Analytical results in type I, II and III intermittency theory

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Abstract. The concept of intermittency has been introduced by Pomeau and Manneville and are usually classified in three classes called I, II, and III. The main attribute of intermittency is a global reinjection mechanism described by the corresponding reinjection probability density (RPD), that maps trajectories of the system from the chaotic region back into the local laminar phase. We generalize the classical RPD for Type-I, II, or III intermittency. As a consequence, the classical intermittency theory is a particular case of the new one. We present an analytical approach to the noise reinjection probability density. It is also important to note that the RPD, obtained from noisy data, provides also a complete description of the noiseless system.

Keywords: Intermittency, chaos, one dimensional map, noise.

1 Introduction

Intermittency is a particular route to the deterministic chaos characterized by spontaneous transitions between laminar and chaotic dynamics. For the first time this concept has been introduced by Pomeau and Manneville in the context of the Lorenz system Manneville[1], Pomeau and Manneville[2]. Later intermittency has been found in a variety of different systems including, for example, periodically forced nonlinear oscillators, Rayleigh-Bénard convection, derivative nonlinear Schrödinger (DNLS) equation, and the development of turbulence in hydrodynamics (see e.g. Refs.Dubois et al.[3], del Rio et al.[4], Stavrinides et al.[5], Krause et al.[6], Sanchez-Arriaga et al.[7]).

Besides this, there are other types of intermittencies such as type V, X, on-off, eyelet and ring Kaplan[8], Price and Mullin[9], Platt et al.[10], Pikovsky et al.[11], Lee et al.[12], Hramov et al.[13]. A more general case of on-off intermittency is the so-called in-out intermittency. A complete review of on-off and in-out intermittencies can be found in Stavrinides and Anagnostopoulos[14].

Proper qualitative and quantitative characterizations of intermittency based on experimental data are especially useful for studying problems with partial

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or complete lack of knowledge on exact governing equations, as it frequently happens e.g. in Economics, Biology, and Medicine (see e.g. Refs. Zebrowski and Baranowski[15], Chian [16]).

It is interesting to note that the most of the above cited references are devoted to system having more than one dimension. Spite of this, they can be described by one dimensional map. This phenomenon is typical of systems that contract volume in phase space Ott[17].

All cases of Pomeau and Manéville intermittency has been classified in three types called I, II, and III Schuster and Just[18]. The local laminar dynamics of type-I intermittency evolves in a narrow channel, whereas the laminar behaviour of type-II and type-III intermittencies develops around a fixed point of its generalized Poincare maps.

Another characteristic attribute of intermittency is the global reinjection mechanism that maps trajectories of the system from the chaotic region back into the local laminar phase. The reinjection mechanism from the chaotic phase into laminar region dependent on the chaotic phase behaviour, so it is a global property, hence the probability density of reinjection (RPD) of the system back from chaotic burst into points in laminar zone is determined by the dynamics in the chaotic region. Only in a few case it is possible to get an analytical expression for PRD, let say $\phi(x)$. It is also difficult to get PRD experimentally or numerically, because the large number of data needed to cover each small subset of length $\Delta x$ which belong to the reinjection zone. Because of all this, different approximations have been used in literature to study the intermittency phenomenon. The most common approximation is to consider PRD uniform and thus independent of the reinjection point Manneville[19], Dubois et al.[3], Pikovsky[20], Kim et al.[21], Kim et al.[22], Kim et al.[23], Cho et al.[24], Schuster and Just[18].

We described here an overview of a recent theory on the intermittency phenomenon based on a new two-parameter class of PRDs appearing in many maps with intermittency (see for instance: del Rio and Elaskar[25], Elaskar et al.[26], del Rio et al.[27] and del Rio et al.[28]) and the noise effect on this PRD. For a specific values of the parameters, we recover the classical theory developed for uniform PRD.

Firstly let us briefly describe the theoretical framework that accounts for a wide class of dynamical systems exhibiting intermittency. We consider a general 1-D map

$$x_{n+1} = G(x_n), \quad G : \mathbb{R} \to \mathbb{R}$$

(1)

which exhibits intermittency. Note that the map (1) can be coming, for instance, from a Poincare map of a continuous dynamical system. Let us introduce the dynamics corresponding to the three types of intermittencies around the unstable fixed point. The local laminar dynamics of type-I intermittency determined by the Poincare map in the form:

$$x_{n+1} = \epsilon + x_n + a x_n^p$$

(2)

where $a > 0$ accounts for the weight of the nonlinear component and $\epsilon$ is a controlling parameter ($\epsilon \ll 1$). The laminar behavior of type-II and type-III
intermittencies develops around a fixed point of generalized Poincare maps:

\[ x_{n+1} = (1 + \varepsilon)x_n + ax_n^p \quad \text{Type-II} \]  
\[ x_{n+1} = -(1 + \varepsilon)x_n - ax_n^p \quad \text{Type-III} \]

where \( a > 0 \) accounts for the weight of the nonlinear component and \( \varepsilon \) is a controlling parameter (\(|\varepsilon| \ll 1\)). For \( \varepsilon \gtrsim 0 \), the fixed point \( x_0 = 0 \) becomes unstable, and hence trajectories slowly escape from the origin preserving and reversing orientation for type-II and type-III intermittencies. In some pioneer papers devoted to type-I intermittency, the nonlinear component in Eq. (2) is quadratic, (i.e. \( p = 2 \)) and cubic for type-II and type-III, i.e. \( p = 3 \) in Eq. (3) and Eq. (4) but actually this restriction is not necessary. In any case, for \( \varepsilon > 0 \), there is a unstable fixed point at \( x = 0 \) for type-II and type-III and there is not a fixed point at \( x = 0 \) for type-I, and hence, the trajectories slowly move along the narrow channel formed with the bisecting line as illustrates Fig. 1 where there are indicated two LBR corresponding with two reinjected mechanisms according with the values of \( \gamma \) of Eq. (6).

Figure 2 illustrates a map having type-II intermittency given by the equation

\[ x_{n+1} = G(x_n) \equiv \begin{cases} 
F(x_n) & x_n \leq x_r \\
(F(x_n) - 1)^\gamma & x_n > x_r 
\end{cases} \]  

Here \( F(x) = (1 + \varepsilon)x + ax^p \) with \( a = 1 - \varepsilon \) and \( x_r \) is the root of the equation \( F(x_r) = 1 \). Note that the map (5) is a generalisation of the map used by Manneville[19], that is, for \( \gamma = 1 \) the map (5) can be write as \( x_{n+1} = (F(x_n) \mod 1) \) and if \( p = 2 \) we recover the Manneville map. Three reinjected mechanisms are also indicated in Fig. 2 depending on the values of the parameter \( \gamma \). For \( \varepsilon > 0 \), an iterated points \( x_n \) of a starting point \( x_0 \) closed to the origin, increases in a process driven by parameters \( \varepsilon \) and \( p \) as it is indicated in Fig. 2. When \( x_n \) becomes larger than \( x_r \), a chaotic burst occurs that will be interrupted when \( x_n \) is again mapped into the laminar region, from the region labelled with heavy black segments. This reinjection process is indicated by a big arrow in Fig. 2.
The next modification of the map 5 illustrates the type I intermittency (see Fig. 1)

\[ x_{n+1} = G(x_n) = \begin{cases} 
\varepsilon + x_n + a|x_n|^p & \text{if } x_n < x_r \\
1 - \hat{x} \left( \frac{x_n - x_r}{1-x_r} \right)^\gamma + \hat{x} & \text{otherwise}
\end{cases} \]

where \( x_r \) is the root of the equation \( \varepsilon + x_n + x_r^p = 1 \) and the parameter \( \gamma > 1 \) driven the nonlinear term of the reinjection mechanism. The parameter \( \hat{x} \) correspond with the so called lower boundary reinjection point (LBR) and it indicates the limit value for the reinjection form the chaotic region into the laminar one.

Note that \( \varepsilon \) and \( p \) modified the duration of the laminar phase where the dynamics of the system look like periodic and \( x_n \) is less than some value, let said \( c \). Note that the function PRD will strongly depend on parameter \( \gamma \), that determines the curvature of the map in region marked by heavy black segment in Fig. 2. Only points in that region will be mapped inside of the laminar region. Note that when \( \gamma \) increases, also increases the number of points that will be mapped around the unstable fixed point \( x = 0 \), hence we expect that the classical hypothesis of uniform RPD used to develop the classical intermittency theory does not work. In the next section we study a more general RPD.

### 2 Assessment of reinjection probability distribution function from data series

The RPD function, determines the statistical distribution of trajectories leaving chaotic region. The key point to solve the problem of model-fitting is to introduce the following integral characteristic:

\[ M(x) = \begin{cases} 
\int_{x_s}^{x} \phi(\tau) d\tau & \text{if } \int_{x_s}^{x} \phi(\tau) d\tau \neq 0 \\
0 & \text{otherwise}
\end{cases} \]

where \( x_s \) is some “starting” point. The interesting property of the function \( M(x) \) is that it is a linear function for a wide class of maps, hence the function \( M(x) \) is an useful tool to find the parameters determining the RPD. Setting a constant \( c > 0 \) that limits the laminar region we define the domain of \( M \), i.e. \( M : [x_0 - c, x_0 + c] \to \mathbb{R} \), where \( x_0 \) is the fixed point of the map.

As \( M(x) \) is an integral characteristic, its numerical estimation is more robust than direct evaluation of \( \phi(x) \). This allows reducing statistical fluctuations even for a relatively small data set or data with high level of noise.

#### 2.1 Fitting linear model to data series

To approximate numerically \( M(x) \), we notice that it is an average over reinjection points in the interval \( (x_s, x) \), hence we can write

\[ M(x) \approx M_j = \frac{1}{j} \sum_{k=1}^{j} x_k, \quad x_{j-1} < x \leq x_j \]
where the data set \((N\) reinjection points) \(\{x_j\}_{j=1}^N\) has been previously ordered, i.e. \(x_j \leq x_{j+1}\).

For a wide class of maps exhibiting type-I, type-II or type-III intermittency the numerical and experimental data show that \(M(x)\) follows the linear law

\[
M(x) = \begin{cases} 
  m(x - \hat{x}) + \hat{x} & \text{if } x \geq \hat{x} \\
  0 & \text{otherwise}
\end{cases}
\]

where \(m \in (0, 1)\) is a free parameter and \(\hat{x}\) is the lower boundary of reinjections (LBR), i.e. \(\hat{x} \approx \inf \{x_j\}\). Then using (7) we obtain the corresponding RPD:

\[
\phi(x) = b(\alpha)(x - \hat{x})^{\alpha}, \quad \text{with} \quad \alpha = \frac{2m - 1}{1 - m}
\]

where \(b(\alpha)\) is a constant chosen to satisfy \(\int_{-\infty}^{\infty} \phi(x) \, dx = 1\). At this point, we note that the linear approximation (9) for the numerical or experimental data determines the RPD given by 10. Figure (3) displays different RPD depending on the exponent \(\alpha\) for \(\hat{x} = 0\) and \(c = 0.5\). It is also shown how the free parameter \(\alpha\) depends on the slope \(m\) according with Eq. (10). For \(m = 1/2\) we recover the most common approach with uniform RPD, i.e. \(\phi(x) = \text{cnst}\), widely considered in the literature. For \(m < 1/2\) we have \(\alpha < 0\) and the RPD increases without bound for \(x \to 0\) as it is shown in Fig. (3). In the opposite case \(m > 1/2\) we have \(\phi(0) = 0\). In this last case, the two possibilities for the RPD, concave or convex are separated by the slope \(m = 2/3\) (see Fig. 3). The RPD (10) has two limit cases:

\[
\phi_0(x) = \lim_{m \to 0} \phi(x) = \delta(x - \hat{x})
\]

\[
\phi_1(x) = \lim_{m \to 1} \phi(x) = \delta(x - c)
\]

(note that \(b(\alpha) \to 0\) in these cases).

From the mathematical RPD shape it is possible to analytically estimate the fundamental characteristic of the intermittency, that is the probability density of the length of laminar phase \(\psi(l)\), depending on \(l\), that approximates the number of iterations in the laminar region, i.e. the length of the laminar
difficulties. Note that the function $\psi(l)$ can be estimated from time series, as it is usual to characterize the intermittency type. The characteristic exponent $\beta$, depending on $\psi(l)$, defined through the relation $l \to \varepsilon^{-\beta}$, is also a good indicator of the intermittency behavior.

