessful attempts, Abraham created a fourth structure in December 1914, which was put in operation from February to October 1915 (Fig. 5).

Figure 5: Picture of the original lamp T.M. made by Abraham (1915).

The original of this valve called "Abraham lamp" is still in the Arts et Métiers museum to this day (Fig. 5). It has a cylindrical structure, which appears to have been designed by Abraham. In November 1917, Abraham consequently invented with his colleague Eugene Bloch (1878-1944) a device able to measure wireless telegraphy emitter frequencies: the so-called *multivibrator* (see Ginoux [13, 16, 18, 19]).

Wireless telegraphy development, spurred by war effort, went from craft to full industrialization. The triode valves were then marketed on a larger scale. More reliable and stable than the *singing arc*, the consistency of the various components used in the triode allowed for exact reproduction of experiments, which facilitated research on sustained oscillations.

4.1 Janet's analogy

In April 1919, the French scientist Paul Janet (1863-1937) published an article entitled "Sur une analogie électrotechnique des oscillations entretenues" [24] which was of considerable importance on several levels. Firstly, it underscored the technology transfer taking place, consisting in replacing an electromechanical component (*singing arc*) with what would later be called an electronic tube. This represented a true revolution since the *singing arc*, because of its structure it made experiments complex and tricky, making it almost impossible to recreate. Secondly, it revealed "technological analogy" between *sustained oscillations* produced by a *series dynamo machine* like the one used by Gérard-Lescuyer [20, 21] and the oscillations of the *singing arc* or a three-electrode valve (*triode*). Janet [24, p. 764] wrote:

> "It seemed to me interesting to mention the unexpected analogies of this experiment with the sustained oscillations so widely used to-day in wireless telegraphy, for example, those produced in Duddell's arc or in the lamp with three-electrodes lamps used as oscillators... Producing and sustaining oscillations in these systems mostly depends on the presence, in the oscillating circuit, of something comparable to a negative resistance. The dynamo-series acts as a negative resistance, and the engine with separated excitation acts as a capacity."

Thus, Janet considered that in order to have analogies in the effects, i.e. in order to see the same type of oscillations in the *series-dynamo machine*, the *triode* and the *singing arc*, there must be an analogy in the causes. Therefore, since the *series-dynamo machine* acts as a *negative resistance*, responsible for the oscillations, there is indeed an analogy. Consequently, only one equation must correspond to these devices. In this article, Janet provided the nonlinear differential equation characterizing the oscillations noted during Gérard-Lescuyer's experiment:

$$L\frac{d^2i}{dt^2} + \left[R - f'\left(i\right)\right]\frac{di}{dt} + \frac{k^2}{K}i = 0 \tag{3}$$

where R corresponds to the resistance of the series dynamo machine, L is the self-induction of the circuit and K/k^2 is analogous to a capacitor and $f(i)$ is the electromotive force of the *series-dynamo machine*. However, as recalled by Janet [24, p. 765], its mathematical modeling was also out of reach at that time.

> "But the phenomenon is limited by the characteristic's curvature, and regular, non-sinusoidal equations actually occur. They are governed by the equation (3), which could only be integrated if we knew the explicit for of the function $f(i)$."

By replacing in Eq. (3) i with x, R with ρ, $f'(i)$ with $\theta(x)$, and k^2/K with H, one find again Poincaré's *singing arc* equation (2). Thus, both ordinary differential equations are analogous but are not of the same order. Nevertheless, it appeared that Janet did make no connection with Poincaré's works.

4.2 Blondel's triode equation

According to the historiography, it is common knowledge the Dutch physicist Balathasar Van der Pol is credited for having stated the differential equation of the triode in his famous publication entitled "On relaxation oscillations" published in 1926 [45]. However, it was proved by Ginoux [13, 16, 17] on the one hand that the triode equation was actually stated by Van der Pol in 1920 in a publication entitled: "A theory of the amplitude of free and forced triode vibrations," [40] and on the other that the French engineer André Blondel sated the triode equation one year before him.

As previously pointed out, the main problem of these three devices was the mathematical modeling of their *oscillation characteristics*, i.e., the e.m.f. of the *series-dynamo machine*, of the *singing arc* and of the *triode*.

Thus, in a note published in the *Comptes Rendus* of the *Académie des Sciences* on the 17^{th} of November 1919, Blondel proposed to model the *oscillation characteristic* of the triode as follows [3]:

$$i = b_1 (u + k\nu) - b_3 (u + k\nu)^3 - b_5 (u + k\nu)^5 ... \tag{4}$$

Then, substituting i by its expression in the triode equation, neglecting the internal resistors and integrating once with respect to time, he obtained

$$C\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \left(b_1 h - 3b_3 h^3 u^2 - ...\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{u}{L} = 0 \tag{5}$$

Let's notice that this equation is perfectly equivalent to those obtained by Poincaré and Janet. Nevertheless, if Blondel solved the problem of the mathematical modeling of the *oscillation characteristic* of the *triode* he did make no connection with Poincaré's works despite of the fact that he knew him personally.

4.3 Pomey's contribution

Less than on year later, the French engineer Jean-Baptiste Pomey (1861-1943) proposed a mathematical modeling of the e.m.f. of the *singing arc* in his entitled: " Introduction à la théorie des courants téléphoniques et de la radiotélégraphie " and published on June 28^{th} 1920 (this detail would be of great importance in the following). Pomey [39, p. 375] wrote:

> "For the oscillations to be sustained it is not enough to have a periodic motion, it is necessary to have a stable motion."

Then, he proposed the following "law" for the e.m.f. of the *singing arc*:

$$E = E_0 + ai - bi^3 \tag{6}$$

and posing $i = x'$ (like Poincaré) he provided the nonlinear differential equation of the *singing arc*:

$$Lx'' + Rx' + \frac{1}{C}x = E_0 + ax' - bx'^3 \tag{7}$$

By posing $H = 1/C$, $\rho = R$ and $\theta(x') = -E_0 - ax' + bx'^3$ it is obvious that Eq. (1) and Eq. (7) are completely identical[2]. Moreover, it is striking to observe that Pomey has used exactly the same variable x' as Poincaré to represent the current intensity. Here again, there is no reference to Poincaré. This is very surprising since Pomey was present during the last lecture of Poincaré at the École Supérieure des Postes et Télégraphes in 1912 whose

[2] For more details see Ginoux [16, 17, 18, 19].

he had written the introduction. So, one can imagine that he could have attended the lecture of 1908.

At the same time, Van der Pol [40] proposed the following mathematical modeling of the *oscillation characteristic* of the *triode* in an article published on July 17, 1920:

$$i = \psi\left(kv\right) = \alpha v + \beta v^2 + \gamma v^3 \tag{8}$$

Van der Pol [40, p. 704] precised that, by symmetry consideration, one can choose $\beta = 0$ and provided the *triode* equation:

$$C\frac{d^2v}{dt^2} - \left(\alpha - 3\gamma v^2\right)\frac{dv}{dt} + \frac{1}{L}v = 0 \tag{9}$$

Taking into account that β can be chosen as equal to zero, one finds no difference between the Eq. (6) and the Eq. (8). Nevertheless, nothing proves that Van der Pol had read Pomey's book.

Five years later, on September 28^{th} 1925, Pomey wrote a letter to the mathematician Élie Cartan (1869-1951) in which he asked him to provide a condition for which the oscillations of an electrotechnics device analogous to the *singing arc* and to the *triode* whose equation is exactly that of Janet (3) are sustained. Within ten days, Élie Cartan and his son Henri sent an article entitled: "Note sur la génération des oscillations entretenues" [4] in which they proved the existence of a periodic solution for Janet's equation (3). In fact, their proof was based on a diagram which corresponds exactly to a "first return map" diagram introduced by Poincaré in his memoir "Sur les Courbes définies par une équation différentielle" [34, p. 251].

4.4 Van der Pol's relaxation oscillations

Van der Pol's most famous publication is probably that entitled "On relaxation oscillations" [45]. However, what is least well-known is that he published four different versions of this paper in 1926 in the following order:

1. Over Relaxatietrillingen [42] (in Dutch);
2. Over Relaxatie-trillingen [43] (in Dutch);
3. Über Relaxationsschwingungen [44] (in German);
4. On relaxation-oscillations [45] (in English).

In these four articles, Van der Pol presents the following generic dimensionless nonlinear differential equation for *relaxation oscillations* which is neither attached to the *triode*, nor to any other device (*series-dynamo machine* or *singing arc*):

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0. \tag{10}$$

Early on, Van der Pol [40, p. 179] realized that the equation (10) was not analytically integrable:

> "It has been found to be impossible to obtain an approximate analytical solution for (10) with the supplementing condition ($\varepsilon \ll 1$), but a graphical solution may be found in the following way."

So, he used the *isoclynes* method to graphically integrate the nonlinear differential equation (10) for the *relaxation oscillations*.

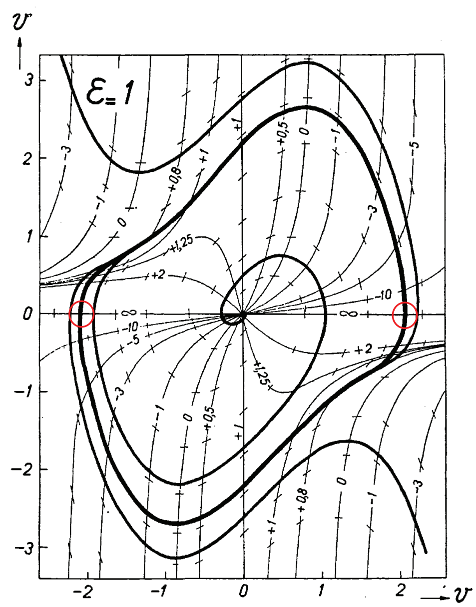

Figure 6: Graphical integration of equation (10)

Obviously, the solution plotted on this figure is nothing else but a *limit cycle* of Poincaré. Nevertheless, contrary to a widespread view, Van der Pol didn't recognize this signature of a *periodic solution* and did make no connection with Poincaré's works till 1930! On the occasion of a series of lectures that he made at the École supérieure d'Électricité on March 10^{th} and 11^{th} 1930, Van der Pol wrote [49]:

> "Note on each of these three figures a closed integral curve, which is an example of what Poincaré called a limit cycle, because the neighboring integral curves are approaching asymptotically."

Moreover, let's notice that he didn't make any reference to Poincaré's works but to Andronov's article [2].

4.5 Liénard's riddle

On May 1928, the French engineer Alfred Liénard (1869-1958) published an article entitled "Étude des oscillations entretenues" in which he studied the solution of the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} + \omega f(x) \frac{dx}{dt} + \omega^2 x = 0 \tag{11}$$

Such an equation is a generalization of the well-known Van der Pol's equation and of course of Janet's equation (4). Under certain assumptions on the function $F(x) = \int_0^x f(x)\, dx$ less restrictive than those chosen by Cartan [4] and Van der Pol [45], Liénard [26] proved the existence and uniqueness of a periodic solution of Eq. (11). Then, Liénard [26, p. 906] plotted this solution (Fig. 7) and wrote:

> "All integral curves, interior or exterior, traveled in the direction of increasing time, tend asymptotically to the curve D, we say that the corresponding periodic motion is a stable motion."

Then, Liénard [26, p. 906] explained that the condition for which the "periodic motion" is stable is given by the following inequality:

$$\int_\Gamma F(x)\, dy > 0 \tag{12}$$

By considering that the trajectory curve describes the closed curve clockwise in the case of Poincaré and counter clockwise in the case of Liénard, it is easy to show that both conditions (2) and (12) are completely identical[3] and represents an analogue of what is now called "orbital stability". Again, one

[3]For more details see Ginoux [13, 16, 17, 18, 19].

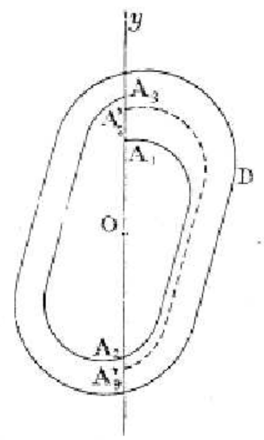

Figure 7: Closed curve solution of Eq. (10), Liénard [26].

can find no reference to Poincaré's works in Liénard's paper. Moreover, it is very surprising to observe that he didn't used the terminology "limit cycle" to describe its periodic solution. All these facts constitutes the Liénard's riddle.

4.6 Andronov's note at the *Comptes Rendus*

On Monday 14 October 1929, the French mathematician Jacques Hadamard (1865-1963) presented to the Académie des Sciences de Paris a note which was sent to him by Aleksandr Andronov and entitled "Poincaré's limit cycles and the theory of self-sustained oscillation". In this work, Andronov [2] proposed to transform the second order nonlinear differential equation modeling the sustained oscillations by the *series-dynamo machine*, the *singing arc* or the *triode* into the following set of two first order differential equations:

$$\frac{dx}{dt} = P(x, y) \quad ; \quad \frac{dy}{dt} = Q(x, y) \tag{13}$$

Then, he explained that the periodic solution of this system (13) is expressed in terms of Poincaré's limit cycles:

> "This results in self-oscillations which emerge in the systems characterized by the equation of type (13) corresponding mathematically to Poincaré's stable limit cycles."

It is important to notice that due to the imposed format of the *Comptes Rendus* (limited to four pages), Andronov did not provide any demonstration. He just claimed that the periodic solution of a non-linear second order differential equation defined by (13) "corresponds" to Poincaré's stable limit cycles. Then, Andronov provided a stability condition for the stability of the limit cycle:

$$\int_0^{2\pi} \left[f_x \left(R\cos\xi, -R\sin\xi; 0 \right) \cos\xi + g_y \left(R\cos\xi, -R\sin\xi; 0 \right) \sin\xi \right] d\xi < 0 \tag{14}$$

In fact, this condition is based on the use of *characteristic exponents* introduced by Poincaré in his so-called *New Methods on Celestial Mechanics* [38, Vol. I, p. 161] and after by Lyapounov in his famous textbook *General Problem of Stability of the Motion* [27]. That's the reason why Andronov will call later the stability condition (14): stability in the sense of Lyapounov or Lyapounov stability. It has been stated by Ginoux [12, 13, 16, 18, 19] that both stability condition of Poincaré (2) and of Andronov (14) are totally identical. Thus by comparing Andronov's previous sentence with that of Poincaré (see above), it clearly appears that Andronov has stated the same correspondence as Poincaré twenty years after him. Nevertheless, it seems that Andronov may not have read Poincaré's article since at that time even if the first volume of his complete works had been already published it didn't contained Poincaré's lectures on *Wireless Telegraphy.*

4.7 The first "lost" International Conference on Nonlinear Oscillations

From 28 to 30 January 1933 the first International Conference of Nonlinear Oscillations was held at the Institut Henri Poincaré (Paris) organized at the initiative of the Dutch physicist Balthasar Van der Pol and of the Russian mathematician Nikolaï Dmitrievich Papaleksi. This event, of which virtually no trace remains, was reported in an article written in Russian by Papaleksi at his return in USSR. This document, recently rediscovered by Ginoux [14], has revealed, on the one hand, the list of participants who included French mathematicians: Alfred Liénard, Élie and Henri Cartan, Henri Abraham, Eugène Bloch, Léon Brillouin, Yves Rocard . . . and, on the other hand the content of presentations and discussions. The analysis of the minutes of this conference highlights the role and involvement of the French scientific community in the development of the theory of nonlinear oscillations[4].