The next section is devoted to evaluate the RPD, that is the key point to determine the rest of the properties associated with a specific intermittency.

3 Length of laminar phase and characteristic exponent

The probability of finding a laminar phase of length between $l$ and $l + dl$ is $dl\psi(l)$, where the $\psi(l)$ is the duration probability density of the laminar phase. It is useful to characterize the type of intermittency to compare the analytical prediction for $\psi(l)$ with numerical or experimental evaluation of it. We explain how the RPD of Eq.(10) can be modified to the classical result about $\psi(l)$. The method used is similar for the three types of intermittencies studied here, however, whereas for type-II and type-III it is possible to find the analytical solution, for type-I it is not possible in the general case.

Firstly we study type-II. To do this, we introduce the next continuous differential equation to approximate the dynamics of the local map (3) in the laminar region

$$\frac{dx}{dl} = \varepsilon x + a x^p$$

where $l$ approximates the number of iterations in the laminar region, i.e. the length of the laminar phase. After integration it yields

$$l(x, c) = \frac{1}{\varepsilon} \left[ \ln \left( \frac{x}{\hat{x}} \right) - \frac{1}{p-1} \ln \left( \frac{ae^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon} \right) \right].$$

(14)

Note that Eq. (14) refers to a local behavior of the map in the laminar region and it determines the length of the laminar phase. After integration it yields

$$l(x, c) = \frac{1}{\varepsilon} \left[ \ln \left( \frac{x}{\hat{x}} \right) - \frac{1}{p-1} \ln \left( \frac{ae^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon} \right) \right].$$

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$$l(x, c) = \frac{1}{\varepsilon} \left[ \ln \left( \frac{x}{\hat{x}} \right) - \frac{1}{p-1} \ln \left( \frac{ae^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon} \right) \right].$$

(15)

where $X(l)$ is the inverse function of $l(x, c)$ and we have used the Eq. (16).

Note that $\psi(l)$ depends on the local parameters $\varepsilon$ and $p$, and on the global parameters $\alpha$ and $\hat{x}$ determined by the linear function $M(x)$ according with Eq. (10).

Concerning with type-III intermittency, in the laminar region the sign $x_n$ change in each mapping. However, $|x|$ can be approximated by Eq. 16, consequently the previous values of $\beta$ reported for type-II intermittency can be applied also in the case of type-III.

Let us consider now the case of type-I intermittency. In this case, the equivalent to Eq. (16) for type-I is

$$\frac{dx}{dl} = \varepsilon + a x^p,$$

(16)
from which we obtain $l = L(x,c)$ as a function of $x$

$$L(x,c) = \frac{c}{\varepsilon} \frac{2 F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ac}{\varepsilon}\right)}{2 F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ax}{\varepsilon}\right)}$$

(17)

in terms of the Gauss hypergeometric function $2 F_1(a, b; c; z)$ Abramowitz and Stegun[29]. In the case of $p = 2$, $L(x,c)$ can be given by

$$L(x,c) = \frac{1}{\sqrt{a} \varepsilon} \left[ \tan^{-1}\left(\sqrt{\frac{a}{\varepsilon}} c\right) - \tan^{-1}\left(\sqrt{\frac{a}{\varepsilon}} x\right) \right].$$

(18)

In the case of type-I intermittency, the Eq. 15 transforms into the follow

$$\psi(l) = \phi(X(l,c)) \frac{dX(l,c)}{dl} = \phi(X(l,c)) |aX(l,c)|^p + \varepsilon|$$

(19)

It is interesting to observe that if $\alpha > 0$ we have $\psi(l_{\text{max}}) = 0$ and the graphs of $\psi(l)$ given by Eq.(19) are very different from the obtained for the classical $\psi(l)$ that can be seen in Schuster and Just[18] and Hirsch et al.[30], for instance. The reader can find all possible shapes for the $\psi(l)$ in del Rio et al.[28]. Two of this graphs are displayed in Figs. (4) and (5). Note that $\psi(l)$ in Fig. (4) has a local maximum, what is a remarkable characteristic does not given by the classical theory on type-I intermittency. We will come back to this point in the noise section.

### 3.1 Characteristic relations

Let us described the how the characteristic exponent is affected by the RPD of Eq. (10). This exponent, $\beta$, defined by the characteristic relation

$$I \propto \frac{1}{\varepsilon^\beta}$$

(20)

describes, for small values of $\varepsilon$, how fast the length of the laminar phase grows while $\varepsilon$ decreases. Traditionally is admitted a single value depending on the intermittency type Schuster and Just[18]. The mean value of $l$ is defined by

$$I = \int_0^\infty s \psi(s) ds.$$  

(21)
Taking into account the function $\psi$, depending on the parameter $\hat{x}$ and $\alpha$, (or $m$) we found that the characteristic exponent $\beta$ is not a single value as it is usually established. According with Eq. (21), intermittencies type-II and type-III have the same characteristic exponent that are summarized in Table 1. In a similar way, for type-I intermittency we find the cases described in Table 2.

**Table 1.** The characteristic exponent $\beta$ for types II and III.

<table>
<thead>
<tr>
<th>$\hat{x}$</th>
<th>$m \in (0, 1 - \frac{1}{p})$</th>
<th>$\beta = \frac{a+2-2p}{1-p} = \frac{1-p^m}{1-p(1-m)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x} \approx x_0$</td>
<td>$m \in [1 - \frac{1}{p}, 1)$</td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>$\hat{x} &gt; x_0$</td>
<td>$m \in (0, 1)$</td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>$\hat{x} &lt; x_0$</td>
<td>$m \in (0, 1)$</td>
<td>$\beta = \frac{p-2}{p-1}$</td>
</tr>
</tbody>
</table>

**Table 2.** The characteristic exponent $\beta$ for types I.

<table>
<thead>
<tr>
<th>$\hat{x}$</th>
<th>$m \in (0, 1 - \frac{1}{p})$</th>
<th>$\beta = \frac{a-\alpha-2}{p} = 1 - \frac{1}{1-m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x} \approx x_0$</td>
<td>$m \in [1 - \frac{1}{p}, 1)$</td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>$\hat{x} &gt; x_0$</td>
<td>$m \in (0, 1)$</td>
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<tr>
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<td>$m \in (\frac{1}{2}, 1)$</td>
<td>$\beta = \frac{p-2}{p-1}$</td>
</tr>
<tr>
<td>$\hat{x} &lt; x_0$</td>
<td>$m \in (0, \frac{1}{2})$</td>
<td>$\beta = \frac{p-1}{p}$</td>
</tr>
</tbody>
</table>

### 4 Effect of noise on the RPD

In previous section we have used the function $M(x)$ as a useful tool to study the RPD. In the noisy case, we also use this function to investigate the new noisy RPD, let say NRPD, in systems with intermittency. Figure 6 shows the noise effect on a point near the maximum for the next map having type-III intermittency,

$$x_{n+1} = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n) + \sigma \xi_n,$$  \hspace{1cm} (22)

where $-(1 + \varepsilon) x_n - a x_n^3$ ($a > 0$) is the standard local map for type-III intermittency, whereas the term $d x_n^6 \sin(x_n)$ ($d > 0$) provides the reinjection mechanism into the laminar region around the critical point $x_0 = 0$. In the map (22) $\xi_n$ is a noise with $<\xi_n, \xi_n> = \delta(m-n)$ and $<\xi_n> = 0$ and $\sigma$ is the noise strength. As Fig. 6 illustrates, the RPD corresponding to the noiseless map is generated around the maximum and minimum of the map by a mechanism that is robust against noise. Following this argument we can obtain the NRPD, let say $\Phi(x)$, from the noiseless RPD according to the convolution

$$\Phi(x) = \int \phi(y) g(x - y, \sigma) dy,$$  \hspace{1cm} (23)
where \( g(x, \sigma) \) is the probability density of the noise term \( \sigma \xi_n \) in Eq.(22) (see del Rio et al.[27]).

![Noisy map with type-III intermittency. Dashed line between the two solid lines indicate the effect of the noiseless map on a point near the maximum. These solid lines indicate the effect of the noisy map on the same point, that will be mapped on the region shows by a heavy line on the graph of the map. The dashed circle with radius \( c \) indicates the laminar region.](image)

In the case of uniform distributed noise, after some algebraic manipulation we get the NRPD as

\[
\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + K \sigma)^{1+\alpha} - S(|x| - K \sigma)||x| - K \sigma|^{1+\alpha}}{2K \sigma}.
\]

where we denote by \( S(x) \) the sign function that extracts the sign from its argument. In Eq. (24), the factor \( K \) is due to the length amplification indicates in Fig. 6 where the interval of length equal to \( l \) is mapped into a new interval of length \( Kl \). We emphasize that, according with Eq. (24), the factor \( K \) produces an amplification of the effect of the noise. Note that \( K \) should be equal to one in the case on direct reinjection from the maximum or minimum point, as in the case on type-I and II shown in Fig. 1 and Fig. 2. Figure 7 shows in dashed lines a typical noiseless RPD (with \( \alpha < 0 \)) for map of Eq. (22)) with \( \sigma = 0 \), whereas the solid line corresponds with noisy case according with Eq. (24). Some consequences can be derived from the NRPD of Eq. (24). Firstly, for \( |x| >> K \sigma \) the NRPD approaches to the noiseless RPD and second, for \( x \approx x_0 \)
Fig. 7. Comparison between nosily and noiseless case for the RPD and $M(x)$. Dashes arrows connect different regions of the nosily RPD with the corresponding zone of the $M(x)$.

(note that in this example we set $x_0 = 0$) we have a constant function, that is uniform reinjection. The described consequences of Eq. (24) for the NRDP can be better investigated by using the $M(x)$. Figure 7 shows typical shapes of $M(x)$ for noiseless and nosily cases as indicates. The uniform reinjection case with $m = 1/2$ is indicate by dots line. In this figure, dashed line correspond with dashed RPD. Note that now, the noisy $M(x)$ look like a piece linear function with two slopes. The first one corresponding to the noiseless RPD is observed far from the $x_0$, that is, on the right side a given value $\chi$ in Fig. 7. The second slope approaches to 1/2 corresponding to uniform reinjection and is observed on the left side of $\chi$. This means that, by the analysis of the noisy data, we can predict the RPD function for the noiseless case. To do this, we proceed like in the noiseless case already explained in the previous sections, but considering only the data on the right side of $\chi$ in Fig. 7. That is, by least mean square analysis we can calculate the slope $m$ in Eq.(10), that determines the reinjection function in the noiseless case. Note that now, $K\sigma$ is the single free parameter in Eq. (24).

It is important to note that whereas the noise is applied to the whole map, the function $M(x)$ evidences that, on the right side of $\chi$, the reinjection function is robust against the noise but on the left side of $\chi$, the noise changes the RPD approaching it to the uniform reinjection, at least locally around $x = 0$.

Concerning with the uniform RPD, note that in this case the piece linear function approximation of $M(x)$ shows in Fig. 7 becomes a linear approximation because the two slopes meet in a single one. This meas that the effect of noise on the RPD is not too important for uniform reinjection. Due to this fact, many researches devoted to the noise on the local Poincar map have been published so far, there are only a few study focused on the effect of noise on the RPD. We will find a similar scenario type-II intermittencies.

The case of type-I can be investigated in a similar way, but this type of intermittency presents a different behavior Krause et al.[31]. To illustrate this case, let us consider the map of Eq.(6) with $p = 2$ and a noise perturbation, that is

$$x_{n+1} = G(x_n) = \begin{cases} 
\varepsilon + x_n + ax_n^2 + \sigma \xi_n & \text{if } x_n < x_r \\
(1 - \hat{x}) \left( \frac{x_n - x_r}{1 - x_r} \right)^\gamma + \hat{x} + \sigma \xi_n & \text{otherwise}
\end{cases}$$ (25)
An important difference with Eq. (22), now the reinjection is not symmetric hence the effect of the noise is to shift the LBR from $\hat{x} - \sigma$. Other important consequence of the no-symmetric reinjection is that the convolution (23) gives a different results depending on the relation between reinjection parameters. For the simplest case, we have

$$\Phi(x) = \frac{b}{2\sigma(\alpha + 1)} \left( [x - (\hat{x} - \sigma)]^{\alpha + 1} - \Theta[x - (\hat{x} + \sigma)][x + (\hat{x} + \sigma)]^{\alpha + 1} \right) \quad (26)$$

where $\Theta[\cdot]$ represent the Heaviside step function.

Note that in Eq. (14) the position of the LBR is shifted to a new position given by $\hat{x} - \sigma$. In view of this, we split our analysis in two cases according to $\hat{x} - \sigma > -c$ or $\hat{x} - \sigma < -c$. In the first case all points are reinjected directly into the laminar zone and the function $M(x)$ can be approximated by linear function as Fig. 7 shows. This shape is a consequence of Eq. (26). Note that for $x < \hat{x} - \sigma$ the Heaviside function is zero and we recover for $\Phi(x)$ the same power law that for $\phi(x)$ but the parameters are shifted from $x$ to $\hat{x} - \sigma$ and from $\alpha$ to $\alpha + 1$, consequently, the Eq. (9) now can be written as

$$M(x) = m_1(x - \hat{x}_1) + \hat{x}_1 \quad (27)$$

On the other hand, for $x > x + \sigma$, and for small values of $\sigma$ we can approximate $\Phi(x)$ in Eq. (26) by

$$\Phi(x) \approx b \frac{d}{dx} (x - \hat{x})^{\alpha + 1} \quad (28)$$

hence in that region the exponent of $\Phi(x)$ approximates to the exponent of the noiseless density. Note that according to Eq. (10), the two slopes of $M(x)$, $m_1$ and $m_2$, corresponding to the regions with exponents $\alpha + 1$ and $\alpha$ respectively, are related by

$$m_1 = \frac{1}{2 - m_2} \quad (29)$$

## 5 Conclusions and discussion

In this work an overview of type-I, II and III intermitemcies and a recent method to investigate it are reported.

The main point to described the intermity behavior is to determine the probability density of reinjection (RPD). Through the use of $M(x)$ studied in section 2, we have set a way to obtain an analytical description for the RPD, the density of laminar length and the characteristic relations.

The quantity $M(x)$ has a more reliable numerical and experimental access than $\phi(x)$. In a number of cases the linear approximation $M(x) \approx m(x - \hat{x}) + \hat{x}$ fits very well the numerical or experimental data. According with this approximation we have $\phi(x) = b(x - \hat{x})^{\alpha}$, hence we have found a rich variety of possible profiles for the function $\psi(l)$. Note that the new RPD is a generalization of the usual uniform reinjection approximation which correspond to $\alpha = 0$ or $m = 1/2.$
Because the probability density of the length of laminar phase $\psi(l)$ depends on the RPD, the $\psi(l)$ shapes are qualitatively different from the classical one.

Also it is extended the characteristic relation for type-I, II and III intermit-
tencies. Now, the critical exponent $\beta$ is determined, through the quantities $m$, $\hat{x}$ and $p$ as is reported in section 3.1, hence very different RPDs can lead to the same characteristic exponent $\beta$.

It is worthy to recall that for $m = 0.5$, the classical uniform reinjection is recovered, together with its corresponding characteristic relation.

Even though, there is certainly many papers devoted to the analysis of the effect of noise on the laminar region, the effect of noise on the reinjection probability density has not been fully considered. Note that the noise effect on the uniform RPD can be neglected if does not change the uniform distribution, however it is not the case for a more general RPD. In section 4, we propose an analytical description of the noisy RPD (NRPD) valid for type-I, II and type-III intermittency. We start making a numerical evaluation of the function $M(x)$. From this knowledge, we obtain the reinjection probability density corresponding to the noiseless map, that is generated around the maximum and minimum of the map. It is also important to note that from the RPD, obtained from noisy data, we have a complete description of the noiseless system.

References

Electrohydrodynamic Stability of an Electrified Jet

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Abstract. The axisymmetric stability of a straight jet in electrospinning process is examined for a Newtonian fluid using the leaky dielectric model. While the previous studies consider cylindrical jet of uniform radius as the base-state, in the present study the thinning jet profile obtained as the steady-state solution of the 1D slender filament model is treated as the base-state. The linear stability of the thinning jet is analyzed for axisymmetric disturbances, which are believed to be responsible for the bead formation. The eigen-spectrum of the disturbance growth rate is constructed from the governing equations discretized using the Chebyshev collocation method. The most unstable growth rate for thinning jet is significantly different from that for the uniform jet. For the same electrospinning conditions, the thinning jet is found to be stable whereas the uniform radius cylindrical jet is unstable to capillary mode driven by surface tension. The dominant mode for the thinning jet is believed to be an oscillatory conductive mode driven by the accumulation of the surface charge on the perturbed jet. The role of various material and process parameters in the stability behavior is also investigated.

Keywords: Electrospinning, Electrohydrodynamic instability, Linear stability theory.

1 Introduction

In electrospinning process, nano fibers are produced by subjecting fluid to a very high potential difference. The external electric field acting on the charges located at the fluid surface generates a tangential force leading to an electrified
jet with strong thinning. The solid fibers, so produced, present tremendous potential for technological applications leading to strong interest in the electrospinning process. Many efforts to produce very thin fibers of size below 100 nm suffer from the jet breakup due to surface tension driven capillary instability. In real electrospinning, this instability manifests in the form of bead formation along the fibers. The stability analysis of the electrified jet provides insightful understanding of the conditions under which the instability can be observed.

The early analyses of stability of an electrified cylinder consider either an uncharged jet in an axial electric field \([?]\) or a perfectly conducting jet with a uniform surface charge density but in the absence of an external electric field \([?]\). In electrospinning, the jet possesses both the surface charge and the tangential electric field which significantly alters the dynamics of the jet due to tangential electric stress on the jet surface. Hohman et al. \([?]\) showed that a new mode of instability attributed to the field-charge coupling is introduced for a charged cylinder in the presence of a tangential electric field. This mode, referred to as the conductive mode, is qualitatively different from the surface tension driven Rayleigh-Plateau mode modified by the presence of an electric field. In particular, while increasing the strength of electric field tends to stabilize the capillary mode of instability, it renders the conductive mode unstable. The dominant mode depends strongly on the applied field, surface charge density, jet radius as well as the rheology of the fluid.

Carroll and Joo \([?]\) carried out theoretical and experimental investigation of the axisymmetric instability of an electrically driven viscoelastic jet. Using an Oldroyd-B model to describe the fluid viscoelasticity, linear stability analysis was carried out to obtain growth rate for the axisymmetric instability. The stabilizing role of fluid elasticity has been observed, much in agreement with experiments. However, in all previous studies, the stability is analyzed for a charged cylinder of uniform radius, whereas in electrospinning the charged jet undergoes significant stretching and thinning. While cylindrical jet as base-state simplifies the calculation of the disturbance growth rate, as imposed perturbations can be assumed periodic in axial direction, this simple base-state ignores the variation in radius, and hence the extensional strain rate developed in the fluid. The strong extensional flow in the jet is believed to influence the stability behavior due to the viscous stresses. In the present analysis, we consider the actual thinning jet as the base-state, taking into account the
variation in jet radius, velocity, electric field as well as surface charge density along the axial direction. The nonlinear coupling of these jet variable with the disturbance can alter the stability behavior of, an otherwise, cylindrical jet.

2 Problem formulation

We analyze the straight jet emanating from the nozzle in the presence of an axial electric field. The jet is modeled as 1D slender filament. The variables are radius, \( R \), velocity, \( v \), surface change density, \( \sigma \) and electric field within the jet, \( E \), made non-dimensionalized using nozzle radius, \( R_0 \), velocity at the nozzle, \( v_0 = Q/ (\pi R_0^2) \), \( \sigma_0 = \epsilon E_0 \), and \( E_0 = I/(\pi R_0^2 K) \). Here, \( Q \) is the volumetric flow rate, \( I \) is the current passing through the jet, \( \epsilon \) is the air permittivity, and \( K \) is the electrical conductivity of the fluid. Additionally, time is non-dimensionalized by \( R_0/v_0 \) and stress in the fluid by \( \rho v_0^2 \), \( \rho \) being the fluid density. In real electrospinning, there exists a non-zero charge density on the surface of the jet, and also the axial-electric field, leading to a strong electric tangential shear force which is responsible for thinning of the jet. The electrical forces in the fluid with finite conductivity is described using the leaky dielectric model. The dimensionless governing equations describing the electrohydrodynamics of the jet are [? ? ]:

\[
\frac{\partial R}{\partial t} + \frac{R \partial R}{\partial z} = 0, \quad (1)
\]
\[
\frac{\partial v}{\partial t} + \frac{v \partial v}{\partial z} = \frac{3}{Re} \frac{1}{R^2} \frac{\partial^2 v}{\partial z^2} + \frac{1}{We} \left( \frac{1}{R^2} \frac{\partial R}{\partial z} + \frac{\partial^3 R}{\partial z^3} \right) + \frac{1}{Fr} + \frac{\alpha}{R} \left( \frac{\partial \sigma}{\partial z} + \beta \epsilon \frac{\partial E}{\partial z} + \frac{2 E \sigma}{R} \right), \quad (2)
\]
\[
\frac{\partial (R \sigma)}{\partial t} + \frac{\partial}{\partial z} \left( E R^2 + Pe R \sigma \right) = 0, \quad (3)
\]
\[
E = E_\infty - \ln(\chi) \left( \frac{d(\sigma R)}{dz} - \frac{\beta}{2} \frac{d^2(E R^2)}{dz^2} \right). \quad (4)
\]

Here, equation (1) is the mass conservation equation; equation (2) represents the conservation of momentum; equation (3) is the conservation equation for the electrical charge; and equation (4) is the governing equation for the axial-electric field within the jet. The definitions of various dimensionless numbers are given in Table 1.
Table 1. List of dimensionless groups

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Dimensionless number</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pe</td>
<td>Electric Peclet No.</td>
<td>( \frac{Ze_{00}}{R_0} )</td>
</tr>
<tr>
<td>Fr</td>
<td>Froude No.</td>
<td>( \frac{v_0^2}{g R_0} )</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds No.</td>
<td>( \frac{\rho v_0^2 R_0}{\eta} )</td>
</tr>
<tr>
<td>We</td>
<td>Weber No.</td>
<td>( \frac{\rho v_0^2 R_0}{\gamma} )</td>
</tr>
<tr>
<td>E_0</td>
<td>Initial electric field</td>
<td>( \frac{1}{(\pi R_0^2 \kappa)} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Alpha</td>
<td>( \frac{\epsilon E_0^2}{\rho v_0^2} )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Relative permittivity</td>
<td>( \epsilon - 1 )</td>
</tr>
<tr>
<td>E_\infty</td>
<td>Imposed potential difference</td>
<td>( \frac{(\Delta V/d)}{E_0} )</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Jet aspect ratio</td>
<td>( d/R_0 )</td>
</tr>
</tbody>
</table>

The electric forces due to Maxwell stresses on the slender filament are obtained using the jump conditions:

\[
||\epsilon E_n|| = \bar{\epsilon} \bar{E}_n - \epsilon E_n = \sigma, \quad (5)
\]

\[
||\epsilon E_t|| = \bar{E}_t - E_t = 0, \quad (6)
\]

where the overbar signifies the parameter for the ambient air outside the jet.

2.1 Base-state

In previous studies \([? ]\), the base-state has been considered to be a cylindrical jet of uniform radius. However, in the present study, we perform linear stability analysis of the thinning jet, representing the actual profile during electrospinning. Therefore, the base-state for the stability analysis is the steady-state solution of the governing equations (1-4), which are solved numerically to obtain the steady profile. The governing equations are supplemented with the following boundary conditions at the top \((z = 0)\):

\[
R(0) = 1, \quad v(0) = 1. \quad (7)
\]

The surface charge density at the nozzle-exit, generally, depends upon the geometry of the top electrode. The simple 1D model employed cannot capture the details of the charge distribution near the electrode. We assume that near the nozzle the free charges are distributed within the bulk of the fluid and hence following boundary condition is enforced \([? ]\):

\[
\sigma(0) = 0. \quad (8)
\]
As the jet travels towards the bottom electrode, the free charges migrate to the surface of the jet (fluid-air interface) and \( \sigma(z) \) becomes non-zero short distance from the nozzle.