According to Papaleksi [29, p. 211], during his talk, Liénard recalled the main results of his study on sustained oscillations:

> "Starting from its graphical method for constructing integral curves of differential equations, he deduced the conditions that must satisfy the nonlinear characteristic of the system in order to have periodic oscillations, that is to say for that the integral curve to be a closed curve, i.e. a limit cycle."

This statement on Liénard must be considered with great caution. Indeed, one must keep in mind that Papaleksi had an excellent understanding of the work of Andronov [2] and that his report was also intended for members of the Academy of the USSR to which he must justified his presence in France at this conference in order to show the important diffusion of the Soviet work in Europe. Despite the presence of MM. Cartan, Lienard, Le Corbeiller and Rocard it does not appear that this conference has generated, for these scientists, a renewed interest in the problem of sustained oscillations and limit cycles.

5 The triode: *from limit cycle to "bizarre" solutions*

At the end of the First World War, the development of wireless telegraphy led the engineers and scientists to turn to the study of self-sustained oscillations in a three-electrode lamp subjected to a periodic "forcing" or a "coupling'". According to Mrs. Mary Lucy Cartwright [9]:

> "The non-linearity [in the Van der Pol equation] may be said to control the amplitude in the sense that it allows it to increase when it is small but prevents it becoming too large. The general solution cannot be obtained by the combination of two linearly independent solutions and similar difficulties arise when we add a forcing term to this equation. This was brought out very clearly by the work of van der Pol and Appleton, partly in collaboration, and partly independently, in a series of papers on radio oscillations published between 1920 and 1927. To me the work of the radio engineers is much more interesting and suggestive

[4]For more details see Ginoux [13, 16, 14, 18, 19].

than that of the mechanical engineers. The radio engineers want their systems to oscillate, and to oscillate in a very orderly way, and therefore they want to know not only whether the system has a periodic solution, but whether it is stable, what its period and amplitude and harmonic content are, and how these vary with the parameters of the equation, and they sometimes want the period to be determined with a very small error. In the early days they wanted to explain why the amplitude was limited in a certain way and why in some cases the period lengthened as the harmonic content increased and not in others. The desire to know why and the insistence on how the various quantities such as amplitude and frequency vary with the parameters of the equation over fairly wide ranges meant that numerical and graphical solutions either failed to provide the answer or were far too cumbersome. Further, unless one knows something about the general behavior of the solutions, the numerical work, which is only approximate, may be misleading."

Thus, in the beginning of the 1920s, Van der Pol [40] studied the oscillations of a forced triode, i.e. a triode powered by a voltage generator with an f.e.m. of type $v\left(t\right) = E_s in\left(\omega_1 t\right)$ the equation of which reads then:

$$\ddot{v} - \alpha\left(1 - v^2\right)\dot{v} + \omega_0^2 v = \omega_1^2 E_s in\left(n\omega_1 t\right) \quad \text{with} \quad \varepsilon = \frac{\alpha}{\omega_0} \ll 1 \qquad (15)$$

Four years later, while using the method of "slowly-varying amplitude" that he had developed, Van der Pol [41] was thus able on the one hand to obtain more directly the various approximations of the amplitude of this forced system, and on the other hand, to construct a solution to the equation more easily than by using the classical Poincar-Lindstedt or Fourier methods[5]. In this paper, Van der Pol [47] highlights the fact that when the difference in frequency of the two signals is inferior to this value an *automatic synchronization* phenomenon occurs and the two circuits oscillate with the same frequency. This led him to evidence the phenomenon of *frequency entrainment*, which he defined thus:

"Hence the free frequency undergoes a correction in the direction of the forced frequency, giving the impression as if the free frequency were being attracted by the forced frequency."

[5]The English version of this article was published in 1927. See Van der Pol [47].

In 1927, Van der Pol and his colleague Jan Van der Mark [48] published an article titled "Frequency Demultiplication," in which they again studied the forced oscillations of a triode, but in the field of *relaxation oscillations*. Then, they explained that the *automatic synchronization* phenomenon, observed in the case of the forced oscillations of a triode, can also occur for a range of the parameter corresponding to the *relaxation oscillations*, i.e. for $\varepsilon \gg 1$, but in a much wider frequency field. They also reported that the *resonance* phenomenon is almost non-existent in forced relaxation oscillations, and that consequently, the sinusoidal e.m.f. inducing the forcing influences the period (or frequency) of the oscillations more than it does their amplitude, and added:

> "It is found that the system is only capable of oscillating with *discrete frequencies*, these being determined by whole *sub-multiples of the applied frequency*."

In their article, Van der Pol and Van der Mark [48] proposed, in order to evidence the *frequency demultiplication* phenomenon, the following construction (see Fig. 8) on which we can see a "jump" of the period for each increase in the value of the capacitor's capacitance.

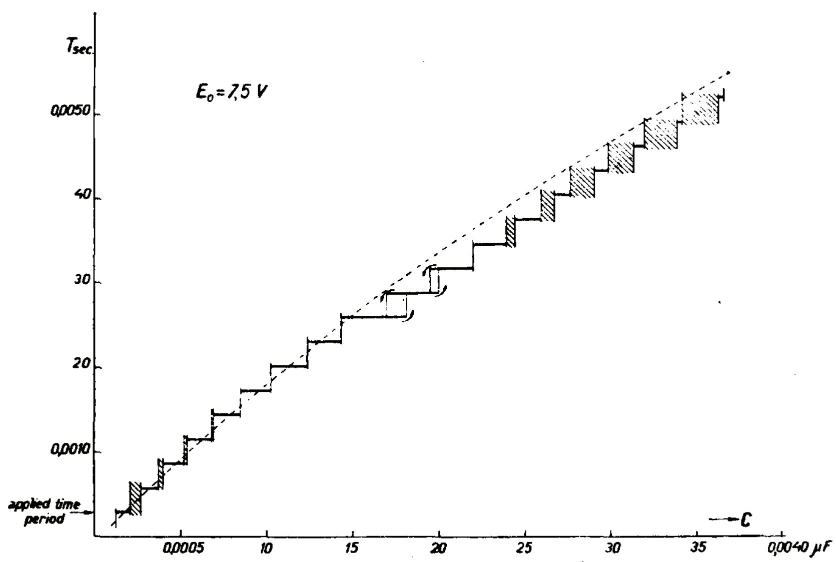

Figure 8: Representation of the phenomenon of *frequency demultiplication*, from Van der Pol et Van der Mark [48, p. 364].

In order to evidence this *frequency demultiplication* phenomenon, Van der Pol and Van der Mark used a phone. They then described the phenomenon what they heard in the receiver:

> "Often an irregular noise is heard in the telephone receivers before the frequency jumps to the next lower value. However, this is a subsidiary phenomenon, the main effect being the regular frequency multiplication."

This irregular noise they heard was actually the sound manifestation of the transition which was taking place. Indeed, as the frequency varied, the solution to the differential equation (15), which had been until now represented by a *limit cycle*, i.e. by a *periodic attractor*, would draw a "strange attractor" transcribing the *chaotic behavior* of the solution. Van der Pol seemed to have reached the limits of deterministic physics with how far he went in the exploration of nonlinear and non-autonomous systems. He "flirted", as Mary Lucy Cartwright and John Edensor Littlewood [5, 6, 7, 8] did twenty years later with the first signs of chaos, when they called "bizarre" the behavior of the solution to the differential equation (15) for specific values of the parameters. Indeed, according to Guckenheimer *et al.* [22]:

> "Van der Pol's work on nonlinear oscillations and circuit theory provided motivation for the seminal work of Cartwright and Littlewood. In 1938, just prior to World War II, the British Radio Research Board issued a request for mathematicians to consider the differential equations that arise in radio engineering. Responding to this request, Cartwright and Littlewood began studying the forced van der Pol equation and showed that it does indeed have bistable parameter regimes. In addition, they showed that there does not exist a smooth boundary between the basins of attraction of the stable periodic orbits. They discovered what is now called chaotic dynamics by detailed investigation of this system."

6 Conclusion

Thus, the analysis of the research performed on the following three devices: the *series-dynamo machine*, the *singing arc* and the *triode*, over a period ranging from the end of the XIXth century till the end of the Second World War, has enabled to reconstruct the historical road leading from nonlinear oscillations to chaos theory. The *series-dynamo machine* has highlighted a new kind of oscillations generated by the presence of a nonlinear component in the circuit, i.e. a *negative resistance*. Poincaré's work on the *singing arc* has provided an analytical condition for the sustaining of these oscillations, i.e. for the existence of a stable limit cycle. Moreover, this has proved that Poincaré has established twenty years before Andronov the correspondence between periodic solution and stable limit cycle. In his research on the *triode*, Blondel has solved the question of the mathematical modeling of its *oscillation characteristic*, i.e. of its *negative resistance* and stated thus, one year before Van der Pol, the *triode*'s equation. Then, Janet highlighted an analogy between the oscillations sustained by the *series-dynamo machine*, the *singing arc* and the *triode* and Van der Pol deduced that they were belonging to the same oscillatory phenomenon that he called *relaxation oscillations*. Though he plotted the solution of the equation that now bears his name, he didn't recognize that it was obviously a Poincaré's limit cycle. Thereafter, Cartan and then Liénard proved the existence and uniqueness of this periodic solution but did not make either a connection with Poincaré's works. Immediately after Andronov established this connection, Van der Pol and Papaleksi organized the first *International Conference on Nonlinear Oscillations* in Paris. Nevertheless, this meeting did not lead to any development or research in this field. At the same time, Van der Pol and Van der Mark highlighted that the forced triode was the source of a strange phenomenon that they called *frequency demultiplication*. At the end of the Second World War, Cartwright and Littlewood investigated this system and considered its oscillations as "bizarre". Many years later, it appeared that they had actually observed the first *chaotic behavior*.

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Variable Elasticity of Substitution in the Diamond Model: Dynamics and Comparisons

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Abstract. We study the dynamics shown by the discrete time Diamond overlapping-generations model with the VES production function in the form given by Revankar[10] and compare our results with those obtained by Brianzoni *et al.*[2] in the Solow model. We prove that, as in Brianzoni *et al.*[2], unbounded endogenous growth can emerge if the elasticity of substitution is greater than one; moreover, differently from Brianzoni *et al.*[2], the Diamond model can admit two positive steady states. We also prove that complex dynamics occur if the elasticity of substitution between production factors is less than one, confirming the results obtained by Brianzoni *et al.*[2]. Numerical simulations support the analysis.
Keywords: Variable Elasticity of Substitution, Diamond Growth Model, Fluctuations and Chaos, Bifurcation in Piecewise Smooth Dynamical Systems.

1 Introduction

The elasticity of substitution between production factors plays a crucial role in the theory of economic growth, it being one of the determinants of the economic growth level (see Klump and de La Grandville[6]).

Within the Solow model (see Solow[11], and Swan[12]) it was found that a country exhibiting a higher elasticity of substitution experiences greater capital (and output) per capita levels in the equilibrium state (see Klump and de La Grandville[6], Klump and Preissler[7], and Masanjala and Papageorgiou[8]). More recently, the role of the elasticity of substitution between production factors in the long run dynamics of the Solow model was investigated both considering the Constant Elasticity of Substitution production function (CES) (see Brianzoni *et al.*[1]) and the Variable Elasticity of Substitution production function (VES) (see Brianzoni *et al.*[2]). The results obtained demonstrate

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that fluctuations may arise if the elasticity of substitution between production factors falls below one.

Miyagiwa and Papageorgiou[9] moved the attention to the Diamond overlapping-generations model (Diamond[4]) while proving that, differently from the Solow setup, "if capital and labor are relatively substitutable, a country with a greater elasticity of substitution exhibits lower per capita output growth in both transient and steady state". To reach this conclusion they considered the normalized CES production function.

In the present work we consider the Diamond overlapping-generations model with the VES production function in the form given by Revankar[10] (see also Karagiannis *et al.*[5]). Our main goal is to study the local and global dynamics of the model to verify if the main result obtained by Brianzoni *et al.*[2] in the Solow model, i.e. cycles and complex dynamics may emerge if the elasticity of substitution between production factors is sufficiently low, still holds in the Diamond framework.

To summarize, the qualitative and quantitative long run dynamics of the Diamond growth model with VES production function are studied, to show that complex features can be observed and to compare the results obtained with the ones reached while considering the CES technology or the Solow framework.

2 The economic setup

Consider a discrete time setup, $t \in \mathbb{N}$, and let $y_t = f(k_t)$ be the production function in intensive form, mapping capital per worker k_t into output per worker y_t. Following Karagiannis *et al.*[5] we consider the Variable Elasticity of Substitution (VES) production function in intensive form with constant return to scale, as given by Revankar[10]:

$$y_t = f(k_t) = Ak_t^a[1 + bak_t]^{(1-a)}, \quad k_t \geq 0 \tag{1}$$

where $A > 0, \quad 0 < a < 1, \quad b \geq -1$; furthermore $1/k_t \geq -b$, in order to assure that $f(k_t) > 0$, $f'(k_t) > 0$ and $f''(k_t) < 0$, $\forall k_t > 0$, where

$$f'(k_t) = Aak_t^a(1 + abk_t)^{1-a}[k_t^{-1} + (1-a)b(1 + abk_t)^{-1}]$$

and

$$f''(k_t) = A\frac{a(a-1)(1 + abk_t)^{-a-1}}{k_t^{2-a}}.$$

The elasticity of substitution between production factors is then given by

$$\sigma(k_t) = 1 + bk_t$$

hence $\sigma \geq (<)1$ iff $b \geq (<)0$. Thus the elasticity of substitution varies with the level of capital per capita, representing an index of economic development. Observe that, while the elasticity of substitution for the CES is constant along an isoquant, in the case of the VES it is constant only along a ray through the origin.