In real electrospinning the straight jet undergoes whipping motion after certain distance. Since, we examine only the straight jet, the boundary conditions at the end of the straight jet are naturally unknown. However, far away from the electrode, the electric field may be assumed to reach its imposed value:

\[
E(\chi) = E_{\infty}.
\]  

[?] derived the asymptotic thinning condition considering that the radius of the jet in the exit condition is very small and the electric forces are comparable to the inertial forces, giving rise to following condition to be imposed at \( z = \chi \):

\[
R + 4z \frac{dR}{dz} = 0.
\]  

The steady-state solution of the nonlinear governing equations (1-4) is obtained using the relaxation method. Figure 1 shows the profiles of jet radius \( (\bar{R}(z)) \), velocity \( (\bar{v}(z)) \), charge density \( (\bar{\sigma}(z)) \) and electric field \( (\bar{E}(Z)) \) for a set of dimensionless parameters corresponding to a Newtonian jet of glycerol.
3 Linear stability analysis

3.1 Stability analysis of a uniform jet

For stability analysis, the disturbance can be imposed on a cylindrical jet of uniform radius, as done by [? ] and others. In this case, the normal mode disturbance of the following form is superimposed on the steady-state jet variable:

\[ \phi(z, t) = \bar{\phi} + \epsilon \phi_\epsilon e^{ikz + i\omega t}, \] (11)

where \( \phi \) represents the generic jet variable \( \phi = [R, v, \sigma, E]^T \), and \( \bar{\phi} \) denotes its steady-state value. \( k \) is the axial wavenumber of the disturbance and \( \omega \) is the temporal growth/decay rate of the imposed disturbance. The steady-state jet radius \( \bar{R} \) may be taken as unity, representing jet radius near the capillary or the radius of the thinned jet near the bottom collector plate. For stability analysis of a jet of uniform property, we consider the steady-state variables \( \bar{\phi} = [\bar{R}, \bar{v}, \bar{\sigma}, \bar{E}]^T \) corresponding to the thinned jet, i.e. \( \bar{\phi} = \phi|_{z=\chi} \), as shown in Figure 1. After substituting the superposition equation (11) in the governing equations (1) - (4) and linearizing about the base-state, using \( \epsilon \) as a small parameter, the algebraic equations for the disturbance dynamics are obtained. The non-trivial solution for disturbance variables \( \phi_\epsilon \) results into a dispersion relation for the disturbance growth rate, \( \omega = \omega(k) \).

For the perturbations imposed on a cylindrical jet, the base-state profile \( \bar{\phi}(z) \) is taken as \( \bar{\phi}(\chi) \), a uniform value corresponding to the end-value of the jet variable \( \phi \). For the steady-state profiles shown in Section 2.1, the base-state variables are \( \bar{R} = 1.65 \times 10^{-2} \), \( \bar{E} = 50 \), and \( \bar{\sigma} = 1630 \). Considering the reference frame moving with the cylindrical jet, we take \( \bar{v} = 0 \). For this jet profile, the dispersion relation provides the growth-rate as a function of disturbance wavenumber, as shown in Figure 2. The cylindrical electrified jet is predicted to be unstable with maximum growth rate corresponding to wavenumber \( k \approx 0.05 \), made non-dimensionalized with capillary radius, \( R_0 \).

3.2 Stability analysis for a thinning jet

Considering the base-state as a cylindrical jet of uniform radius is not appropriate as the jet undergoes strong thinning during the electrospinning. The uniform radius jet ignores the stretching and hence the axial strain rate that is developed in the electrified jet. Since the viscous stresses are important in the
jet dynamics, the oversimplification of uniform jet neglects the role of the viscous stresses on the jet instability. The nonlinear coupling of the steady-state extension rate and the disturbance of jet radius is believed to play an important role in the stability behavior. In the present study, we consider the thinning profile $\tilde{\phi}(z)$ as the base-state for the linear stability analysis. The generic variable is expanded as steady-state profile superposed with infinitesimal amplitude non-periodic disturbance as follows:

$$\phi(z,t) = \tilde{\phi}(z) + \epsilon \tilde{\phi}(z) e^{i\omega t},$$  \hspace{1cm} (12)$$

where $\tilde{\phi}(z)$ is the steady-state jet profile and $\tilde{\phi}(z)$ denotes the disturbance profile. Upon substituting above superposition in the conservation equations and linearizing to $O(\epsilon)$ terms result into the disturbance governing equations. For the form of non-periodic disturbance imposed, $\tilde{\phi}(z)$, we need to identify boundary conditions for the disturbance variables. The boundary conditions are:

$$\bar{R}(0) = 0, \quad \tilde{v}(0) = 0,$$

$$\bar{E}(0) = 0, \quad \tilde{\sigma}(0) = 0.$$  \hspace{1cm} (13)$$
Fig. 3. Eigenspectrum of disturbance growth rate: real part against imaginary part of the growth rate for uniform jet superimposed with non-periodic perturbations. Parameters: $\chi = 75$, $\beta = 50$, $Re = 10^{-3}$, $We = 10^{-3}$, $Fr = 10^{-3}$, $Pe = 10^{-5}$, $\alpha = 0.01$, and $E_\infty = 50$.

At lower end of the jet, $z = \chi$, we consider following conditions:

$$\tilde{R}(\chi) = 0 \quad \tilde{E}(\chi) = 0.$$  \hfill (15)

The disturbance equations are discretized using the Chebyshev collocation technique resulting into a generalized eigenvalue problem of the form:

$$\mathbf{A}\tilde{\phi} = \omega\mathbf{B}\tilde{\phi},$$  \hfill (16)

where $\mathbf{A}$ and $\mathbf{B}$ are matrices of size $4N \times 4N$, with $N$ being the number of collocation points in the domain $z = (0, \chi)$. The spectrum of complex eigenvalues is obtain using LAPACK numerical libraries.

In order to validate the numerical scheme, we first obtain the eigenspectrum for the jet of uniform radius, studied in previous section. Figure 3 plots the eigenspectrum showing the real and imaginary parts of the discrete eigenvalues, $\omega_r$ and $\omega_i$ respectively. As seen, the eigenspectrum is unaffected by the number of Chebyshev collocation points, $N$, thus eliminating the possibility of any spurious eigenvalues. The most unstable eigenvalue has growth rate $\omega_r \approx 6.04$, which is similar to the maximum $\omega_r$ obtained earlier using periodic
Fig. 4. Eigenspectrum of disturbance growth rate: real part against imaginary part of the growth rate for a thinning jet superimposed with non-periodic perturbations. Parameters: $\chi = 75$, $\beta = 50$, $Re = 10^{-3}$, $We = 10^{-3}$, $Fr = 10^{-3}$, $Pe = 10^{-5}$, $\alpha = 0.01$, and $E_{\infty} = 50$.

So far, we have used the end-values of the jet profile when the jet has sufficiently thinned far away from the capillary, as the base-state upon which the infinitesimal amplitude disturbances are imposed. Thus, considering $\bar{\phi}(z) = \bar{\phi}(\chi)$ in equation (12) ignores the entire thinning profile of the steady-state jet. Next, the disturbances are superimposed on the thinning profile $\bar{\phi}(z)$ taking into account the role of extensional rate in stability behavior. Figure 4 shows the eigenspectrum for the thinning jet using the same set of parameters as used for the cylindrical jet. The eigenspectrum is found to be independent of the discretization points, $N$. Comparing with Figure 3 for the cylindrical jet, the thinning jet is found to be stable as the real part of the growth rate $\omega_r$ is negative, $\omega_r \approx -2.1$, for the leading eigenvalue. Therefore, the viscous stresses as well as the variation in the surface charge density along the fiber render stability to the jet.

The effect of various parameters on the leading growth rate is shown in Figure 5. On decreasing the surface tension, i.e. increasing Weber number, the
real part of the leading growth rate is found to be nearly unaffected, as shown in Figure 5(a). The insensitivity of surface tension to the disturbance growth rate indicates that the leading eigenvalue corresponds to the conductive mode of instability. This instability is driven by the electric field in the presence of non-zero charge density on the jet surface \([?, \text{?}]\). To further confirm the type of instability mode, the influence of external electric field, \(E_\infty\), is shown in Figure 5(b). As the strength of external field increases, the leading growth rate, \(\omega_r\), increases, even though remaining negative. Thus, the electric field tends to weaken the stability of the jet. For the set of parameters employed, the growth rate of the leading disturbance remains negative for a range of electric field strength studied.

Finally, we examine the effect of electrical conductivity of the fluid on the leading growth rate. As seen in Table 1, the conductivity, \(K\), appears in the electric Péclet number, \(Pe\) and the definition of initial electric field \(E_0\), which in turn, affects dimensionless numbers \(\alpha\) and \(E_\infty\). Hence, to study the effect of variation in fluid conductivity, three dimensionless parameters, viz. \(Pe\), \(\alpha\) and \(E_\infty\) are varied, in accordance with their definitions. Figure 6 shows the influence of electrical conductivity of the fluid on the leading growth rate. It should be noted that in addition to Péclet number, \(\alpha\) and \(E_\infty\) are also varied so that the variation in \(K\) is captured keeping other parameters unchanged. With decrease in conductivity (increase in \(Pe\)), the surface charge density decreases. Since the leading mode is conductive mode, its growth rate is significantly
Fig. 6. Effect of electrical conductivity of the fluid on the leading growth rate. Parameters: $\chi = 75$, $\beta = 50$, $Re = 10^{-3}$, $We = 10^{-3}$, $Fr = 10^{-3}$, $\alpha = 0.01$, and $E_{\infty} = 50$.

affected by the surface charge density. Thus, the leading growth rate is found to decrease with increase in Péclet number.

4 Conclusion

The stability of a charged fluid jet under axial electric field is analyzed to understand the bead formation during electrospinning process. Contrary to previous studies in which the jet has been considered cylindrical with uniform radius, the present analysis considers the actual thinning jet as the base-state for stability analysis. Taking into consideration, the gradient of jet radius and other variables along the axial-direction is found to significantly influence the stability behavior of the jet. In particular, we find the thinning profile renders the flow stable to axisymmetric disturbances. Under the same operating and material parameters, while the uniform jet has positive growth rate, the thinning jet is found to be stable with negative growth rate. The leading growth rate appears to be a conductive mode, such that an increase in applied voltage or increase in current tends to have destabilizing effect. However, the growth rate remains negative for the range of parameters studied.
Comparison of Nonlinear Dynamics of Parkinsonian and Essential Tremor

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Abstract. It is known that wrong clinical diagnosis of Parkinson’s disease is about 20% among patients suffering from pathological tremor. That is why the search of new possibilities to improve the diagnostics has high priority. The aim of the work is to answer the question whether the methods of nonlinear dynamics can be used for the guaranteed differential diagnostics of two main types of pathological tremor (parkinsonian and essential ones). We have analyzed tremor determined as fast involuntary shaking and arising during the performance of the motor task by healthy subjects and two groups of patients with parkinsonian syndrome. The first group has the primary Parkinson’s disease and the second group has the essential tremor as finger’s shaking during the some movements as the main symptom. Using the wavelet transform modulus maxima method, the calculation of the Hölder exponents as well as the detection of unstable periodic orbits and surrogate data we demonstrate the statistically confirmed differences in dynamical complexity, multifractality degree and number of unstable periodic orbits for the two groups of patients. The results give the positive answer the question rose in the work.

Keywords: Dynamical Complexity, Unstable Periodic Orbits, Multifractality, Parkinson’s disease, Essential Tremor.

1 Introduction

In spite of enormous number of works [1, 2] devoted to the study of pathological tremor the topic is of immediate interest because of large number of clinical errors connected with wrong administration of antiparkinsonian drugs for subjects having tremor symptoms but not having Parkinson’s disease. For example, parkinsonian tremor and so called essential tremor (or action tremor) when the body parts are involved into involuntary shaking during the movement performance differ by frequency. The frequency in essential tremor, however, declines with age in the side of the parkinsonian tremor frequency [3] so that oldest patients can be objects of clinical errors.
The aim of the work is to answer the question whether the methods of nonlinear dynamics can be used for the guaranteed differential diagnostics of two main types of pathological tremor (parkinsonian and essential ones). We studied involuntary shaking (tremor) of fingers accompanied the performance of the motor task such as sustaining the given effort of human hands under isometric conditions (without finger movement in space). For estimating the tremor features we used the methods of nonlinear dynamics such as the wavelet transform and multifractal analysis as well as recurrence plot technique for detecting unstable periodic orbits and surrogate data. We demonstrate the use of these methods for a diagnostics of the human motor dysfunction.

2 The experimental procedure

We used the results of testing 10 healthy subjects aged 47-54 years, 6 parkinsonian patients with bilateral akinesis and tremor aged 45–62 years and 7 subjects with syndrome of essential tremor and without other symptoms of Parkinson’ disease. The motor task was to control the isometric muscle effort with the strength of muscle contraction shown by the positions of marks on a monitor. The subjects sat in front of a monitor standing on a table and pressed on platforms containing stress sensors with their fingers. The sensors transformed the pressure strength of the fingers of each hand into an electric signal. The rigidity of the platforms made it possible to record the effort in the isometric mode, i.e., without noticeable movement of fingers at the points of contact with the sensors. The isometric effort was recorded for 50 s. The subject’s fingers sustained an upward muscle effort, with the back of each hand pressing against the base of the platform.

The patients with Parkinson’s disease did not take any drugs before the test on the day of testing. Usually, these patients received nakom, an antiparkinsonian preparation three times a day to compensate for dopamine deficiency. The subjects with syndrome of essential tremor did not have tremor medication.

The recorded trajectory of isometric effort consisted of a slow trend and a fast involuntary component (tremor), which was isolated from the recorded trajectory using the MATLAB software.

3 Wavelet transform and multifractality

3.1 Estimation the global wavelet spectrum of the tremor

To evaluate the difference between physiological and pathological tremors, we used the wavelet transform modulus maxima (WTMM) method [4] based on the continuous wavelet transform of a time series describing the examined tremor $x(t)$:
\[
W(a,t_0) = a^{-1/2} \int_{-\infty}^{\infty} x(t) \psi^*(\frac{(t-t_0)}{a}) \ dt,
\]

where \( a \) and \( t_0 \) are the scale and space parameters, \( \psi((t-t_0)/a) \) is the wavelet function obtained from the basic wavelet \( \psi(t) \) by scaling and shifting along the time, symbol * means the complex conjugate. As the basic wavelet we use the complex Morlet wavelet:

\[
\psi_0(t) = \pi^{-1/4} \exp(-0.5t^2) \left( \exp(i\omega_0 t) - \exp(-0.5\omega_0^2) \right),
\]

where the second component in brackets can be neglected at \( \omega_0=2\pi>0 \), the multiplier factor \( \exp(i\omega_0 t) \) is a complex form of a harmonic function modulated by the Gaussian \( \exp(-0.5t^2) \), the coefficient \( \pi^{-1/4} \) is necessary to normalize the wavelet energy. The value \( \omega_0=2\pi \) gives the simple relation \( f=1/a \) between the scale \( a \) and the frequency \( f \) of the Fourier spectrum. Then expression has the form:

\[
W(f,t_0) = \pi^{-1/4} \sqrt{F} \int_{-\infty}^{\infty} x(t) \exp(-0.5(t-t_0)^2 f^2) \exp(-i2\pi(t-t_0)f) \ dt.
\]

The modulus of the wavelet spectrum \( |W(f, t_0)| \) characterizes the presence and intensity of the frequency \( f \) at the moment \( t_0 \) in the signal and \( |W(f, t_0)|^2 \) describes the instantaneous distribution of the tremor energy over frequencies, that is, the local spectrum of the signal energy at the time \( t_0 \).

The value

\[
E(f) = \int |W(f, t_0)|^2 \ dt
\]

determines the global wavelet spectrum, i.e., the integral distribution of the wavelet spectrum energy over frequency range on the time interval \([t_1, t_2]\).

### 3.2 Estimation the tremor multifractality

Information about possible multifractal feature of the signal and its localization \( t_0 \) reflects in the asymptotic behavior of coefficients \( |W(a, t_0)| \) at small \( a \) values and large \( f \) values, respectively. Abnormal small decrease of the wavelet coefficients at \( a\to0 \) in a neighborhood of the point \( t_0 \) testifies about singularity of the signal at the point. Thus, the rate of the change of the modulus of the wavelet coefficients enables to analyze the presence or absence of singularities of the signal.

The degree of singularity of the signal \( x(t) \) at the point \( t_0 \) is described by the Hölder exponent, \( h(t_0) \), the largest exponent such that the analyzed signal in a neighborhood of the point \( t_0 \) can be represented as the sum of the regular component (a polynomial \( P_n(t) \) of order \( n < h(t_0) \)) and a member describing a non-regular behavior [4]:

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\[ x(t) = P(t) + c(t - t_0)^{h(t_0)}. \]

The value \( h(t_0) \) is the measure of singularity of the signal at the point \( t_0 \) since the smaller \( h(t_0) \) value, the more singular the signal. The Hölder exponents characterize the presence of correlations of different types in the analyzed process, e.g., anti-correlated \((h < 0.5)\) or correlated \((h > 0.5)\) dynamics or absence of correlations \((h = 0.5)\).

The Hölder exponents are found on the basis of statistical description of local singularities by partition functions \([5]\). The algorithm consists of the following procedures.

1) The continuous wavelet transform of the time series is used.
2) A set \( L(a) \) of lines of local modulus maxima of the wavelet coefficients is found at each scale \( a \).
3) The partition functions are calculated by the sum of \( q \) powers of the modulus maxima of the wavelet coefficients along each line at the scales smaller than the given value \( a \):
\[
Z(q,a) = \sum_{l(a)} \left( \sup_{a' < a} \left| W(a', t_l(a')) \right|^q \right),
\]

\( t_l(a^*) \) determines the position of the maximum corresponding to the line \( l \) at this scale.
4) The partition function is \( Z(q,a) \sim a^\tau(q) \) at \( a \to 0 \) \([5]\), therefore, the scaling exponent can be extracted as
\[
\tau(q) \sim \log_{10} Z(q,a) / \log_{10} a.
\]
5) Choosing different values of the power \( q \) one can obtain a linear dependence \( \tau(q) \) with a constant value of the Hölder exponent
\[
h(q) = d \tau(q) / dq = \text{const}
\]

for monofractal signals and nonlinear dependence
\[
\tau(q) = q h(q) - D(h)
\]

with large number of the Hölder exponents for multifractal signals.
6) The singularity spectrum (distribution of the local Hölder exponents) is calculated from the Legendre transform \([5]\):
\[
D(h) = q h(q) - \tau(q).
\]

Using the global wavelet spectra and the WWTM algorithm for different tremor recordings we obtain the maximum of the global tremor energy \( (E_{\text{max}}) \) and two multifractal parameters: a) the width of the singularity spectrum
\[
\Delta h = h_{\text{max}} - h_{\text{min}},
\]

where \( h_{\text{max}} \) and \( h_{\text{min}} \) are the maximal and minimal values of the Hölder exponent corresponding to minimal и maximal tremor fluctuation, respectively; b) the asymmetry of the singularity spectrum
\[
\Delta = | \Delta_2 - \Delta_1 |,
\]

where \( \Delta_1 = h_{\text{max}} - h_0 \) and \( \Delta_2 = h_0 - h_{\text{min}} \), \( h_0 = h \ (q = 0) \).

Smaller \( \Delta h \) indicates that the time series tends to be monofractal and larger \( \Delta h \) testifies the enhancement of multifractality. The asymmetry parameter \( \Delta \)
characterizes where, in the region of strong singularities \((q > 0)\) or in the region of weak singularities \((q < 0)\), the singularity spectrum is more concentrated.

To compare the mean values in each of the examined group of subjects the Student criterion was applied.

### 4 Recurrence plot and localization of unstable periodic orbits

The set of unstable periodic orbits (UPOs) which form the skeleton of the chaotic attractor can be found by the recurrence quantification analysis (RQA) [6]. The calculation for the RQA was performed using the CRP Toolbox, available at tocsy.pik-potsdam.de/crp.php.

A recurrence plot (RP) is a graphical representation of a matrix defined as

\[
R_{ij}(m,\varepsilon) = \Theta(\varepsilon - \|y_i - y_j\|)
\]

where \(\varepsilon\) is an error (threshold distance for RP computation), \(\Theta(\cdot)\) is the Heaviside function, \(\|\cdot\|\) denotes a norm and \(y\) is a phase space trajectory in a \(m\)-dimension phase space [7]. The trajectory can be reconstructed from a time series by using the delay coordinate embedding method [8].

The values \(R_{ij} = 1\) and \(R_{ij} = 0\) are plotted as gray and white dots, reflecting events that are termed as recurrence and nonrecurrence, respectively.

The recurrence time is defined as the time needed for a trajectory of a dynamical system to return into a previously visited neighborhood [9].

The pattern corresponding to periodic oscillations (periodic orbits) is reflected in the RP by noninterrupted equally spaced diagonal lines. The vertical distance between these lines corresponds to the period of the oscillations. The chaotic pattern leads to the emergence of diagonals which are seemingly shorter. The vertical distances become irregular. When the trajectory of the system comes close to an unstable periodic orbit (UPO), it stays in its vicinity for a certain time interval, whose length depends on how unstable the UPO is [9, 10]. Hence, UPOs can be localized by identifying such windows inside the RP, where the patterns correspond to a periodic movement. If the distance between the diagonal lines varies from one chosen window to the other then various UPOs coexist with different periods.

The period of UPO can be estimated by the vertical distances between the recurrence points in the periodic window multiplied by the sampling time of the data series [9, 11].

The algorithm for finding UPOs consists of the following procedures.

1. A phase space trajectory \(y(t)\) is reconstructed from a measured time series \(\{x(t)\}\) by the delay coordinate embedding method:

\[
y(t) = (x(t), x(t+d), \ldots, x(t+(m-1)d)),
\]

where \(m\) is the embedding dimension and \(d\) is the delay time. Parameters \(m=5\) and \(d=2\) were chosen on the basis of first minimum of the mutual information function and the false nearest neighbor method [12].

2. To identify unstable periodic orbits a recurrence plot
is constructed with the threshold distance $\epsilon$ equal to 1% of the standard deviation of the data series.

3. The recurrence times of second type [10] are found for the recurrence neighbourhood of radius $\epsilon$. The values of recurrence periods are determined as recurrence times multiplied by the sampling time of the data series. The values are recorded in a histogram. The periods of UPOs are the maxima of the histogram of the recurrence periods.

4. To exclude the noise influence the obtained UPOs are tested for statistical accuracy. For this purpose the procedure is repeated for 30 surrogates obtained as randomized versions of the original data. In the surrogate data the time interval sequences are destroyed by randomly shuffling the locations of the time intervals of original data [13].

The statistical measure of the presence of statistically significant UPOs in the original time series is given by the ratio

$$k = (A - \bar{A})/\sigma,$$

where $A$ is the value of maximum of the histogram, $\bar{A}$ is the mean of $A$ for surrogates and $\sigma$ is a standard deviation. The value of $k$ characterizes the existence of statistically significant UPOs in the original data in comparison with its surrogate (noisy) version. The value $k>2$ means the detection of UPOs with a greater than 95% confidence level.

5 Results and discussion

Examples of fast component of the isometric force trajectory of the human hand (tremor) for the healthy subject, the patient with Parkinson disease and for the subject with essential tremor as well as their global wavelet spectra are given in Fig.1. The healthy and pathological tremors differ by spectra maxima. The maximum ($E_{\text{max}}$) of the physiological tremor spectrum is in the frequency range of the alpha rhythm [8, 14] Hz. For the pathological tremor $E_{\text{max}}$ is shifted in the theta range [4, 7.5] Hz and it increases in ten times in the parkinsonian tremor and in five times in the essential one as compared with the healthy tremor. The essential tremor spectrum has two peaks as opposed to the parkinsonian tremor but the values of the peaks do not differ significantly. Figure 2 illustrates the differences in the singularity spectra $D(h)$ for the same subjects. The form of spectrum testifies the multifractality of both physiological and parkinsonian tremor but the spectra differ for the three examples.
Fig. 1 Examples of healthy, parkinsonian and essential tremors (left column) and their global wavelet spectra $E(f)$ (right column).