In the Diamond[4] overlapping-generations model a new generation is born at the beginning of every period. Agents are identical and live for two periods. In the first period each agent supplies a unit of labor inelastically and receives a competitive wage:

$$w_t = f(k_t) - k_t f'(k_t),$$

thus, taking into account the specification of the production function in (1), we obtain

$$w_t = Ak_t^a \frac{(1+2abk_t)(1-a)}{(1+abk_t)^a}. \tag{2}$$

As in Miyagiwa and Papageorgiou[9] we assume that agents save a fixed proportion $s \in (0,1)$ of the wage income to finance consumption in the second period of their lives. All savings are invested as capital to be used in the next period's production, so that the evolution of capital per capita is described by the following map

$$k_{t+1} = \phi(k_t) = \frac{s}{1+n} w_t = \frac{sA}{1+n} k_t^a \frac{(1+2abk_t)(1-a)}{(1+abk_t)^a}, \tag{3}$$

where $n > 0$ is the exogenous labor growth rate and capital depreciates fully.

As in Brianzoni *et al.*[2] we distinguish between the following two cases.

(a) If $b > 0$ the elasticity of substitution between production factors is greater than one and the standard properties of the production function are verified $\forall k_t > 0$; in this case k_t evolves according to (3).We do not consider the case $b = 0$ as $\sigma(k_t)$ becomes constant and equal to one, $\forall k_t \geq 0$, thus obtaining a particular case of the CES production function.

(b) If $b \in [-1, 0)$ the elasticity of substitution between production factors is less than one and the standard properties of the production function are verified for all $0 < k_t < -\frac{1}{b}$; in this case k_t evolves according to (3) iff $k_t \in [0, -1/b]$ while, following Karagiannis *et al.*[5] and Brianzoni *et al.*[2], if $k_t > -1/b$ then $k_t = \phi(-1/b)$.

3 Local and Global Dynamics

3.1 Elasticity of Substitution Greater than One

Let $b > 0$. Then the discrete time evolution of the capital per capita k_t is described by the continuous and differentiable map (3).

The establishment of the number of steady states is not trivial to solve, considering the high variety of parameters. As a generale result, the map ϕ always admits one fixed point characterized by zero capital per capita, i.e. $k = 0$ is a fixed point for any choice of parameter values. Anyway steady states which are economically interesting are those characterized by positive capital per worker. In order to determine the positive fixed points of ϕ, let us define the following function:

$$G(k) = \frac{1-a}{k^{1-a}} \frac{1+2abk}{(1+abk)^a}, k > 0 \tag{4}$$

where

$$G'(k) = \frac{(1-a)}{k^{2-a}(1+abk)^{1+a}}[(a-1)+(2a-1)abk], \qquad (5)$$

then solutions of $G(k) = \frac{1+n}{sA}$ are positive fixed points of ϕ.

The following proposition establishes the number of fixed points of map ϕ.

Proposition 1 *Let ϕ given by (3).*

(i) Assume $b > 0$ and $a \le \frac{1}{2}$. Then:
   (a) if $\frac{1+n}{sA} > (ab)^{1-a}2(1-a)$, ϕ has two fixed points given by $k_t = 0$ and $k_t = k^* > 0$;
   (b) if $0 < \frac{1+n}{sA} \le (ab)^{1-a}2(1-a)$, ϕ has a unique fixed point given by $k_t = 0$.

(ii) Assume $b > 0$ and $a > \frac{1}{2}$ and let $k_m = \frac{1-a}{ab(2a-1)}$. Then:
   (a) if $\frac{1+n}{sA} < (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$, ϕ has a unique fixed point given by $k_t = 0$;
   (b) if $\frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$, ϕ has two fixed points given by $k_t = 0$ and $k^* = k_m$;
   (c) if $(\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a)$, ϕ has three fixed points given by $k_t = 0$, $k_t = k_1$ and $k_t = k_2$, where $0 < k_1 < k_m < k_2$;
   (d) if $\frac{1+n}{sA} \ge (ab)^{1-a}2(1-a)$, ϕ has two fixed points given by $k_t = 0$ and $k^* > 0$, where $0 < k^* < k_m$.

Proof. $k_t = 0$ is a solution of equation $k_t = \phi(k_t)$ for all parameter values hence it is a fixed point. Function (4) is such that $G(k_t) > 0$ for all $k_t > 0$, furthermore $\lim_{k_t \to 0^+} G(k_t) = +\infty$ while $\lim_{k_t \to +\infty} G(k_t) = (ab)^{1-a}2(1-a)$.

(i) Observe that if $b > 0$ and $a \le \frac{1}{2}$, $G(k)$ is strictly decreasing $\forall k_t > 0$ since $G'(k) < 0$. Hence $G(k_t)$ intersects the positive and constant function $g = \frac{1+n}{sA}$ in a unique positive value $k_t = k^*$ iff $\frac{1+n}{sA} > (ab)^{1-a}2(1-a)$.

(ii) Assume $a > \frac{1}{2}$ and $b > 0$ then G has a unique minimum point $k_m = \frac{1-a}{ab(2a-1)}$ where $G(k_m) = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$. Hence, if $(\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a}) < \frac{1+n}{sA} < (ab)^{1-a}2(1-a)$, then equation $G(k_t) = \frac{1+n}{sA}$ admits two positive solutions. Similarly, it can be observed that if $\frac{1+n}{sA} = (\frac{a^2b}{1-a})^{1-a}(\frac{1-a}{a})$ or $\frac{1+n}{sA} \ge (ab)^{1-a}2(1-a)$ then $\phi(k_t)$ admits a unique positive fixed point. Trivially, for the other parameter values, equation $G(k_t) = \frac{1+n}{sA}$ has no positive solutions.

For what it concerns the local stability of the steady states the following proposition holds.

Proposition 2 *Let ϕ be as given in (3).*

(i) The equilibrium $k_t = 0$ is locally unstable for all parameter values.

(ii) If ϕ admits two fixed points then the equilibrium $k_t = k^* > 0$ is locally stable.

(iii) If ϕ admits three fixed points, then the equilibrium $k_t = k_1$ is locally stable while the equilibrium $k_t = k_2$ is locally unstable.

Proof. Firstly notice that function ϕ may be written in terms of function G being:

$$\phi(k) = \frac{sA}{1+n} kG(k) \tag{6}$$

hence $\phi'(k) = \frac{sA}{1+n}[G(k) + kG'(k)]$.

(i) Since $\lim_{k_t \to 0^+} G(k_t) = +\infty$ and $\lim_{k_t \to 0^+} kG'(k_t) = +\infty$, then $\phi'(0) = +\infty$ and consequently the origin is a locally unstable fixed point for map ϕ.

(ii) Assume that ϕ admits two fixed points. After some algebra it can be noticed that

$$\phi'(k) = \frac{a(1+a)sA}{1+n} \frac{(1+abk)^{-1-a}}{k^{1-a}} [2ab^2k^2 + 2b(1+a)k + 1] > 0 \qquad \forall k > 0 \tag{7}$$

hence $\phi(k)$ is strictly increasing and consequently k^* is locally stable. In the particular case in which $\frac{1+n}{sA} = (\frac{a^2 b}{1-a})^{1-a}(\frac{1-a}{a})$ then $k^* = k_m$ is a non hyperbolic fixed point: a tangent bifurcation occurs at which k^* is locally stable.

(iii) Assume that ϕ has three equilibria. Since $\phi'(k) > 0 \; \forall k > 0$ then point *(iii)* is easily proved.

The results concernig the existence and number of fixed points and their local stability when the elasticity of substitution between production factors is greater than one, are resumed in Fig. 1. We fix all the parameters but s and we show that, as s is increased, we pass from two to three and, finally, to one fixed point. Hence it can be observed that unbounded growth can emerge if the propensity to save in sufficiently high.

As in Brianzoni *et al.*[2], if the elasticity of substitution between the two factors is greater than one ($b > 0$), then unbounded endogenous growth can be observed but only simple dynamics can be produced. Anyway, differently from Brianzoni *et al.*[2], the growth model can exhibit two positive steady states so that the final outcome of the economy depends on the initial condition (in fact if $k_0 \in (0, k_2)$ then the convergence toward k_1 is observed while if $k_0 > k_2$ then unbounded endogenous growth is exhibited).

3.2 Elasticity of Substitution Less than One

Let $b \in [-1, 0)$. Then the discrete time evolution of the capital per capita k_t is described by the following continuous and piecewise smooth map:

$$k_{t+1} = F(k_t) = \begin{cases} \phi(k_t) & \forall k_t \in \left[0, -\frac{1}{b}\right] \\ \phi\left(-\frac{1}{b}\right) & \forall k_t > -\frac{1}{b} \end{cases} . \tag{8}$$

As it is easy to verify, F is non-differentiable in the point $k_t = -\frac{1}{b}$, which separates the state space into two regions $R_1 = \{(k) : 0 \leq k < -\frac{1}{b}\}$ and $R_2 = \{(k) : k > -\frac{1}{b}\}$. Furthermore, the map F is constant for $k_t > -\frac{1}{b}$ and non-linear for $0 \leq k_t \leq -\frac{1}{b}$. The following proposition describes the number of fixed points when $b \in [-1, 0)$.

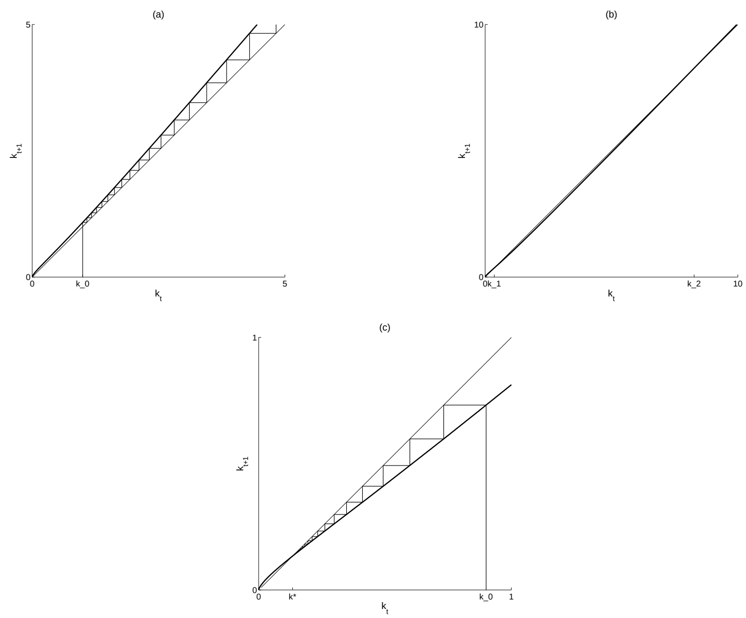

Fig. 1. Map ϕ, its fixed points and their stability for $b = 1$, $a = 0.7$, $A = 3$ and $n = 0.1$. (a) $s = 0.8$, (b) $s = 0.7$ and (c) $s = 0.6$.

Proposition 3 *Let F be as given in (8) and $b \in [-1, 0)$.*

(i) Assume $a > \frac{1}{2}$. Then F has two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{2ab})$.

(ii) Assume $a \leq \frac{1}{2}$ and $M = \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$. Then:
  (a) if $\frac{1+n}{sA} \geq M$ there exist two fixed points given by $k = 0$ and $k^* \in (0, -\frac{1}{b}]$;
  (b) if $\frac{1+n}{sA} < M$ there exist two fixed points given by $k = 0$ and $k^* = F(-\frac{1}{b})$.

Proof. It is easy to see that $k = 0$ is a fixed point for any choice of the parameter values.

(i) Firstly notice that $\phi \geq 0$ iff $k \in [0, -\frac{1}{2ab}]$ and $\phi(0) = \phi(-\frac{1}{2ab}) = 0$, so values of $k > -\frac{1}{2ab}$ are not economically significant. Moreover ϕ has a unique maximum point given by $k_M = \frac{-1-a+\sqrt{1+a^2}}{2ab}$ with $\phi(k_M) = \frac{sA}{1+n}\left(\frac{\sqrt{1+a^2}-1-a}{ab\sqrt{1+a^2}+1-a}\right)^a (1-a)(\sqrt{1+a^2}-a)$. Finally $\lim_{k\to 0^+} \phi'(k) = \infty$. Hence equation $\phi(k) = k$ has always a unique positive solution given by $k^* \in (0, -\frac{1}{2ab})$.

(ii) The positive fixed points of F such that $k \leq -\frac{1}{b}$ are given by the solutions of equation $G(k) = \frac{1+n}{sA}$ with $G(k)$ as given in (4) and $G > 0$ defined in $(0, -\frac{1}{b}]$. Being $G'(k) = \frac{(1-a)}{k^{2-a}(1+abk)^{1+a}}[(a-1) + (2a-1)abk]$, G is strictly decreasing $\forall k \in (0, -\frac{1}{b}]$ with minimum point in $k_m = -\frac{1}{b}$ and $G(k_m) = G(-\frac{1}{b}) = \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} = M$. Hence $G(k) = \frac{1+n}{sA}$ has a unique positive solution $k^* \in (0, -\frac{1}{b}]$ iff $\frac{1+n}{sA} \geq \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$. Differently, the unique fixed point of F such that $k > -\frac{1}{b}$ is defined by $k^* = F(-\frac{1}{b}) = \phi(-\frac{1}{b})$ and it exists iff $F(-\frac{1}{b}) > -\frac{1}{b}$, which is equivalent to require $\frac{1+n}{sA} < \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}}$.

Let us move to the study of the local stability of the fixed points. Since

$$\phi'(k) = \frac{a(1+a)sA}{1+n}\frac{(1+abk)^{-1-a}}{k^{1-a}}[2ab^2k^2 + 2b(1+a)k + 1]$$

then $\lim_{k\to 0^+} \phi'(k) = +\infty$, so that the equilibrium characterized by zero capital-per capita is always locally unstable.

We firstly focus on the case with $a > \frac{1}{2}$. As it has been discussed, the related one dimensional map is continuous and differentiable in its domain $[0, -\frac{1}{2ab}]$. Furthermore, $\phi(0) = \phi(-\frac{1}{2ab}) = 0$ and $\phi''(k) < 0$ $\forall k \in (0, -\frac{1}{2ab})$, i.e. it is strictly concave. As a consequence map ϕ behaves as the logistic map, that is it exhibits the standard period doubling bifurcation cascade as one parameter is moved (see Devaney[3]).

The period doubling bifurcation cascade is observed, for instance, if A is increased. In fact it can be easily observed that $\phi(k_M)$ increases as A increases so that $\exists \bar{A}$ such that $\phi(k_M) > -\frac{1}{2ab}$ $\forall A > \bar{A}$, i.e. almost all trajectories are unfeasible. At $A = \bar{A}$ a final bifurcation occurs (the origin is a pre-periodic fixed point and ϕ is chaotic in a Cantor set) while $\forall A \in (0, \bar{A})$ the period doubling bifurcation cascade is observed (see Fig. 2 (a),(b) and (e)). Notice also that the situation presented in panel (b) becomes simpler if a greater value of b is considered (see Fig. 2 (c)), proving that in order to have complex dynamics b must be sufficiently low (as also showed in panel (d)).

In order to study the local stability of the positive fixed point when $a \leq \frac{1}{2}$ and $b \in [-1, 0)$ we observe that function F has a non differentiable point given by

$$P = \left(-\frac{1}{b}, F(-\frac{1}{b})\right), \tag{9}$$

where $F(-\frac{1}{b}) = \frac{sA}{1+n}(-b)^{-a}(1-a)^{1-a}(1-2a)$.

Notice that if P is above the main diagonal, the fixed point k^* is superstable being $F'(k^*) = 0$ and no complex dynamics can be exhibited. This case occurs, for instance, if A is great enough and the related situation is presented in Fig. 3 (a). If $\frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} = \frac{1+n}{sA}$ we get that $k^* = -\frac{1}{b}$, then a border collision of the superstable fixed point occurs.

If P is below the main diagonal then k^* may be locally stable or unstable and complex dynamics may arise.

The following Proposition states a sufficient condition for the existence of a stable 2-period cycle $\{k_1, k_2\}$ such that $k_i \in R_i, (i = 1, 2)$.

Proposition 4 *Let $b \in [-1, 0)$. For all b in the region defined as*

$$\Omega = \left\{b : F^2(-\frac{1}{b}) > -\frac{1}{b} \cap \frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} < \frac{1+n}{sA}\right\} \tag{10}$$

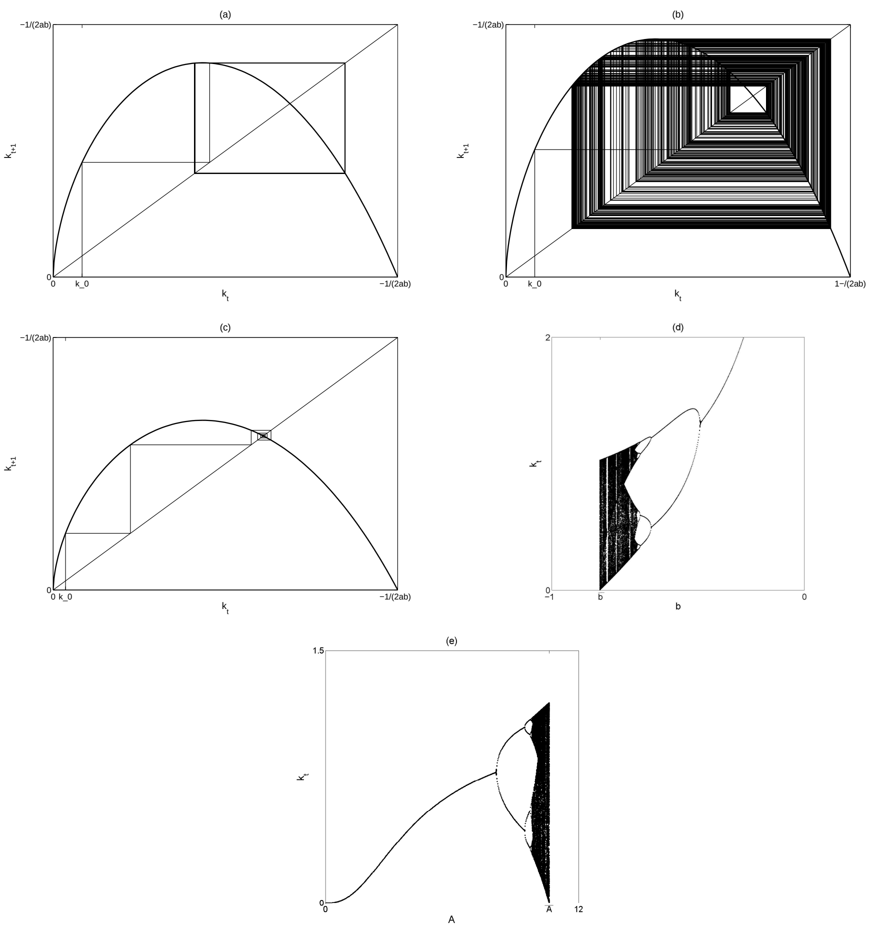

Fig. 2. $a = 0.6$, $n = 0.1$, $s = 0.7$. (a) If $b = -0.7$ and $A = 9$ a stable two period cycle is presented, while (b) if $A = 10$ complexity emerges. (c) Locally stable fixed point for $A = 10$ and $b = -0.3$. (d) Bifurcation diagram w.r.t. b. (e) Bifurcation diagram w.r.t. A.

map F admits a superstable 2-period cycle defined as $C_2 = \{F(-\frac{1}{b}), F^2(-\frac{1}{b})\}$.

Proof. A 2-cycle for map F is given by $\{k_1, k_2\}$ with $F(k_1) = k_2$ and $F(k_2) = k_1$. Let $k_0 > -\frac{1}{b}$ with $k_0 \in R_2$, then $k_1 = F(-\frac{1}{b})$ belongs to R_1 (being the point P below the main diagonal) and $k_2 = F(k_1) = F(F(-\frac{1}{b})) = F^2(-\frac{1}{b})$. If $F^2(-\frac{1}{b}) > -\frac{1}{b}$, then $F^2(-\frac{1}{b}) \in R_2$ and consequently $F(F^2(-\frac{1}{b})) = F(k_2) = F(-\frac{1}{b}) = k_1$. This proves the existence of a two period cycle. Moreover, the eigenvalue of such cycle is zero, since $F'(k_2) = 0$, therefore it is a superstable two period cycle.

Notice that in $F^2(-\frac{1}{b}) = -\frac{1}{b}$ a border collision bifurcation of the superstable 2-period cycle occurs. The superstable two period cycle is depicted in Fig. 3 (b).

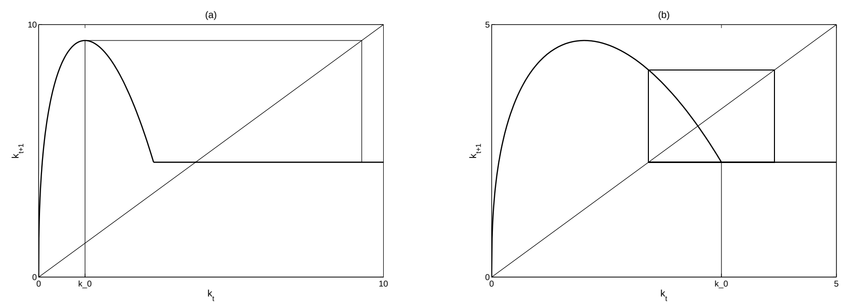

Fig. 3. $a = 0.4$, $n = 0.1$, $s = 0.7$. (a) If $b = -0.3$ and $A = 30$ the positive steady state is superstable. (b) The superstable two period cycle for $b = -0.3$ and $A = 15$.

In order to describe how complex dynamics may emerge if $a \leq \frac{1}{2}$, we recall that F is unimodal and $k_M = \frac{-1-a+\sqrt{1+a^2}}{2ab}$ is its maximum point.

If $k^* \in (0, k_M)$ (i.e. point $(k_M, F(k_M))$ is below the main diagonal), then k^* is globally stable $\forall k_0 \neq 0$. On the contrary, if $F(k_M) > k_M$ (i.e. point $(k_M, F(k_M))$ is above the main diagonal), then its eigenvalue is negative and k^* may lose stability only via a period-doubling bifurcation. Therefore, a necessary condition for a flip bifurcation is that that point $(k_M, F(k_M))$ is above the main diagonal.

To recap, as in Brianzoni *et al.*[2], our model can exhibit cycles or more complex dynamics iff P is below the main diagonal while the maximum point k_M is above the main diagonal. In this case all positive initial conditions produce trajectories converging to an attractor belonging to a trapping interval J defined in the following proposition.

Proposition 5 *Let* $\frac{(-b)^{1-a}(1-2a)}{(1-a)^{a-1}} < \frac{1+n}{sA}$ *and* $F(k_M) > k_M$*. Then the one-dimensional map* F *admits a trapping interval* J*, where* J *is defined as follows:*

1. $J = [F(-\frac{1}{b}), F(k_M)]$ *if* $F(k_M) \geq -\frac{1}{b}$*,*
2. $J = [F^2(k_M), F(k_M)]$ *if* $F(k_M) < -\frac{1}{b}$*.*

Proof. If the one-dimensional map F has a maximum point k_M above the main diagonal and point P is below the main diagonal, then through the graphical analysis it is possible to see that when the image of k_M belongs to $R_2 \cup \{-\frac{1}{b}\}$, then $J = [F(-\frac{1}{b}), F(k_M)]$ is mapped into itself; otherwise $J = [F^2(k_M), F(k_M)]$ is mapped into itself by F.

Since every initial condition $k_0 \neq 0$ creates bounded trajectories converging to an attractor included into the trapping interval J, it can be noticed that if $F(k_M) \geq -\frac{1}{b}$, the flat branch of map F is involved. Moreover, since all the points mapped in R_2 have the same trajectory of point $F(-\frac{1}{b})$, then the attractor will be a cycle. The transition from $F(k_M) \geq -\frac{1}{b}$ to $F(k_M) < -\frac{1}{b}$ corresponds to a border collision bifurcation.

In order to describe the qualitative dynamics occurring on set J, we consider the situation in which $k^* \in R_1$ is locally stable (as in Fig. 4 (a)), for instance b is close to zero. Then, as b decreases, k^* becomes unstable via flip bifurcation and a period doubling route to chaos occurs till a border collision bifurcation emerges at $F(k_M) = -\frac{1}{b}$. This bifurcation occurs at $b = b_c$ and a point of the attractor of F collides with point P. In Fig. 4 (b) and (c) the situations immediately before and after the border collision bifurcation occurring at $b_c \simeq -0.315$ are presented. Notice that after this bifurcation the qualitative dynamics drastically changes, passing from a complex attractor to a locally stable 5-period cycle. The related bifurcation diagram is presented in Fig. 4 (d).

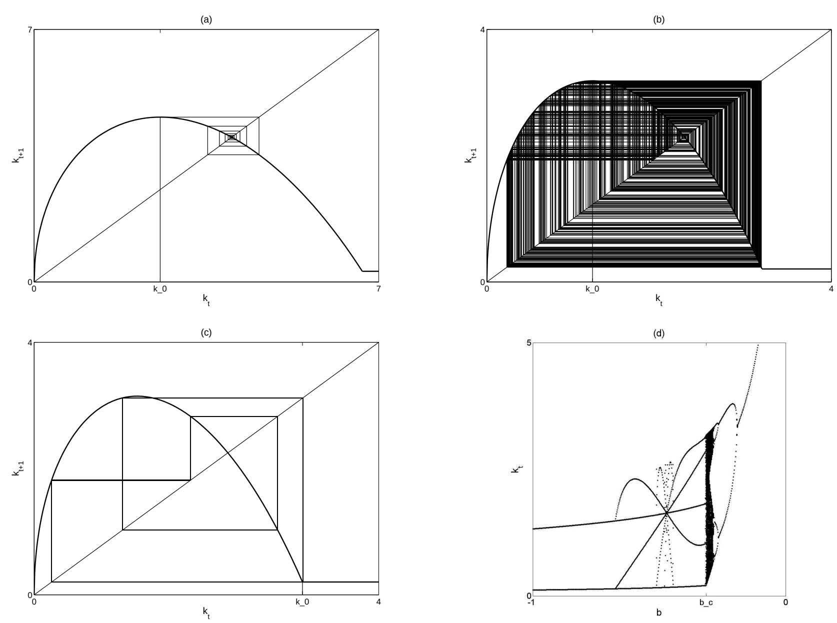

Fig. 4. $A = 10$, $a = 0.49$, $n = 0.1$, $s = 0.9$. (a) If $b = -0.15$ the fixed point is locally stable. (b) Situation before the border collision bifurcation, i.e. $b = -0.314$. (c) Situation immediately after the border collision bifurcation, i.e. $b = -0.316$. (d) Bifurcation diagram w.r.t. b.

As in Brianzoni *et al.*[2] if elasticity of substitution between production factors in less then one, then the system becomes more complex as b decreases since cycles or more complex features may be exhibited.

4 Conclusions

In this paper we considered the Diamond overlapping-generations model with the VES production function in the form given by Revankar[10]. We examined existence and stability conditions for steady state and the results of our analysis show that fluctuation or even chaotic patterns can be exhibited. As in Brianzoni *et al.*[2], cycles or complex dynamics can emerge if the elasticity of substitution between production factors is low enough. Moreover, unbounded endogenous growth can be observed. A new feature is due to the fact that, if elasticity of substitution is greater then one, then up to three fixed point can be exhibited.

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Self-organization and fractality created by gluconeogenesis in the metabolic process

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Abstract. Within a mathematical model, the process of interaction of the metabolic processes such as glycolysis and gluconeogenesis is studied. As a result of the running of two opposite processes in a cell, the conditions for their interaction and the self-organization in a single dissipative system are created. The reasons for the appearance of autocatalysis in the given system and autocatalytic oscillations are studied. With the help of a phase-parametric diagram, the scenario of their appearance is analyzed. The bifurcations of the doubling of a period and the transition to chaotic oscillations according to the Feigenbaum scenario and the intermittence are determined. The obtained strange attractors are created as a result of the formation of a mixing funnel. Their complete spectra of Lyapunov indices, KS-entropies, "horizons of predictability," and the Lyapunov dimensions of strange attractors are calculated. The conclusions about the reasons for variations of the cyclicity in the given metabolic process, its stability, and the physiological state of a cell are made.

Keywords: *Gluconeogenesis, glycolysis, metabolic process, self-organization, fractality, strange attractor, Feigenbaum scenario.*

1 Introduction

Gluconeogenesis is a biochemical process of formation of glucose from hydrocarbonless predecessors such as pyruvates, aminoacids, and glycerin. The biosynthesis of glucose runs analogously to glycolysis, but in the reverse direction. Gluconeogenesis is realized by means of the inversion of seven invertible stages of glycolysis. Three remaining stages of glycolysis are exergenous and, therefore, irreversible. They are replaced by three "by-pass reactions" that are thermodynamically gained for the synthesis of glucose. Since gluconeogenesis uses the same invertible reactions, as glycolysis does, its biochemical evolution occurred, apparently, jointly with glycolysis. Maybe, the symbiosis of these biochemical processes arose else in protobionts 3.5 bln years in Earth's oxygenless atmosphere. It can be considered as one of the primary open nonlinear biochemical systems, being far from the equilibrium. The self-organization of the given biochemical system resulted in the appearance of a stable dissipative system independent of other biochemical processes of a primary broth. The directedness of the running of a reaction in it was determined by the energy-gained balance. The organic molecule ATP, which was formed as a result of glycolysis, became the principal carrier of the energy consumed in all other biochemical processes. This created the conditions of self-

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organization of other biochemical processes that used ATP as the input product of a reaction. But if the need in glucose arose in other biochemical processes, the directedness of biochemical reactions in the system was changed by the opposite one. In the course of the subsequent biochemical evolution, the given dissipative system is conserved and is present in cells of all types, which indicates their common prehistory.