Fig. 2 Examples of the singularity spectra $D(h)$ for the different tremors (left column) and intervals between local maxima of the tremor data (right column).
The healthy tremor is characterized by the largest width $\Delta h$ of the singularity spectrum and, therefore, by the significant degree of multifractality. The decline in the width of the spectrum shows a fall in the multifractality degree. It means a reduction of nonuniformity of the pathological tremors. We illustrate it in the right column of Figure 2 where intervals between local maxima of the tremor data are depicted.

The parkinsonian tremor is characterized by the smallest width of the singularity spectrum and its smallest asymmetry ($\Delta$). The values of $\Delta h$ and $\Delta$ for the essential tremor are larger than for the parkinsonian one but they do not exceed the values for healthy tremor.

The decrease of both parameters in pathological tremor is due to decreasing contribution of weak fluctuations (for $q < 0$). These fluctuations lead to the expansion of the singularity spectrum and emergence of both anticorrelated (for $h < 0.5$) and correlated (for $h > 0.5$) dynamics of sequent intervals between local maxima of the tremor data.

The recurrence plots depicted in Figure 3 exhibit non-homogeneous but quasi-periodic recurrent structures reflecting in that the distances between the diagonal lines vary in all the three considered tremors. The RP of the healthy

Fig. 3. Examples of recurrence plots for the different tremors (left column) and histograms of recurrence periods for tremor data and their surrogates (right column, solid and dash-and-dot lines, respectively).

Parameters: the embedding dimension $m = 5$, the delay time $d = 2$, the threshold distance $\epsilon = 1\%$ of the standard deviation of the data series.

The recurrence plots depicted in Figure 3 exhibit non-homogeneous but quasi-periodic recurrent structures reflecting in that the distances between the diagonal lines vary in all the three considered tremors. The RP of the healthy
tremor is characterized by small black rectangles, whereas the RPs from the pathological tremors show larger rectangles. These rectangles may reflect time intervals when the trajectory is travelling near the corresponding UPOs [10].

The recurrence times obtained from the RP given in the Figure 3 are clustered in the intervals around the value $i=24$ for the healthy tremor, around $i=36$ and 72 for the parkinsonian one and around $i=28$, 84 and 168 for the essential tremor. Taking into account the value of the sampling rate value $dt=0.005(s)$ the recurrence periods are equal to 0.12 (s) for the healthy data, 0.18 (s) and 0.36 (s) for the parkinsonian data and 0.14 (s), 0.42 (s) and 0.84 (s) for the essential data. These recurrence periods were extracted as peaks of the histograms given in the right column of Figure 3 (solid lines). The periods obtained can be used for localization of UPOs.

Testing surrogate data we excluded the values 0.12 (s) and 0.36 (s) since the statistical measure $k<1$ in both cases. For other recurrence periods extracted from Figure 3 the value $k>2$ that supports the detection of UPOs with a greater than 95% confidence level. Thus, for the healthy tremor data represented in Figure 3 there are no statistically significant UPOs. By contrast, the UPO of period 1 (0.18 s) is found for the parkinsonian tremor and the UPOs of periods 1, 3 and 6 are obtained for the essential tremor (0.42/0.14=3, 0.84/0.14=6).

The similar dynamics of the wavelet and multifractal parameters as well as UPOs localization is observed for all the examined subjects. It enables us to use the common practice of averaging the recordings of all subjects for testing significant variations among the groups.

The values of $E_{max}$, $\Delta h$, $\Delta$ and statistical measures $k$ for UPOs of various periods averaged by subjects in every group are given in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>hand</th>
<th>healthy</th>
<th>parkinsonian</th>
<th>essential</th>
</tr>
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<tbody>
<tr>
<td>$E_{max}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>left</td>
<td>0.029±0.001</td>
<td>0.45±0.02</td>
<td>0.25±0.01</td>
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<tr>
<td>right</td>
<td>0.037±0.003</td>
<td>0.56±0.04</td>
<td>0.31±0.02</td>
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</tr>
<tr>
<td>$\Delta h$</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>left</td>
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<td>0.22±0.02</td>
<td>0.49±0.05</td>
<td></td>
</tr>
<tr>
<td>right</td>
<td>0.76±0.09</td>
<td>0.27±0.02</td>
<td>0.42±0.04</td>
<td></td>
</tr>
<tr>
<td>$\Delta$</td>
<td></td>
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<tr>
<td>left</td>
<td>0.46±0.04</td>
<td>0.09±0.01</td>
<td>0.27±0.03</td>
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<tr>
<td>right</td>
<td>0.38±0.03</td>
<td>0.12±0.01</td>
<td>0.20±0.02</td>
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<tr>
<td>$k(p_1)$</td>
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<td></td>
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<tr>
<td>left</td>
<td>&lt;1</td>
<td>4.9±0.8</td>
<td>5.7±0.9</td>
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<tr>
<td>right</td>
<td>&lt;1</td>
<td>3.8±0.6</td>
<td>4.5±0.8</td>
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<tr>
<td>$k(p_2)$</td>
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<td>&lt;1</td>
<td>&lt;1</td>
<td></td>
</tr>
<tr>
<td>right</td>
<td>&lt;1</td>
<td>2.1±0.6</td>
<td>&lt;1</td>
<td></td>
</tr>
<tr>
<td>$k(p_3)$</td>
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</tr>
<tr>
<td>left</td>
<td>&lt;1</td>
<td>&lt;1</td>
<td>2.1±0.3</td>
<td></td>
</tr>
<tr>
<td>right</td>
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<td>&lt;1</td>
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<tr>
<td>$k(p_6)$</td>
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<td>&lt;1</td>
<td>3.8±0.4</td>
<td></td>
</tr>
<tr>
<td>right</td>
<td>&lt;1</td>
<td>&lt;1</td>
<td>4.1±0.4</td>
<td></td>
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</tbody>
</table>

Table 1. Comparison of the mean values of wavelet and singularity spectra characteristics and statistical measure of UPOs (averaging over subjects inside the every examined group).
The significant distinctions between the states (pathological or physiological tremor) are identified by all the parameters ($p<0.03$). The values for the essential and parkinsonian tremors also differ ($p<0.05$). The results serve one more verification for the decline of dynamical complexity of time intervals in pathological tremor. It exhibits in the decrease of the multifractality degree, disappearance of long–range correlations and transitions to strongly periodic dynamics including the emergence of unstable periodic orbits in involuntary oscillations of the human hand.

**Conclusions**

Our study of differences in involuntary oscillations arising during the maintenance of isometric force by the human hand of a subject suffering from Parkinson’ disease and a subject having tremor symptoms but not having the disease demonstrates that the multifractal characteristics and number of UPOs can serve useful indicators of a dysfunctional network in the central nervous system.

**References**

The Spectral Chaos in a Spherically Centered Layered Dielectric Cavity Resonator

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Abstract. This paper deals with the study of chaotic spectral wave properties of a cavity sphere layered central-symmetric dielectric resonator. The analytical and numerical research was carried out. It is determined that resonant frequencies of a given layered resonator accurately coincide with the resonant frequencies of inhomogeneous resonator with specified oscillation indices if the radius of inner sphere is much less than the outer resonator radius. Increasing the radius of inner sphere these resonant frequencies shift to smaller values and new additional resonances appear, which cannot be identified by the same oscillation indices and it can be considered as possible chaotic presentation. The probability of inter-frequency interval distribution has signs of spectral chaos in studied structure.

Keywords: Sphere dielectric central-symmetric resonator, spectral wave properties, resonant frequencies, oscillation indices, signs of spectral chaos, probability of inter-frequency interval distribution.

1 Introduction

Our aim is to study the chaotic properties of a layered spherical dielectric cavity resonator with a inner centered spherical dielectric sphere. Dielectric resonators are known to be widely used in optics, laser technology, solid-state electronics (see, for example, Refs. [1,2]). The change of the oscillation spectrum of such resonators strongly depends on both inhomogeneities in the dielectric filling and the resonator shape. For practical applications it is extremely important to know the degree of regularity or randomness of the frequency spectrum. The detailed analysis of the spectrum chaotic properties for different resonant systems can be found, for instance, in [3].

The resonators with electromagnetic wave oscillations are often similar to classical dynamic billiards. Spectral properties of classical dynamical billiards have been thoroughly studied to date (see, e.g., the book [4]). The spectral properties of wave billiard systems are the subject of study by the relatively young field of physics, called “quantum (or wave) chaos” [5,6]. Using the terminology given in paper [7], such systems can be called composite billiards.
It is necessary to underline that the presence of additional spatial scale in wave billiards — the wavelength $\lambda$ — results in serious limitations when trying to describe the chaotic properties of their spectra using the ray approach. In particular, there exist the ray splitting on the interface of different edges in the composite billiards [8,9], which cannot be captured by the classical dynamics. Thus, the ray approach is not well-suited to wave billiard-type systems, so their chaotic properties have to be studied, in general, applying of wave equations.

Statistic analysis of the wave system spectrum is mainly based on the methods used in the classical chaos dynamics, for instance, on the study of inter-frequency interval distribution, spectral rigidity and so on [5,6,10]. The goal of the present work is to investigate spectral properties of layered cavity resonators starting from electromagnetic wave approach. To reach this objective we apply the calculation technique consisting of rigorous splitting of oscillation modes by means of the operational method. This technique was used previously for inhomogeneous waveguides and resonators with bulk and surface inhomogeneities [11–14]. The result of the mode splitting in such complicated and conventionally non-integrable systems is the appearance of specific potentials of operator nature in the wave equation. The structure of these potentials gives rise to the possibility of studying the oscillation spectrum both numerically and analytically.

The spectrum of spherical resonator with homogeneous dielectric inside is strongly degenerate due to the central symmetry. The degeneracy leads to the clustering of the probability distribution maximum for inter-frequency intervals near zero value. It is quite natural to expect that when the spherical resonator becomes layered due to the spherical inner dielectric the spectrum degeneracy is removed. This is strongly expected to be so at least in the case of the symmetry violation.

In the present work we attempt to answer the following questions. What is the type of the probability distribution for inter-frequency intervals in the case of composite (layered) spherical resonator with and without the spatial symmetry? What is the qualitative nature of deformation of the probability distribution when spatial symmetry is violated? What are the signatures of classical chaos in this distribution?

2 Problem statement and basic relationships

We are interested in eigen-oscillations of an electromagnetic resonator taken in the form of ideal conducting sphere of radius $R_{out}$ filled with homogeneous dielectric of permittivity $\varepsilon_{out}$, in which a centered inner dielectric sphere of smaller radius $R_{in}$ is placed, whose permittivity is $\varepsilon_{in}$ (see Fig. 1).

The electromagnetic field inside the resonator can be expressed through electrical and magnetic Hertz functions, $U(\mathbf{r})$ and $V(\mathbf{r})$ [15]. Using these functions, we can go over to Debye potentials $\Psi_{U,V}(\mathbf{r})$ $\Psi_{U}(\mathbf{r}) = r^{-1}U(\mathbf{r})$ and
\( \Psi_V(r) = r^{-1} V(r) \) [15,16] both obeying the same Helmholtz equation,

\[
[\Delta + k^2 \varepsilon(r)] \Psi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \varphi^2} + k^2 \varepsilon(r) \Psi = 0
\] (1)

(\( \vartheta \) and \( \varphi \) are polar and azimuthal angle variables), but different independent boundary conditions,

\[
\left. \frac{\partial}{\partial r} (r \Psi_V) \right|_{r=R_{\text{out}}} = 0 , \quad \Psi_V \left. \right|_{r=R_{\text{out}}} = 0 .
\] (2a) (2b)

The first condition belongs to the class of so-called Robin’s boundary conditions (see, e.g., Ref. [17]), the second one is the well-known Dirichlet condition. The conditions (2) for the electrical and magnetic Debye potentials allows to find these potentials independently from each other, which may be interpreted as the possibility to separate electrical and magnetic-type oscillation in the inhomogeneous spherical resonator.