Thus, the studies of the reactions of gluconeogenesis are determined in many aspects by the results of studies of glycolysis. The direct sequence of reactions with the known input and output products is studied easier than the reverse one.

The experimental studies of glycolysis discovered autooscillations [1]. In order to explain their origin, a number of mathematical models were developed [2-4]. Sel'kov explained the appearance of those oscillations by the activation of phosphofructokinase by its product. In the Goldbeter--Lefever model, the origin of autooscillations was explained by the allosteric nature of the enzyme. Some other models are available in [5-7].

The present work is based on the mathematical model of glycolysis and gluconeogenesis, which was developed by Professor V.P. Gachok and his coauthors [8-10]. The peculiarity of his model consists in the consideration of the influence of the adeninenucleotide cycle and gluconeogenesis on the phosphofructokinase complex of the given allosteric enzyme. This allowed one to study the effect of these factors as the reason for the appearance of oscillations in glycolysis.

At the present time, this model is improved and studied with the purpose to investigate gluconeogenesis. Some equations were added, and some equations were modified in order to describe the complete closed chain of the metabolic process of glycolysis-gluconeogenesis under anaerobic conditions. The developed complete model allows us to consider glycolysis-gluconeogenesis as a united integral dissipative structure with a positive feedback formed by the transfer of charges with the help of NAD. Glycolysis with gluconeogenesis is considered as an open section of the biosystem, which is self-organized by itself at the expense of input and output products of the reaction in a cell, which is a condition of its survival and the evolution. The appearance of an autocatalytic process in the given dissipative structure can be a cause of oscillatory modes in the metabolic process of the whole cell.

Gluconeogenesis occurs in animals, plants, fungi, and microorganisms. Its reactions are identical in all tissues and biological species. Phototrophs transform the products of the own photosynthesis in glucose with the help of gluconeogenesis. Many microorganisms use this process for the production of glucose from a medium, where they live.

The conditions modeled in the present work are established in muscles after an intense physical load and the formation of a large amount of lactic acid in them. As a result of the running of the reverse reaction of gluconeogenesis, it is transformed again in glucose.

2 Mathematical Model

The given mathematical model describes glycolysis-gluconeogenesis under anaerobic conditions, whose output product is lactate. At a sufficient level of glucose, the process runs in the direct way. At the deficit of glucose, it runs in the reverse one: lactate is transformed in glucose.

The general scheme of the process of glycolysis-gluconeogenesis is presented in Fig.1. According to it, the mathematical model (1) - (16) is constructed with regard for the mass balance and the enzymatic kinetics.

The equations describe variations in the concentrations of the corresponding metabolites: (1) – lactate L; (2) – pyruvate P; (3) - 2-phosphoglycerate ψ_3; (4) – 3-phosphoglycerate ψ_2; (5) - 1,3-diphosphoglycerate (ψ_1); (6) - fructose-1,6-diphosphate (F_2); (7) – fructose-6-phosphate (F_1) ; (8) – glucose G; (9), (10), and (11) - ATP, ADP, and AMP, respectively, form the adeninenucleotide cycle at the phosphorylation; (12) - R_1 and (13) - R_2 (two active forms of the allosteric enzyme phosphofructokinase; (14) - T_1 and (15) - T_2 (two inactive forms of the allosteric enzyme phosphofructokinase; (16) - $NAD \cdot H$ (where $NAD \cdot H(t) + NAD^{+}(t) = M$).

Variations of the concentrations of omitted metabolites have no significant influence on the self-organization of the system and are taken into account in the equations generically. Since glycolysis and gluconeogenesis on seven sections of the metabolic chain are mutually reverse processes, only the coefficients are changed, whereas the system of equations describing glycolysis is conserved [8-10]. The model involves the running of gluconeogenesis on the section: glucose - glucose-6-phosphate. Here in the direct way with the help of the enzyme hexokinase, the catabolism of glucose to glucose-6-phosphate occurs. In the reverse direction with the help of the enzyme glucose-6-phosphatase, glucose is synthesized from glucose-6-phosphate. Thus, the positive feedback is formed on this section.

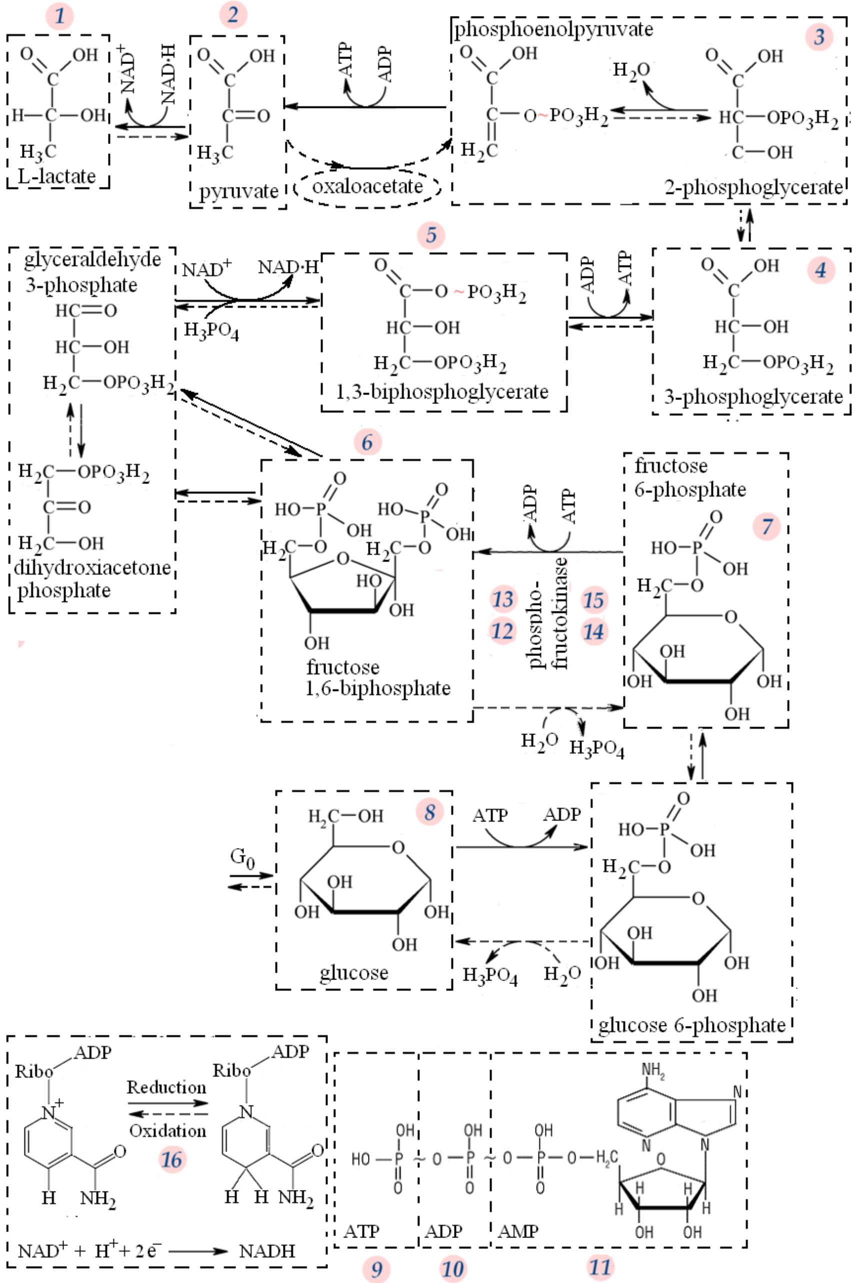

Fig.1. General scheme of the metabolic process of glycolysis-gluconeogenesis.

$$\frac{dL}{dt} = l_7 V(N)V(P) - m_9 \frac{L}{S}, \tag{1}$$

$$\frac{dP}{dt} = l_2 V(\psi_3)V(D) - m_6 \frac{P}{S} - l_7 V(N)V(P), \tag{2}$$

$$\frac{d\psi_3}{dt} = \frac{\psi_2}{S}\frac{m_2}{m_2 + \psi_3} - l_2 V(\psi_3)V(D) - m_4 \frac{\psi_3}{S}, \tag{3}$$

$$\frac{d\psi_2}{dt} = l_6 V(\psi_1)V(D) - m_8 \frac{\psi_2}{S}, \tag{4}$$

$$\frac{d\psi_1}{dt} = \frac{m_5 (F_2 / S)}{S_1 + m_5 (F_2 / S)} - l_6 V(\psi_1)V(D) + m_7 V(M - N)V(P), \tag{5}$$

$$\frac{dF_2}{dt} = l_1 V(R_1)V(F_1)V(T) - l_5 \frac{1}{1 + \gamma A} V(F_2) - m_5 \frac{F_2}{S}, \tag{6}$$

$$\frac{dF_1}{dt} = l_8 V(G)V(T) - l_1 V(R_1)V(F_1)V(T) + l_5 \frac{1}{1 + \gamma A} V(F_2) - m_3 \frac{F_1}{S}, \tag{7}$$

$$\frac{dG}{dt} = \frac{G_0}{S}\frac{m_1}{m_1 + F_1} - l_8 V(G)V(T), \tag{8}$$

$$\begin{aligned}\frac{dT}{dt} = {} & l_2 V(\psi_3)V(D) - l_1 V(R_1)V(F_1)V(T) + l_3 \frac{A}{\delta + A} V(T) - l_4 \frac{T^4}{\beta + T^4} + \\ & + l_6 V(\psi_1)V(D) - l_9 V(G)V(T),\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{dD}{dt} = {} & l_1 V(R_1)V(F_1)V(T) - l_2 V(\psi_3)V(D) + 2 \cdot l_3 \frac{A}{\delta + A} V(T) - \\ & - l_6 V(\psi_1)V(D) + l_9 V(G)V(T),\end{aligned} \tag{10}$$

$$\frac{dA}{dt} = l_4 \frac{T^4}{\beta + T^4} - l_3 \frac{A}{\delta + A} V(T), \tag{11}$$

$$\frac{dR_1}{dt} = k_1 T_1 V(F_1^2) + k_3 R_2 V(D^2) - k_5 R_1 \frac{T}{1 + T + \alpha A} - k_7 R_1 V(T^2), \tag{12}$$

$$\frac{dR_2}{dt} = k_5 R_1 \frac{T}{1+T+\alpha A} - k_3 R_2 V(D^2) + k_2 T_2 V(F_1^2) - k_8 R_2 V(T^2), \tag{13}$$

$$\frac{dT_1}{dt} = k_7 R_1 V(T^2) - k_6 T_1 \frac{T}{1+T+\alpha A} + k_4 T_2 V(D^2) - k_1 T_1 V(F_1^2), \tag{14}$$

$$\frac{dT_2}{dt} = k_6 T_1 \frac{T}{1+T+\alpha A} - k_4 T_2 V(D^2) - k_2 T_2 V(F_1^2) + k_8 R_2 V(T^2), \tag{15}$$

$$\frac{dN}{dt} = -l_7 V(N) V(P) + l_7 V(M-N) V(\psi_1). \tag{16}$$

Here, $V(X) = X/(1+X)$ is the function that describes the adsorption of the enzyme in the region of a local coupling. The variables of the system are dimensionless [8-10]. We take
$l_2 = 0.046;\ l_3 = 0.0017;\ l_4 = 0.01334;\ l_5 = 0.3;\ l_6 = 0.001;\ l_7 = 0.01;$
$l_8 = 0.0535;\ l_9 = 0.001;\ k_1 = 0.07;\ k_2 = 0.01;\ k_3 = 0.0015;\ k_4 = 0.0005;$
$k_5 = 0.05;\ k_6 = 0.005;\ k_7 = 0.03;\ k_8 = 0.005;\ m_1 = 0.3;\ m_2 = 0.15;\ m_3 = 1.6;$
$m_4 = 0.0005;\ m_5 = 0.007;\ m_6 = 10;\ m_7 = 0.0001;\ m_8 = 0.0000171;\ m_9 = 0.5;$
$G_0 = 18.4;\ L = 0.005;\ S = 1000;\ A = 0.6779;\ M = 0.005;\ S_1 = 150;$
$\alpha = 184.5;\ \beta = 250;\ \delta = 0.3;\ \gamma = 79.7.$

In the study of the given mathematical model (1)-(16), we have applied the theories of dissipative structures [11] and nonlinear differential equations [12,13], as well as the methods of mathematical modeling used in author's works [14-34]. In the numerical solution, we applied the Runge--Kutta--Merson method. The accuracy of calculations is 10^{-8}. The duration for the system to asymptotically approach an attractor is 10^6.

3 The Results of Studies

The mathematical model includes a system of nonlinear differential equations (1)-(16) and describes the open nonlinear biochemical system involving glycolysis and gluconeogenesis. In it, the input and output flows are glucose and lactate. Namely the concentrations of these substances form the direct or reverse way of the dynamics of the metabolic process. Both processes are irreversible and are running in the open nonlinear system, being far from the equilibrium. The presence of the reverse way of gluconeogenesis in the glycolytic system is the reason for the autocatalysis in it. In addition, the whole metabolic process of glycolysis is enveloped by the feedback formed by redox reactions with the transfer of electrons with the help of NAD (16) and the presence of the adeninenucleotide cycle (9) – (11) (Fig.1).

We now study the dependence of the dynamics of the metabolic process of glycolysis-gluconeogenesis on the value of parameter l_5 characterizing the activity of gluconeogenesis. The calculations indicate that, as the value of this parameter increases to 0.234, the system passes to the stationary state. As this parameter increases further, the autooscillations of a 1-fold periodic cycle $1 \cdot 2^0$ arise and then, at $l_5 \approx 0.2369$, transit to chaotic ones $- \cdot 2^x$. The analogous behavior of the system is observed at larger values of l_5. As the parameter decreases to 0.43, the system stays in a stationary state. If the parameter l_5 decreases further, the system gradually transits in a 1-fold periodic cycle $1 \cdot 2^0$, and the region of oscillatory dynamics arises.