We will consider the resonator inhomogeneity according to quantum-mechanical perturbation approach. If we take the inhomogeneity as a potential in Schrodinger equation we can write the permittivity in the equation (1) as a “weighted” sum of permittivities of inner and outer dielectric spheres,

\[
\varepsilon(r) = \varepsilon_{\text{in}} \Theta(r \in \Omega_{\text{in}}) + \varepsilon_{\text{out}} \Theta(r \in \Omega_{\text{out}} \setminus \Omega_{\text{in}}) .
\] (3)
Here $\Theta(A)$ stands for the logical theta-function determined as

$$
\Theta(A) = \begin{cases} 
1, & \text{if } A = \text{true} \\
0, & \text{if } A = \text{false} 
\end{cases},
$$

$\Omega_{\text{in}}$ and $\Omega_{\text{out}}$ are the portions of spatial points belonging to inner and outer spheres, respectively. It is convenient to present function (3) as a sum of its spatially averaged part

$$
\varepsilon = \frac{\varepsilon_{\text{in}} V_{\text{in}} + \varepsilon_{\text{out}} (V_{\text{out}} - V_{\text{in}})}{V_{\text{out}}},
$$

with $V_{\text{in/out}} = (4\pi/3)R_{\text{in/out}}^3$ being the volumes of inner and outer spheres, and the summand $\Delta \varepsilon(r)$, the integral of which over the whole resonator volume is equal to zero. The solution to Eq. (1) with exact permittivity value instead of its average one given by (5) will be the starting point to build the constructive perturbation theory.

Equation (1) with coordinate-independent permittivity can be solved by the method described in a number of textbooks (see, e.g., Ref. [18]). The general solution can be presented as an expansion in complete orthogonal eigenfunctions of the Laplace operator, which in spherical coordinates have the form [19,20]

$$
|r; \mu\rangle = \frac{D_{l_n}^{(l)}}{R} \sqrt{\frac{2}{r}} J_{l+\frac{1}{2}} \left( \frac{\lambda_{l_n}^{(l)} r}{R} \right) Y_{l}^{m} (\vartheta, \varphi)
$$

(n = 1, 2, ..., $\infty$; $l = 0, 1, 2, ..., \infty$; $m = -l, -l+1, ..., l-1, l$).

Here, to simplify the equations we introduce the vector mode index $\mu = \{n, l, m\}$, $J_p(u)$ is the Bessel function of the first kind, $Y_l^m(\vartheta, \varphi)$ is the spherical function,

$$
Y_{l}^{m} (\vartheta, \varphi) = (-1)^m \left[ \frac{(2l + 1)}{2} \cdot \frac{(l - m)!}{(l + m)!} \right]^{1/2} P_{l}^{m} (\cos \vartheta) \cdot \frac{e^{im\varphi}}{\sqrt{2\pi}},
$$

$P_{l}^{m}(t)$ is the Legendre function. The coefficients $\lambda_{l_n}^{(l)}$ in the equation (6) are the positive zeros of either the sum $u J_{l+\frac{1}{2}}'(u) + (1/2) J_{l+\frac{1}{2}}(u)$ (if boundary conditions (BC) (2a) is applied), or the function $J_{l+\frac{1}{2}}(u)$ (in the case of BC (2b)), which are numbered by natural index $n$ in ascending order. Normalization coefficient $D_{l_n}^{(l)}$ in relation (6) depends on the particular boundary condition,

$$
D_{l_n}^{(l)} = \begin{cases} 
J_{l+\frac{1}{2}}^{2} (\lambda_{l_n}^{(l)}) + \left[ 1 - \left( \frac{l + 1/2}{\lambda_{l_n}^{(l)}} \right)^2 \right] J_{l+\frac{1}{2}}^{2} (\lambda_{l_n}^{(l)}) \right]^{-\frac{1}{2}} & \text{for BC (2a)}, \\
J_{l+\frac{1}{2}}^{-1} (\lambda_{l_n}^{(l)}) & \text{for BC (2b)}. 
\end{cases}
$$
The eigenvalue of the Laplace operator, that corresponds to eigenfunction (6), is degenerated over azimuthal index \( m \),

\[
E_\mu = -k_\mu^2 = -\left( \frac{\lambda_n^{(l)}}{R} \right)^2 .
\]

with the degeneracy equal to \( 2l + 1 \).

The spectrum of the resonator with nonuniform permittivity (3) can be found through the calculation of density of states \( \nu(k) \) (see, e.g., Ref. [21]). Function \( \nu(k) \) can be expressed through the Green function of wave equation (1) with complex-valued frequency account for dissipation in the resonator,

\[
\nu(k) = \pi^{-1} \text{Im}\{\text{Tr} \hat{G}^{(-)}\} .
\]

Here \( \hat{G}^{(-)} \) is the advanced Green operator corresponding the equation (1) with negative imaginary part in the complex frequency plane. The Green function (considered as the coordinate matrix element of operator \( \hat{G}^{(-)} \)) obeys the equation

\[
[\Delta + \pi k^2 - i/\tau_d - V(\mathbf{r})] G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') ,
\]

where the term \( V(\mathbf{r}) = -k^2 \Delta \varepsilon(\mathbf{r}) \) will be interpreted as the effective potential (in the quantum-mechanical terminology). In comparison with Eq. (1), equation (11) is supplied with imaginary term \( i/\tau_d \) which takes phenomenologically into account the dissipation processes in the bulk and on the surface of the resonator. Strictly speaking, the dielectric loss in the resonator depend on the frequency in the general case. Yet now we will neglect this dependence to simplify further investigations.

For the numerical calculation purposes it is suitable to go over from the coordinate representation of Eq. (11) to the momentum representation. Equation (11) then takes the form of an infinite set of coupled algebraic equations,

\[
(\pi k^2 - k_\mu^2 - i/\tau_d - V_\mu) G_{\mu\mu'} - \sum_{\nu \neq \mu} \mathcal{U}_{\mu\nu} G_{\nu\mu'} = \delta_{\mu\mu'} .
\]

Here the quantities \( V_\mu \) and \( \mathcal{U}_{\mu\nu} \), which we will term the intramode and the intermode potentials, are the matrix elements of potential \( V(\mathbf{r}) \) taken in the basis of functions (6),

\[
\mathcal{U}_{\mu\nu} = \int_{\Omega} d\mathbf{r} \langle \mathbf{r}; \mu | V(\mathbf{r}) | \mathbf{r}; \nu \rangle = -k^2 (\varepsilon_{\text{in}} - \varepsilon_{\text{out}}) I_{\mu\nu} ,
\]

\[
V_\mu = \mathcal{U}_{\mu\mu} = -k^2 (\varepsilon_{\text{in}} - \varepsilon_{\text{out}}) [I_{\mu\mu} - V_{\text{in}}/V_{\text{out}}] ,
\]

\[
I_{\mu\nu} = \int_{\Omega_{\text{in}}} d\mathbf{r} \langle \mathbf{r}; \mu | \mathbf{r}; \nu \rangle .
\]
In the case of strictly centered outer and inner dielectric spheres the integrals in the relationships (13) are calculated rigorously, and the result is as follows,

\[ I_{\mu\nu}(\Omega_{in}) = 2Q \delta_{\mu\nu} \delta_{m_{\mu}m_{\nu}} \frac{D_{n_{\mu}}^l D_{n_{\nu}}^l}{\lambda_{n_{\mu}}^l \lambda_{n_{\nu}}^l} \left[ \lambda_{n_{\mu}}^l J_{\mu} + \frac{3}{2} \left( \lambda_{n_{\mu}}^l Q \right) J_{\mu + \frac{1}{2}} \left( \lambda_{n_{\nu}}^l Q \right) \right] - \lambda_{n_{\mu}}^l J_{\mu + \frac{1}{2}} \left( \lambda_{n_{\mu}}^l Q \right) J_{\mu + \frac{3}{2}} \left( \lambda_{n_{\mu}}^l Q \right) \] (14a)

\[ I_{\mu\mu}(\Omega_{in}) = Q^2 \left[ D_{n_{\mu}}^l \right]^2 \left[ J_{\mu}^2 + \frac{1}{2} \left( \lambda_{n_{\mu}}^l Q \right) J_{\mu + \frac{1}{2}} \left( \lambda_{n_{\mu}}^l Q \right) - J_{\mu - \frac{1}{2}} \left( \lambda_{n_{\mu}}^l Q \right) J_{\mu + \frac{3}{2}} \left( \lambda_{n_{\mu}}^l Q \right) \right]. \] (14b)

Here we have introduced the scale parameter \( Q = R_{in}/R_{out} \leq 1 \) that describes the degree of the resonator geometric inhomogeneity.

### 3 Numerical results and discussion

The set of basic equations (12) can, in principle, be solved analytically using the operator technique of mode separation [14]. Yet, in view of the tediousness of that technique, in this study we examine equations (12) numerically. To obtain the solution we have elaborated programming software that calculate the resonator Green function, determine its maxima locations, and also build the inter-frequency distribution function. It is necessary to accentuate that such a calculation task is quite resource-intensive, and it leads to rigid constraint for the number \( N \) of oscillation modes taken into account. The computational complexity grows much faster than \( N^3 \). Such a dependence on the number of analyzed oscillations can be explained by the complexity of numerical integration of oscillating functions (Bessel functions, spherical Legendre functions) with the growing number of their zeros on the interval of integration. The main numerical calculations were carried out on the computing cluster at the Institute for Radiophysics and Electronics of National Academy of Sciences of Ukraine, which is a part of the infrastructure of the Ukrainian National Grid (UNG). Based on the available computation resources (CPU clock speed 2.5 GHz, RAM 1.5 Gb/core), we were compelled to limit the number of harmonics by 10,000 and no more than 2000 harmonics for an arbitrary value of heterogeneity. The calculation of each harmonics takes from a few seconds for the long-wavelength modes to tens of minutes for short ones. To speed up the calculations and the possibility to operate with a greater number of harmonics, the parallelization of computational algorithm with the use of MPI technology was implemented. Note that the task under consideration is highly scalable. Thus, the parallel computation provides a performance increase. It is almost proportional to the number of computing nodes involved. All calculations were performed in the standard representation for double-precision real numbers. Relative error of calculation does not exceed \( 10^6 \), and the main source of error was the accuracy of numerical integration and calculation of special functions.

From Eqs. (12) we have calculated all diagonal elements of the Green function matrix \( G_{\mu\mu} \). In Fig. 2, the density of states (DoS) of the resonator is presented, which is calculated using the definition (10). It can be seen that the
Fig. 2. The whole frequency spectrum as the frequency dependence of the imaginary part of the sum of diagonal Green functions for the composite cavity resonator with centered dielectric spheres: A — $Q=0$; B — $Q=0.583$; C — $Q=0.897$; D — $Q=0.998$. The permittivities of the inner and outer spheres are $\varepsilon_{in}=2.08$, $\varepsilon_{out}=1.0$. The dissipation value corresponds to $\tau_d=1000$.

DoS graph becomes thicker with growing the radius of inner dielectric sphere. When the inner radius value goes to the outer one, the DoS is getting thinner. In this case the resonator filling tends to become homogenous with the effective permittivity $\varepsilon_{out}$. Thus, the average DoS maximal value is observed at $Q \rightarrow 1$.

To analyze the oscillation spectrum we examine the probability of the inter-frequency intervals (nearest-neighbor spacings, NNS) between adjacent resonances, $P(S)$. Conventionally, the spectrum unfolding is used for this purpose, implying the normalized mean inter-frequency distance to be equal to unity. Fig. 3 demonstrates distribution $P(S)$ for different inner radii and dissipation values. For $\tau_d=100000$ (the loss is practically neglected) and $Q=0$ we have convention with Poisson distribution, $P_p(S) = \exp(-S)$. This suggests the resonance frequencies to be completely uncorrelated. With the increase in the dissipation value (for example, $\tau_d=100$) we obtain the distribution function that tends to Wigner form, $P_w(S) = 0.5\pi S \exp(-\frac{2S^2}{3})$. Thus, we are led to conclude that the presence of dissipation in the resonator results in the chaotic behavior of oscillation modes.

The essential difference between NNS distribution of the chaotic spectrum and the regular one is the presence of mode “repulsion” (the downfall of $P(S)$ at low values of $S$). The repulsion of modes with close frequencies in the chaotic spectrum can be explained as follows. When the resonator infill is homogeneous, different oscillation modes are independent of each other and do not interact with each other even if their own frequencies coincide, i.e. if they are in a degenerate state. Any heterogeneity lifts the degeneracy, and the
natural frequencies of different modes change in different ways, depending on the degree of heterogeneity influence. That is, there is a kind of “repulsion” of oscillations modes. The larger the impact of heterogeneity be, the greater is the repulsion effect.