Let us consider the oscillatory dynamics of this process. We constructed the phase-parametric diagrams, while l_5 varies in the intervals 0.235 – 0.28 and 0.25 – 0.266 (Fig.2,a,b). The diagrams are presented for fructose-6-phosphate F_1. We emphasize that the choice of a diagram for the namely given variable is arbitrary. The diagrams of other components are analogous by bifurcations. We want to show that the oscillations on the section fructose-6-phosphate – fructose-1,6-biphosphate can be explained by the oscillations of fructose-6-phosphate caused by gluconeogenesis, rather than the allosteric property of the enzyme phosphofructokinase.

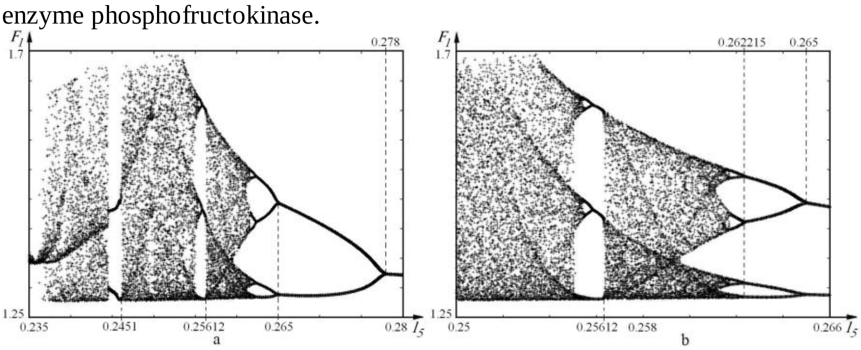

Fig. 2. Phase-parametric diagram of the system for the variable $F_1(t)$:
a - $l_5 \in (0.235, 0.28)$; b - $l_5 \in (0.25, 0.266)$.

The phase-parametric diagrams were constructed with the help of the cutting method. In the phase space, we took the cutting plane at $R_2 = 1.0$. This choice is explained by the symmetry of oscillations $F_1(t)$ relative to this point. At the cross of this plane by the trajectory, we fix the value of each variable. If a multiple periodic limiting cycle arises, we will observe a number of points on the plane, which coincide in the period. If a deterministic chaos arises, the points, where the trajectory crosses the plane, are located chaotically.

Considering the diagram from right to left, we may indicate that, at $l_5^j = 0.278$, the first bifurcation of the period doubling arises. Then at $l_5^{j+1} = 0.265$ and $l_5^{j+2} = 0.262215$, we see the second and third bifurcations, respectively. Further, the autooscillations transit in the chaotic mode due to the intermittence. The obtained sequence of bifurcations satisfies the relation

$$\lim_{t \to \infty} \frac{l_5^{j+1} - l_5^j}{l_5^{j+2} - l_5^{j+1}} \approx 4.668 .$$

This number is very close to the universal Feigenbaum constant. The transition to the chaos has happened by the Feigenbaum scenario [35].

It is seen from Fig.2,a,b that, for $l_5 = 0.25612$ and $l_5 = 0.2451$, the periodicity windows appear. Instead of the chaotic modes, the periodic and quasiperiodic modes are established. The same periodicity windows are observed on smaller scales of the diagram. The similarity of diagrams on small and large scales means the fractal nature of the obtained cascade of bifurcations in the metabolic process created by gluconeogenesis.

As examples of the sequential doubling of a period of autoperiodic modes of the system by the Feigenbaum scenario, we present projections of the phase portraits of the corresponding regular attractors in Fig.3,a-c. In Fig.3,d-f, we show some regular attractors arising in the periodicity windows. For $l_5 = 0.256$, the 3-fold periodic mode $3 \cdot 2^0$ is formed. For $l_5 = 0.2556$, we observe the 5-fold mode. Then, as $l_5 = 0.245$, the 3-fold periodic cycle is formed again.

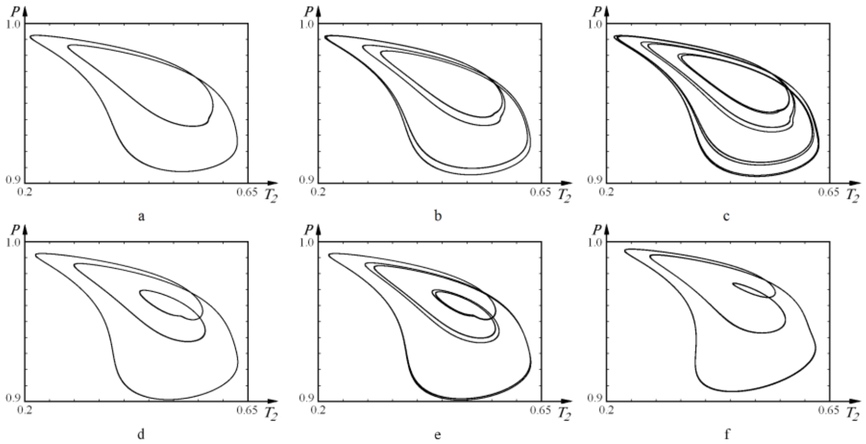

Fig.3. Projections of phase portraits of the regular attractors of the system: a - $1 \cdot 2^1$, for $l_5 = 0{,}268$; b - $1 \cdot 2^2$, for $l_5 = 0.264$; c - $1 \cdot 2^4$, for $l_5 = 0.262$; d - $3 \cdot 2^0$, for $l_5 = 0.256$; e - $5 \cdot 2^0$, for $l_5 = 0.2556$; and f - $3 \cdot 2^0$, for $l_5 = 0.245$.

In Fig. 4,a,b, we give projections of the strange attractor 2^x for $l_5 = 0.25$. The obtained chaotic mode is a strange attractor. It appears as a result of the formation of a funnel. In the funnel, there occurs the mixing of trajectories. At an arbitrarily small fluctuation, the periodic process becomes unstable, and the deterministic chaos arises.

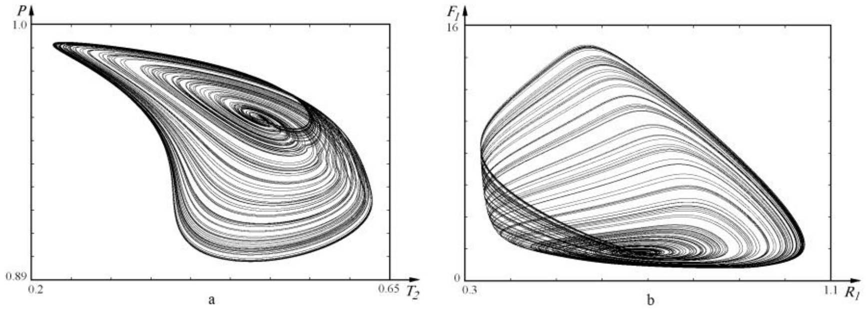

Fig.4. Projections of the phase portrait of the strange attractor 2^x for $l_5 = 0.25$: a – in the plane (T_2, P), b – in the plane (R_1, F_1).

In Fig.5,a,b, we present, as an example, the kinetics of autooscillations of some components of the metabolic process in a 1-fold mode for $l_5 = 0.3$ and in the chaotic mode for $l_5 = 0.25$. The synchronous autooscillations of fructose-6-phosphate and the inactive form T_2 of the allosteric enzyme phosphofructokinase are replaced by chaotic ones.

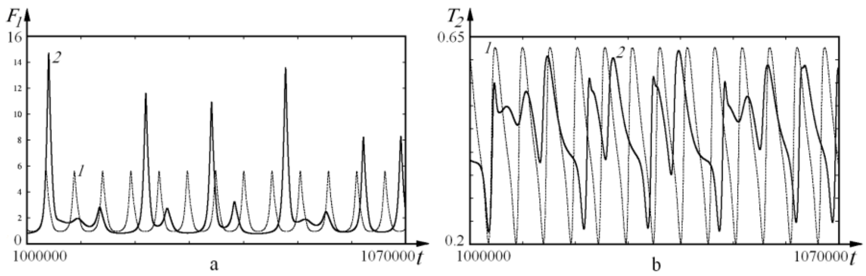

Fig.5. Kinetic curves of the variables: $F_1(t)$ - a and $T_2(t)$ - b in the 1-fold periodic mode for $l_5 = 0.3$ (1) and in the chaotic mode for $l_5 = 0.25$ (2).

While studying the phase-parametric diagrams in Fig.2,a,b, it is impossible beforehand to determine, for which values of parameter l_5 a multiple stable (quasistable) autoperiodic cycle or a strange attractor is formed.

For the unique identification of the type of the obtained attractors and for the determination of their stability, we calculated the complete spectra of

Lyapunov indices $\lambda_1, \lambda_2, .. \lambda_{16}$ for chosen points and their sum $\Lambda = \sum_{j=1}^{16} \lambda_j$. The calculation was carried out by Benettin's algorithm with the orthogonalization of the perturbation vectors by the Gram--Schmidt method [13].

As a specific feature of the calculation of these indices, we mention the difficulty to calculate the perturbation vectors represented by 16×16 matrices on a personal computer.

Below in Table 1, we give several results of calculations of the complete spectrum of Lyapunov indices, as an example. For the purpose of clearness, we show only three first indices $\lambda_1 - \lambda_3$. The values of $\lambda_4 - \lambda_{16}$ and Λ are omitted, since their values are not essential in this case. The numbers are rounded to the fifth decimal digit. For the strange attractors, we calculated the following indices, by using the data from Table 1. With the use of the Pesin theorem [36], we calculated the KS-entropy (Kolmogorov-Sinai entropy) and the Lyapunov index of a "horizon of predictability" [37]. The Lyapunov dimension of the fractality of strange attractors was found by the Kaplan--Yorke formula [38,39]:

By the calculated indices, we may judge about the difference in the geometric structures of the given strange attractors. For $l_5 = 0.25$, the KS-entropy takes the largest value $h = 0.00014$. In Fig.4,a, we present the projection of the given strange attractor. For comparison, we constructed the strange attractors for $l_5 = 0.26$ (Fig.6,a) and $l_5 = 0.237$ (Fig.6,b). Their KS-entropies are, respectively, 0.00008 and 0.00005. The comparison of the plots of the given strange attractors is supported by calculations. The trajectory of a strange attractor (Fig.4,a) is the most chaotic. It fills uniformly the whole projecton plane of the attractor. Two other attractors (Fig.6,a,b) have the own relevant regions of attraction of trajectories. The phase space is divided into the regions, which are visited by the trajectory more or less.

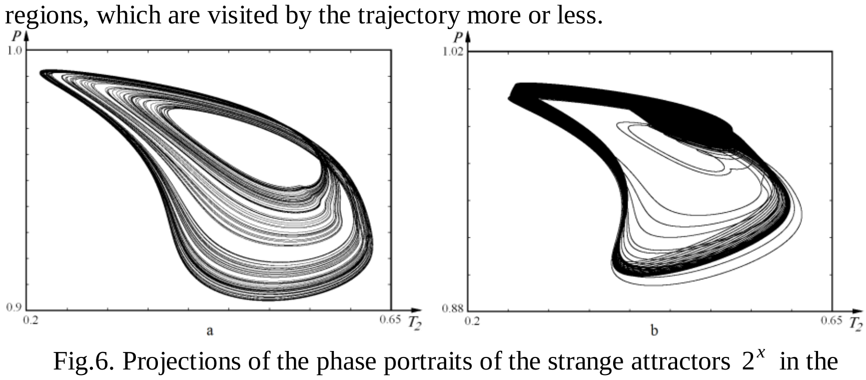

Fig.6. Projections of the phase portraits of the strange attractors 2^x in the plane (T_2, P): a – for $l_5 = 0.26$ and b – for $l_5 = 0.237$.

Table 1. Lyapunov indices, KS-entropy, "horizon of predictability," and the Lyapunov dimension of the fractality of strange attractors calculated for various modes

l_5	Attrac tor	λ_1	λ_2	λ_3	h	$t_{\min}$	D_{Fr}
0.28	$1\cdot 2^0$	.00000	-.00008	-.00010	-	-	-
0.264	$1\cdot 2^1$	.00000	-.00005	-.00008	-	-	-
0.262	$1\cdot 2^2$	.00000	-.00005	-.00009	-	-	-
0.26	2^x	.00008	.00000	-.00008	.00008	12500	4
0.257	2^x	.00007	.00000	-.00008	.00007	14285.7	3.9
0.256	$3\cdot 2^0$	.00000	-.00006	-.00007	-	-	-
0.2556	$5\cdot 2^0$.	.00000	-.00005	-.00009	-	-	-
0.254	2^x	.00009	.00000	-.00009	.00009	11111.1	4
0.252	2^x	.00012	.00000	-.00010	.00012	8333.3	4.2
0.25	2^x	.00014	.00000	-.00009	.00014	7142.9	4.6
0.248	2^x	.00009	.00000	-.00007	.00009	11111.1	4.3
0.247	2^x	.00013	.00000	-.00010	.00013	7692.2	4.3
0.2463	2^x	.00008	.00000	-.00011	.00008	12500	3.7
0.245	$3\cdot 2^0$	.00000	-.00011	-.00011	-	-	-
0.242	2^x	.00010	.00000	-.00008	.00010	10000	4.25
0.24	2^x	.00006	.00000	-.00009	.00006	16666.7	3.7
0.239	2^x	.00006	.00000	-.00010	.00006	16666.7	3.6
0.238	2^x.	.00011	.00000	-.00010	.00011	9090.9	4.1
0.237	2^x	.00005	.00000	-.00008	.00005	20000	3.6

The Lyapunov dimensions of the given strange attractors are changed analogously. We have, respectively: 4.6, 4, and 3.6. These values characterize generally the fractal dimension of the given attractors. If we separate small rectangular area on one of the phase curves in each of the given plots and increase them, we will see the geometric structures of the given strange attractors on small and large scales. Each arisen curve of the projection of a chaotic attractor is a source of formation of new curves. Moreover, the geometric regularity of construction of trajectories in the phase space is repeated for each strange attractor. In the given case, the best geometric self-similarity conserves in the presented strange attractors in the following sequence: Fig.4,a, Fig.6,a, and Fig.6,b.

The value of "horizon of predictability" $t_{\min}$ for the modes presented in the table is the largest for $l_5 = 0.237$ (Fig.6,b). The narrow regions of attraction of the projection of the strange attractor correspond to the most predictable kinetics of the running metabolic process. From all metabolic chaotic modes, this mode is the mostly functionally stable for a cell.

The above-described study of the process of glycolysis-gluconeogenesis with the help of a change of the coefficient of positive feedback l_5 indicates that, in the given metabolic process under definite conditions, the autocatalysis arises. The value of l_5 determines the activity of gluconeogenesis on the section of the transformation of glucose-6-phosphate in glucose. This reaction is catalyzed by the enzyme glucose-6-phosphatase. This phosphatase is magnesium-dependent. If the magnesium balance is violated or some other factors come in play, the rate of this reaction varies. In addition, the absence of some coenzymes in a cell affects essentially also the rates of other enzymatic reactions, which can lead to the desynchronization of metabolic processes. As a result, the autooscillations arise in the metabolic process of glycolysis-gluconeogenesis. The autooscillations can be autoperiodic with various multiplicities or chaotic. Their appearance can influence the kinetics of the metabolic process in the whole cell and its physiological state.

Conclusions

With the help of a mathematical model, we have studied the influence of gluconeogenesis on the metabolic process of glycolysis. The metabolic chain of glycolysis-gluconeogenesis is considered as a single dissipative system arisen as a result of the self-organization, i.e., as a product of the biochemical evolution in protobionts. The reasons for the appearance of autocatalysis in it are investigated. A phase-parametric diagram of autooscillatory modes depending on the activity of gluconeogenesis is constructed. We have determined the bifurcations of the doubling of a cycle according to the Feigenbaum scenario and have shown that, as a result of the intermittence, the aperiodic modes of strange attractors arise. The fractal nature of the calculated cascade of bifurcations is demonstrated. The strange attractors arising as a result of the formation of a mixing funnel are found. The complete spectra of Lyapunov indices for various modes are calculated. For strange attractors, we have calculated the KS-entropies, "horizons of predictability," and the Lyapunov dimensions of the fractality of attractors. The structure of a chaos of the given attractors and its influence on the stability of the metabolic process, adaptation, and functionality of a cell are studied. It is shown that a change of the cyclicity in the metabolic process in a cell can be caused by the violation of the magnesium balance in it or the absence of some coenzymes. The obtained results allow one to study the influence of gluconeogenesis on the self-organization of the metabolic process in a cell and to find the reasons for a change of its physiological state.

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Two parametric bifurcation in LPA model

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Abstract. The structured population LPA model describes flour beetle population dynamics of four stage populations: eggs, larvae, pupae, and adults with cannibalism between these stages. The case of non-zero cannibalistic rates of adults on eggs and adults on pupae and no cannibalism of larvae on eggs. This assumption is necessary to make at least some calculations analytically. It is shown that the behavior can be continued to the generic model with non-zero cannibalistic rate of larvae on eggs. In the model exist both supercritical and subcritical strong 1:2 resonance. The bifurcation responsible for the change of topological type of the strong 1:2 resonance is study. This bifurcation is accompanied by the origination of the Chenciner bifurcation. The destabilization of the system is caused by two parametric bifurcation is study together with its biological consequences.

Keywords: Two parametric bifurcation; LPA model; Tribolium model; strong 1:2 resonance; Chenciner bifurcation.

1 Introduction

This article is based on original work of Robert F. Costantino, Ph.D., Jim Cushing, Ph.D., Brian Dennis, Ph.D., Robert A. Desharnais, Ph.D. and Shandelle Henson, Ph.D. about LPA model (Tribolium model). LPA model is a structured population model that describes flour beetle population dynamics of four stage populations: eggs, larvae, pupae, adults with cannibalism between these stages. Main results of the research were published from the year 1995 to nowadays. In published articles authors concentrate mainly on the chaotic behavior in the system. The nonlinear dynamics of the system associated with the LPA model is rich, there is a lot of studies that deal with basic analysis of equilibria and their stability (e.g. Cushing[6], Cushing[4] or Kuang and Cushing[10]), some works are devoted to one-parameter bifurcations (as Dennis *et al.*[7]) and its route to chaotic dynamics (e.g. Cushing[6], Constantino *et al.*[2], Cushing *et al.*[5], Cushing *et al.*[3]). To our best knowledge, there is not any

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published work about two-parameter bifurcation analysis by far. The original analysis of Chenciner bifurcation and subcritical strong 1:2 resonance were done in our article which is under review in Journal of theoretical biology.

In this article we concentrate on the both supercritical a subcritical strong resonance 1:2 and the bifurcation responsible for the change of topological type of the strong 1:2 resonance, which is accompanied with Chenciner bifurcations. The mathematical background for these bifurcations, their normal forms and analysis can be found in Kuznetsov[11].

The structured population LPA model consists of three stages: larvae L, pupae P and adults A, while the population of eggs as a function of the adult population is not included into the state space. We assume cannibalism between the stages. We have to point out that we concentrate on LPA model with non-zero cannibalistic rates of adults on eggs and adults on pupae and no cannibalism of larvae on eggs. Here this assumption of no cannibalism of larvae on eggs is used only to make the mathematical calculations more easy and clear (a lot of them may be done analytically in this case) and this case was also examined in e.g. Dennis *et al.*[7].

2 Model description and basic analysis

The dynamic of LPA model is (see e.g. Cushing[6] or Cushing[4]):

$$\begin{aligned} L(t+1) &= bA(t)\,\mathrm{e}^{-c_{EL}L(t)-c_{EA}A(t)} \\ P(t+1) &= (1-\mu_L)\,L(t) \\ A(t+1) &= P(t)\,\mathrm{e}^{-c_{PA}A(t)} + (1-\mu_A)\,A(t)\,, \end{aligned} \tag{1}$$

where state variables L, P, A represent number of larvae, pupae and adults in population. Parameter $b > 0$ represents natality. Parameters μ_L and μ_A represent mortality of larvae and adults. We assume natural inequalities $0 < \mu_L < 1$, $0 < \mu_A < 1$ to be satisfied. Parameters c_{EL}, c_{EA}, c_{PA} denote cannibalistic rates. Namely, c_{EL} is the cannibalistic rate of larvae on eggs, c_{EA} is the cannibalistic rate of adults on eggs and c_{PA} is the cannibalistic rate of adults on pupae. We assume $c_{EA} \geq 0, c_{PA} \geq 0$ and $c_{EL} \geq 0$, in this article we consider a special case $c_{EL} = 0$.

There can be two fixed points of the system (1). The trivial fixed point corresponds to extinction of the population, the non-trivial fixed point $[L^*, P^*, A^*]$ satisfies formulas

$$\begin{aligned} L^* &= \frac{b\ln\left(\frac{b(1-\mu_L)}{\mu_A}\right)\mathrm{e}^{-c_{EA}\ln\left(\frac{b(1-\mu_L)}{\mu_A}\right)}}{(c_{PA}+c_{EA})} \\ P^* &= \frac{b\,(1-\mu_L)\ln\left(\frac{b(1-\mu_L)}{\mu_A)}\right)\mathrm{e}^{-c_{EA}\ln\left(\frac{b(1-\mu_L)}{\mu_A}\right)}}{(c_{PA}+c_{EA})} \\ A^* &= \frac{\ln\left(\frac{b(1-\mu_L)}{\mu_A}\right)}{(c_{PA}+c_{EA})}. \end{aligned} \tag{2}$$

We introduce the basic reproduction number $R_0 = \frac{b(1-\mu_L)}{\mu_A}$. The non-trivial fixed point exists for $R_0 > 0$, but for $R_0 \in (0, 1)$ it has no biological meaning. It can be easily shown that the trivial fixed point is stable for $R_0 < 1$, for $R_0 = 1$ $[L^*, P^*, A^*] = [0, 0, 0]$ and is unstable for $R_0 > 1$, while the fixed point $[L^*, P^*, A^*]$ is not stable for all values of parameters. In the words of biology, population will extinct for $R_0 \leq 1$ and can survive for $R_0 > 1$. In the words of the bifurcation theory, $R_0 = 1$ is a critical value of the transcritical bifurcation. The manifold of the transcritical bifurcation is included in $b = \frac{\mu_A}{1-\mu_L}$ of the parameter space. It's good to mention that the transcritical bifurcation does not depend on cannibalistic rates.

The one-parameter bifurcations are already already described in Dennis *et al.*[7]. From the presented work it's clear that the flip bifurcation curve (called there 2-cycles) and Neimark-Sacker bifurcation curve (called there loops) can intersect (see the figure 1 in Dennis *et al.*[7]). In next sections of this paper we will go on with deeper two-parameter bifurcation analysis. All our results are in agreement with the results presented in the paper Dennis *et al.*[7] as well as with sufficient conditions for stability of the non-trivial fixed point that can be found in Kuang and Cushing[10].

3 Routes to two-parameter bifurcations

There are two ways how we receive two-parameter local bifurcations of the fixed point. One of them is that the non-degeneracy conditions of the one-parameter bifurcation is violated. For example the Neimark-Sacker bifurcation non-degeneracy condition is violated in the Chenciner critical points. Qualitative changes in dynamics near the Chenciner bifurcation have globally destabilizing effect to the population and this will be discussed in the next separate section. The other way is that two eigenvalues reach the unit circle. Let's consider this case now. Obviously, the two-parameter bifurcation manifold covers the intersection of one-parameter bifurcation manifolds. In our system there exists thee different one-parameter bifurcation manifolds: transcritical, flip and Neimark-Sacker bifurcation. There are two types of intersections of the flip and the Neimark-Sacker bifurcation manifolds:

(i) $b = \frac{\mu_A e^{\frac{2}{\mu_A}}}{1-\mu_L}, c_{EA} = \frac{(\mu_A+1)c_{PA}}{1-\mu_A}$ and

(ii) $b = \frac{\mu_A e^{\frac{2}{\mu_A}}}{1-\mu_L}, c_{EA} = \frac{(2\mu_A-1)c_{PA}}{5-2\mu_A}$.

The manifold (i) exists for all allowed values of parameters. On the other hand the manifold (ii) exists for $\mu_A > \frac{1}{2}$ only.

In this paper we will focus on manifold (ii). The manifold (ii) corresponds to the strong 1:2 resonance with associated eigenvalues are $-1, -1, \frac{1}{2}$.

For arbitrarily fixed parameters μ_L, μ_A, c_{PA}, the two-parameter bifurcation manifolds correspond to points of intersection of one-parameter bifurcation curves in the two-parameter space c_{EA} versus b. The parameters μ_L, μ_A, c_{PA} are fixed to common values (see e.g. Dennis *et al.*[8]).

4 Strong 1:2 resonance in LPA model

Strong 1:2 resonance is a two-parameter bifurcation that lies in the intersection of flip bifurcation manifold and the Neimark-Sacker bifurcation manifold. In our model two topological types of the strong 1:2 resonance exists: subcritical bifurcations of a node or a focus, supercritical bifurcation of a node or a focus. Then normal form for the supercritical bifurcation is similar to the subcritical, but the time variable has an opposite sign (see e.g. Kuznetsov[11]). Therefore the phase portraits of subcritical and supercritical bifurcations has an opposite stability.

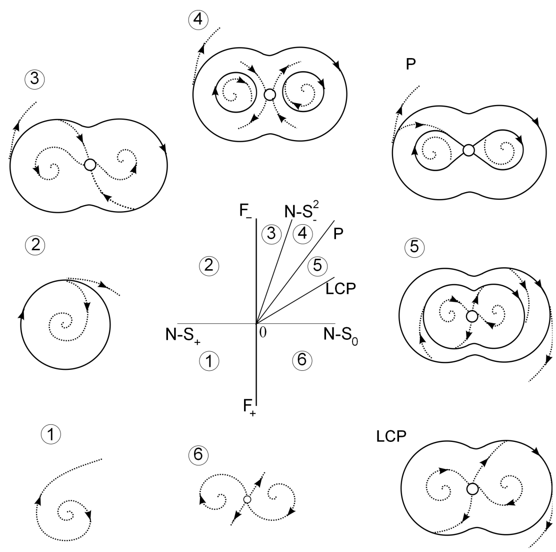


Fig. 1. Subcritical strong 1:2 resonance diagram in a two-parameter space. The N-S$_+$ denotes the subcritical branch of the Neimark-Sacker curve, N-S$_0$ denotes the neutral saddles, F$_+$, F$_-$ denote the flip bifurcation curves, LPC denotes the fold bifurcation of the invariant loop curve, P denotes the saddle separatrix loop curve. The phase portraits in each domain ① - ⑥ are topologically generic. Similarly to the Chenciner bifurcation, a special heteroclinic structure of orbits appears in the neighbourhood of LPC and P. For more details see Kuznetsov[11].

Strong 1:2 resonance points lie in the intersection of Neimark-Sacker and flip manifolds, therefore we expect birth of the limit loop from a fixed point due to N-S bifurcation and split of the fixed point into a 2-cycle nearby the strong 1:2 resonance point. The figure 1 displays the generic transversal two-parameter space section of a canonical subcritical strong 1:2 resonance bifurcation manifold at zero with one-parameter N-S and flip manifolds at the horizontal and vertical axes.

As we move around the strong 1:2 resonance point, the topological structure of the state space change the way that is presented for the canonical form at the figure 1.

5 Chenciner bifurcation in LPA model

Transversal crossing of the Neimark-Sacker bifurcation manifold give rise to an invariant loop around a fixed point that changes its stability. There are two topological types of the Neimark-Sacker bifurcation: supercritical and subcritical. The supercritical type give rise to a stable invariant loop, reversely, the subcritical bring about an unstable loop. The Chenciner bifurcation is a critical change of these two types. There exists an accompanying bifurcation manifold of the Chenciner bifurcation. It is called the fold bifurcation of the invariant loop or the limit point bifurcation of the invariant loop and it gives a birth to the stable and unstable invariant loop around.

The Chenciner bifurcation is found strictly on the one branch of Neimark-Sacker bifurcation near the strong 1:2 resonance. We found even parameter values for which the Chenciner and strong 1:2 resonance bifurcations collide. This collision is responsible for change of topological type of strong 1:2 resonance.

6 Change of topological type of strong 1:2 resonance in LPA model

Both Chenciner bifurcation and subcritical strong 1:2 resonance occur for μ_A sufficiently close to 1 in LPA model (remember that the necessary condition for the strong 1:2 resonance is $\mu_A > \frac{1}{2}$). For μ_A sufficiently close to $\frac{1}{2}$ there exists only supercritical strong 1:2 resonance. The critical change of subcritical and supercritical strong 1:2 resonance gives a birth to the Chenciner bifurcation. Here we present our original analysis of the phenomenon of changing topological type of the strong 1:2 resonance. We will describe the structure by equivalence classes of structurally stable domains with topologically equivalent state spaces for both topological types of the strong 1:2 resonance. The borders of these domains are the one-parameter bifurcation.

The transversal two-dimensional section b versus c_{EA} of supercritical strong 1:2 resonance is taken for fixed parameters $\mu_L = 0.1613; \mu_A = 0.75; c_{PA} = 0.004348$ (which is shown in the picture 2). The dynamic classes I. - VI. are displayed at the figures 3.

The topological structure of the parameter space near Chenciner bifurcation and subcritical strong 1:2 resonance give rise to a complicated state space dynamics with coexistence of different types of invariant sets. The transversal two-dimensional section b versus c_{EA} of both two-parameter manifolds (Chenciner and subcritical strong 1:2 resonance) is taken for fixed parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$. Striped and shadowed domains belong to the basins of attraction corresponding to weak and huge oscillations respectively. White domains belong to a stable fixed point basins of attraction. The two different branches of LPC (fold bifurcation of the invariant loop) collide in a typical V-shape in the cusp point, that is a two-parameter bifurcation point. The cusp point is typically connected with another phenomenon of hysteresis. The parameter space is divided into nine domains where the state

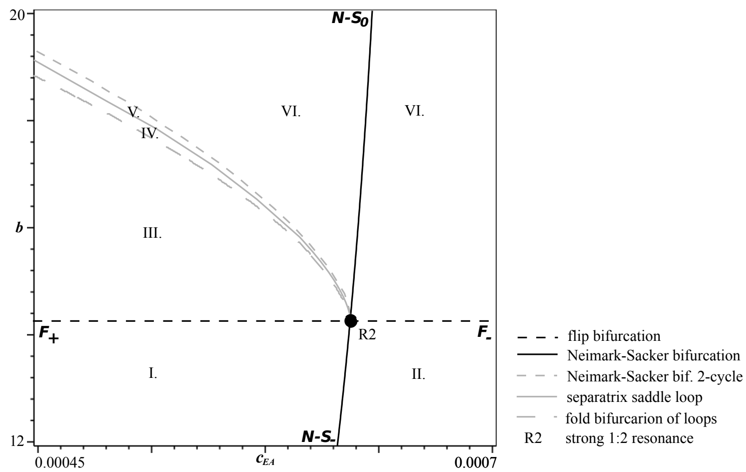
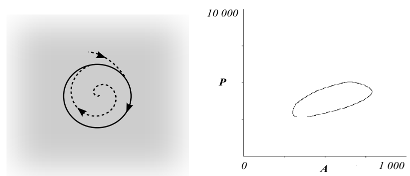


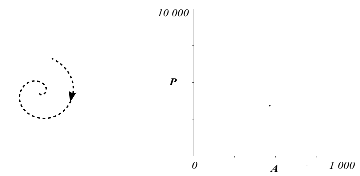
Fig. 2. Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters c_{EA} and b for fixed $\mu_L = 0.1613; \mu_A = 0.75; c_{PA} = 0.004348$.

spaces stay topologically equivalent. All dynamic classes I. - IX. are displayed at the figures 6. We omit the stripe underneath the transcritical bifurcation curve, where the population is dying out. Here the only fixed point is the trivial equilibrium that is globally stable and so the population extincts. For values of b above the transcritical bifurcation curve, the trivial equilibrium is unstable and the orbits can tend to another attractors.

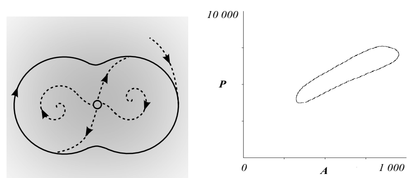
For parameter values near the change of topological type of strong 1:2 resonance the system is locally topologically equivalent to the system displayed in the picture 4. The global behavior is shown in the picture 5.



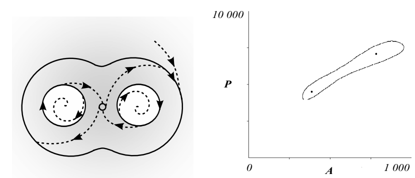
(a) I: stable invariant loop, e.g. $c_{EA} = 0.0013$, $b = 6$



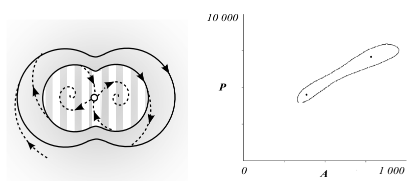
(b) II: stable fixed point, e.g. $c_{EA} = 0.002$, $b = 8$



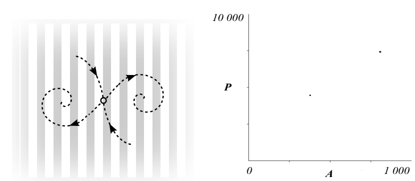
(c) III: stable fixed point, stable invariant loop, e.g. $c_{EA} = 0.0014$, $b = 6$



(d) IV: stable 2-cycle, stable invariant loop, e.g. $c_{EA} = 0.0015$, $b = 9.2$



(e) V: two stable symmetric coupled loops, stable invariant loop, e.g. $c_{EA} = 0.00154$, $b = 9.6$



(f) VI: two stable invariant loops, e.g. $c_{EA} = 0.00172$, $b = 10.05$

Fig. 3. Phase portraits near the strong resonance 1:2 and Chenciner bifurcation in LPA model with parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$ and free parameters c_{EA} a b. In the left column, there are schematic phase portrait for each domain according to the figure 2. In the right column, there are computed stable sets at adults and pupae state variables.

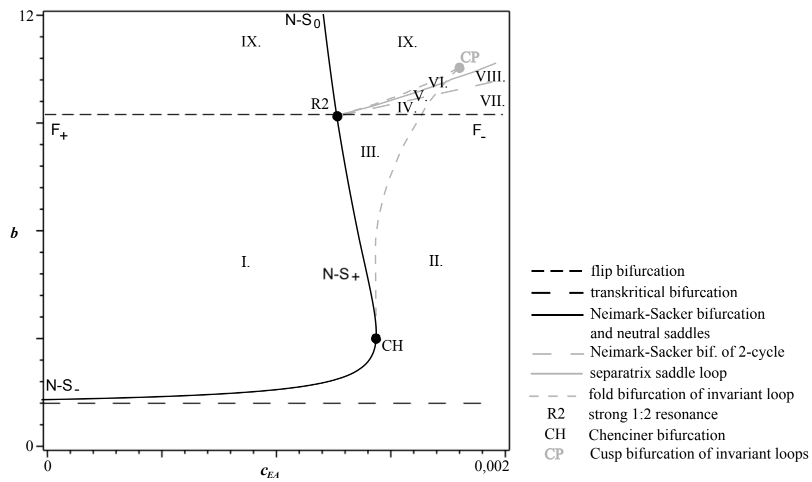


Fig. 4. Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters c_{EA} and b for fixed $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$.

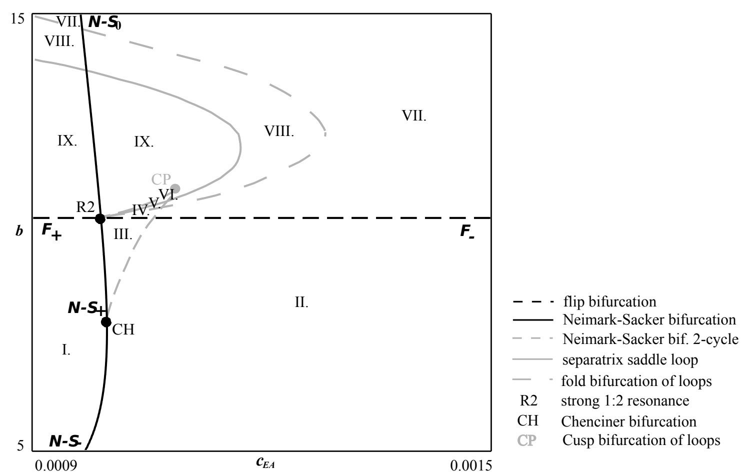
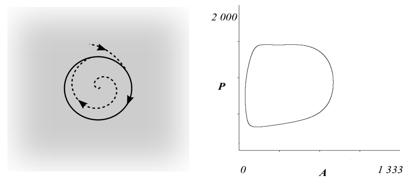
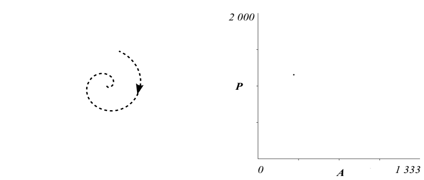


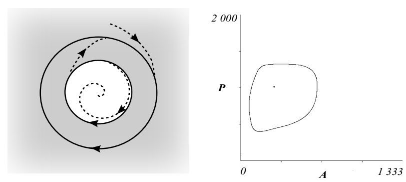
Fig. 5. Strong 1:2 resonance and Chenciner bifurcation diagram. Bifurcation curves in parametric space with free parameters c_{EA} and b for fixed $\mu_L = 0.1613; \mu_A = 0.87; c_{PA} = 0.004348$.



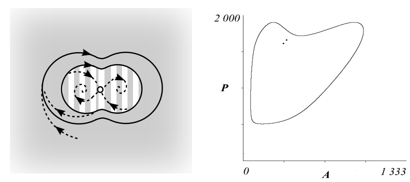
(a) I: stable invariant loop, e.g. $c_{EA} = 0.0013$, $b = 6$



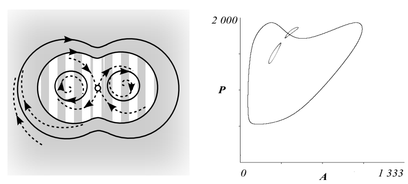
(b) II: stable fixed point, e.g. $c_{EA} = 0.002$, $b = 8$



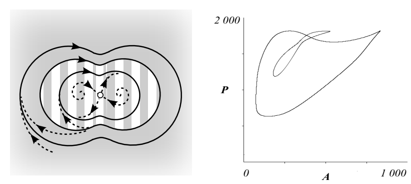
(c) III: stable fixed point, stable invariant loop, e.g. $c_{EA} = 0.0014$, $b = 6$



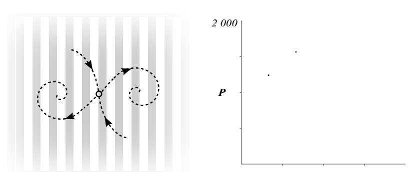
(d) IV: stable 2-cycle, stable invariant loop, e.g. $c_{EA} = 0.0015$, $b = 9.2$



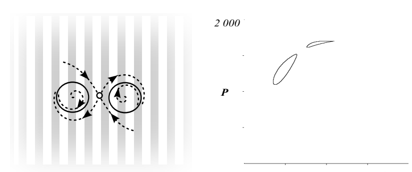
(e) V: two stable symmetric coupled loops, stable invariant loop, e.g. $c_{EA} = 0.00154$, $b = 9.6$



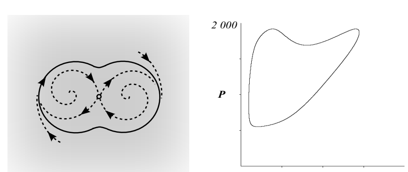
(f) VI: two stable invariant loops, e.g. $c_{EA} = 0.00172$, $b = 10.05$



(g) VII: stable 2-cycle, e.g. $c_{EA} = 0.002$, $b = 10$



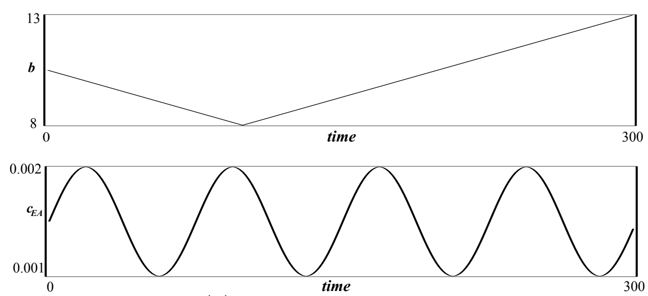
(h) VIII: stable symmetric coupled loops, e.g. $c_{EA} = 0.00185$, $b = 10.2$



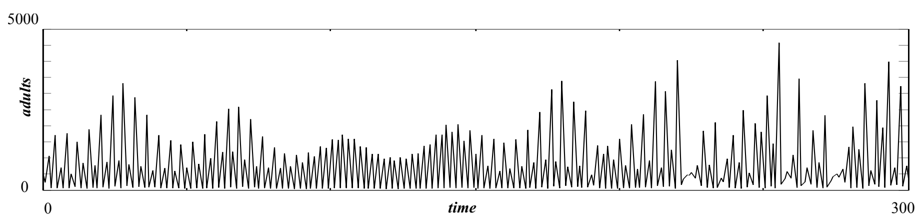
(i) IX: stable invariant loop, e.g. $c_{EA} = 0.0016$, $b = 9.9$

Fig. 6. Phase portraits near the strong resonance 1:2 and Chenciner bifurcation in LPA model with parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$ and free parameters c_{EA} a b. In the left column, there are schematic phase portrait for each domain according to the figure 4. In the right column, there are computed stable sets at adults and pupae state variables.

In our opinion, this complicated structure near the strong resonance 1:2 and the Chenciner bifurcation in LPA model has a very troublesome consequence, since in this quite a big area of parameters it's very hard to compare the simulated and real data. In real experiments, the natality b and the cannibalistic rate c_{EA} as parameters are not strict constants and they can vary during time due to temperature or attainability of other sources of food and other random events, there can be some measure errors also. The real data and simulations may become totally different even in the case of a proper model usage. Even the simulated data may be considered to be chaotic or random by mistake. Imagine parameters b and c_{EA} that vary slowly in their parameter domain near the described phenomenon. The simulated data look as chaotic or random, since they are very sensitive to the parameter changes, see the figure 7.



(a) Parameter changes



(b) "Chaos-like" simulated time series.

Fig. 7. Simulated time series with slowly varying parameters b and c_{EA} for parameters $\mu_L = 0.1613; \mu_A = 0.96; c_{PA} = 0.004348$.

7 Conclusion

We presented a two-parameter bifurcation analysis of LPA model (for parameters c_{EA}, b and μ_A) with zero c_{PA} cannibalistic rate to show complex dynamics in the model of the tribolium population. Here we mention that we did not concerned to the period doubling and chaos, since there is a lot of papers devoted to this topic, but we focused on another bifurcations that were overlooked so far and their destabilization effects were not mentioned yet.

We found strong 1:2 resonance of node or a focus in LPA model and we explained its topological structure. We explained the importance of the bifurcation type of the strong 1:2 resonance bifurcation, because both of the types (subcritical and supercritical) are present. The topological change of the strong 1:2 resonance gives a birth to the Chenciner bifurcation. We expressed the Chenciner bifurcation.

As the most important part of our paper we consider to be the finding of connection between the Chenciner bifurcation and strong 1:2 resonance and setting of the complete two-parameter bifurcation diagram of these manifolds connection (together with the nearby non-local bifurcation manifolds).

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