In Fig. 4, the intensity of a partial Green function $G_{\mu\mu}$ from Eq. (12) on wave number is shown for the particular polar and radial indices and different inner sphere radii $R_{in}$. At $R_{in}=0$ we observe one oscillation mode only. We will call it the main resonance for the selected Green function. With the increase in the inner radius $R_{in}$, additional resonances appear at the frequencies that coincide with main resonances of the rest of radial modes with the definite polar index.

In Fig. 5, the frequency dependence of the imaginary part of the sum of diagonal Green functions for the oscillations with two different polar indices. As the radius $R_{in}$ increases, we observe that the resonances 1 and 2 interchange their relative position. Thus, we see the occasional and unpredictable oscillations moving. We explain this behavior of resonances as a signature of wave chaos arisen due to inhomogeneity of the resonator.

Thus, we have developed the statistical spectral theory of the centrally symmetric layered cavity resonator with homogeneous and inhomogeneous infill. Numerical investigation of the resonator frequency spectrum was also carried out. The signature of chaotic behavior of the resonator spectrum is demonstrated. We have found out that the homogeneous resonator has inter-frequency interval distribution similar to the Poisson distribution typical for the spectrum with uncorrelated inter-frequency intervals. In the presence of dissipation in

\[
\text{Fig. 3. The probability of inter-frequency interval distribution at different dissipation values and inner radii: A — } \tau_d = 100000, \ Q=0; \ B — \tau_d = 100, \ Q=0.67.
\]
Fig. 4. Frequency dependence of the logarithm of partial normalized Green function for different inner radii: 1 — $Q=0$, 2 — $Q=0.448$, 3 — $Q=0.672$, 4 — $Q=0.8968$, 5 — $Q=0.9977$, 6 — $Q=0.9997$. Polar index is 3, radial index is 1. The permittivities of the inner and outer spheres are $\varepsilon_{\text{in}}=2.08$, $\varepsilon_{\text{out}}=1.0$. The dissipation value corresponds to $\tau_d=100000$.

The resonator, the NNS distribution tents to the distribution of Wigner form, which clearly demonstrates the effect of “mode repulsion”.

References

Fig. 5. Frequency dependence of the imaginary part of the sum of diagonal Green functions with fixed polar indices (1 — polar index $l = 1$, 2 — $l = 3$) with different values of radius $R_{in}$: A — $Q=0.19$, B — $Q=0.21$, C — $Q=0.23$. The permittivities of the inner and outer spheres are $\varepsilon_{in}=2.08$, $\varepsilon_{out}=1.0$. The dissipation value corresponds to $\tau_d = 100000$.

Reachable Sets for a Class of Nonlinear Control Systems with Uncertain Initial States

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Abstract. We consider the problem of estimating reachable sets of nonlinear dynamical control systems with uncertainty in initial states when we assume that we know only the bounding set for initial system positions and any additional statistical information is not available. We study the case when the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The problem may be reformulated as the problem of describing the motion of set-valued states in the state space under nonlinear dynamics with state velocities having bilinear-quadratic kind. Using results of the theory of trajectory tubes of control systems and techniques of differential inclusions theory we find set-valued estimates of related reachable sets of such nonlinear uncertain control system. The algorithms of constructing the ellipsoidal estimates for studied nonlinear systems are given.

Keywords: Nonlinear control systems, Bilinear nonlinearity, Quadratic nonlinearity, Set-membership uncertainty, Ellipsoidal calculus, Funnel equations, Trajectory tubes.

1 Introduction

The problem of parameter estimation for control problems and of the evaluation of related estimating sets describing uncertainty is considered in the paper in the case when a probabilistic description of noise and errors is not available, but only a bound on them is known (Bertsekas and Rhodes[1], Kurzhanski and Valyi[14], Milanese et al.[18], Scheppe[20], Walter and Pronzato[22]). Such models may be found in many applied areas ranged from engineering problems in physics to economics as well as to biological and ecological modeling when it occurs that a stochastic nature of the errors is questionable because of limited
data or because of nonlinearity of the model. Unlike the classical estimation approach, set-membership estimation is not concerned with minimizing any objective function and instead of finding a single optimal parameter vector, a set of feasible parameter vectors, consistent with the model structure, measurements and bounded uncertainty characterization, should usually be found.

The solution of many control and estimation problems under uncertainty involves constructing reachable sets and their analogs. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets. We note here two of the most developed approaches to research in this area. The first one is based on ellipsoidal calculus (Chernousko[3], Kurzhanski and Valyi[14]) and the second one uses the interval analysis (Walter and Pronzato[22]).

Many applied problems are mostly nonlinear in their parameters and the set of feasible system states is usually non-convex or even non-connected. The key issue in nonlinear set-membership estimation is to find suitable techniques, which produce related bounds for the set of unknown system states without being too computationally demanding. Some approaches to the nonlinear set-membership estimation problems and discrete approximation techniques for differential inclusions through a set-valued analogy of well-known Euler’s method were developed in Kurzhanski and Varaiya[15], Kurzhanski and Filippova[13], Dontchev and Lempio[6], Veliov[21].

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus (Kurzhanski and Valyi[14], Chernousko[3]) are presented. We study here more complicated case than in Filippova and Matyviychuk[12] and we assume now that the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The problem may be reformulated as the problem of describing the motion of set-valued states in the state space under nonlinear dynamics with state velocities having bilinear-quadratic kind. Using results of the theory of trajectory tubes of control systems and techniques of differential inclusions theory we find set-valued estimates of related reachable sets of such nonlinear uncertain control system. The algorithms of constructing the ellipsoidal estimates for studied nonlinear systems are given. Numerical simulation results related to the proposed techniques and to the presented algorithms are also included.

2 Problem formulation

Let us introduce the following basic notations. Let $R^n$ be the $n$–dimensional Euclidean space, $\text{comp}R^n$ is the set of all compact subsets of $R^n$, $R^{n \times n}$ stands for the set of all $n \times n$–matrices and $x'y = (x, y) = \sum_{i=1}^{n} x_i y_i$ be the usual inner product of $x, y \in R^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$. We denote as $B(a, r)$ the ball in $R^n$, $B(a, r) = \{x \in R^n : \|x - a\| \leq r\}$, $I$ is the identity $n \times n$-matrix. Denote by $E(a, Q) = \{x \in R^n : (Q^{-1}(x - a), (x - a)) \leq 1\}$ the ellipsoid in $R^n$ with a center $a \in R^n$ and a symmetric positive definite
\( n \times n \)-matrix \( Q \), \( Tr(Y) \) denotes the trace of \( n \times n \)-matrix \( Y \) (the sum of its diagonal elements). Consider the following system

\[
\dot{x} = A(t)x + f(x)d + u(t), \quad x_0 \in X_0, \quad t_0 \leq t \leq T,
\]

where \( x, d \in \mathbb{R}^n \), \( \|x\| \leq K \) \((K > 0)\), \( f(x) \) is the nonlinear function, which is quadratic in \( x \),

\[ f(x) = x'Bx, \]

with a given symmetric and positive definite \( n \times n \)-matrix \( B \). Control functions \( u(t) \) in (1) are assumed Lebesgue measurable on \([t_0, T]\) and satisfying the constraint

\[ u(t) \in U, \quad \text{for a.e.} \ t \in [t_0, T], \]

(here \( U \) is a given set, \( U \in \text{comp}\mathbb{R}^m \)). The \( n \times n \)-matrix function \( A(t) \) in (1) has the form

\[ A(t) = A^0 + A^1(t), \]

where the \( n \times n \)-matrix \( A^0 \) is given and the measurable \( n \times n \)-matrix \( A^1(t) \) with elements \( \{a^{(1)}_{ij}(t)\} \) \((i, j = 1, \ldots, n)\) is unknown but bounded

\[ A^1(t) \in \mathcal{A} = \{A = \{a_{ij}\} \in \mathbb{R}^{n \times n} : |a_{ij}| \leq c_{ij}, \ i, j = 1, \ldots, n\}, \ t \in [t_0, T], \]

where \( c_{ij} \geq 0 \) \((i, j = 1, \ldots, n)\) are given.

We will assume that \( X_0 \) in (1) is an ellipsoid, \( X_0 = E(a_0, Q_0) \), with a symmetric and positive definite matrix \( Q_0 \) and with a center \( a_0 \).

Let the absolutely continuous function \( x(t) = x(t; u(\cdot), A^1(\cdot), x_0) \) be a solution to dynamical system (1)–(3) with initial state \( x_0 \in X_0 \), with admissible control \( u(\cdot) \) and with a matrix \( A^1(\cdot) \) satisfying (2)–(3). The reachable set \( X(t) \) at time \( t \) \((t_0 < t \leq T)\) of system (1)–(3) is defined as the following set

\[
X(t) = \{ x \in \mathbb{R}^n : \exists x_0 \in X_0, \ \exists u(\cdot) \in U, \ \exists A^1(\cdot) \in \mathcal{A} \text{ such that } x = x(t) = x(t; u(\cdot), A^1(\cdot), x_0) \}, \ t_0 < t \leq T.
\]

The main problem of the paper is to find the external ellipsoidal estimate \( E(a^+(t), Q^+(t)) \) (with respect to the inclusion of sets) of the reachable set \( X(t) \) \((t_0 < t \leq T)\) by using the analysis of a special type of nonlinear control systems with uncertain initial data.

## 3 Preliminaries

In this section we present some auxiliary results on the properties of reachable sets for different types of dynamical systems which we will need in the sequel.
3.1 Bilinear system

Bilinear dynamic systems are a special kind of nonlinear systems representing a variety of important physical processes. A great number of results related to control problems for such systems has been developed over past decades, among them we mention here Brockett[2], Chernousko[4,5], Polyak et al.[19], Kurzhanski and Varaiya[15], Kurzhanski and Filippova[13], Mazurenko[17], Filippova[7,11]. Reachable sets of bilinear systems in general are not convex, but have special properties (for example, are star-shaped). We, however, consider here the guaranteed state estimation problem and use ellipsoidal calculus for the construction of external estimates of reachable sets of such systems.

Consider the bilinear system

\[
\dot{x} = A(t)x, \quad t_0 \leq t \leq T, \tag{4}
\]

\[
x_0 \in X_0 = E(a_0, Q_0), \tag{5}
\]

where \(x, a_0 \in \mathbb{R}^n\), \(Q_0\) is symmetric and positive definite. The unknown matrix function \(A(t) \in \mathbb{R}^{n \times n}\) is assumed to be of the form (2) with the assumption (3).

The external ellipsoidal estimate of reachable set \(X(T)\) of the system (4)-(5) can be found by applying the following theorem.

**Theorem 1 (Chernousko[4]).** Let \(a^+(t)\) and \(Q^+(t)\) be the solutions of the following system of nonlinear differential equations

\[
\dot{a}^+ = A_0 a^+, \quad a^+(t_0) = a_0, \quad t_0 \leq t \leq T, \tag{6}
\]

\[
\dot{Q}^+ = A_0^T Q^+ + Q^+ A_0^T + q Q^+ + q^{-1} G, \quad Q^+(t_0) = Q_0, \quad t_0 \leq t \leq T, \tag{7}
\]

where

\[
q = (n^{-1} \text{Tr}((Q^+)^{-1} G))^{1/2}, \tag{8}
\]

\[
G = \text{diag}\left\{ (n-v) \left[ \sum_{i=1}^{n} c_{ij} |a_i^+|^2 \right] + \left( \max_{\sigma = \{\pm\}} \sum_{p,q=1}^{n} Q_{pq}^+ c_{jp} c_{jq} |\sigma_{jp} \sigma_{jq}|^{1/2} \right)^2 \right\}, \tag{9}
\]

the maximum in (9) is taken over all \(\sigma_{ij} = \pm 1\), \(i, j = 1, \ldots, n\), such that \(c_{ij} \neq 0\) and \(v\) is a number of such indices \(i\) for which we have: \(c_{ij} = 0\) for all \(j = 1, \ldots, n\). Then the following external estimate for the reachable set \(X(t)\) of the system (4)-(5) is true

\[
X(t) \subseteq E(a^+(t), Q^+(t)), \quad t_0 \leq t \leq T. \tag{10}
\]

**Corollary 1.** Under conditions of the Theorem 1 the following inclusion holds

\[
X(t_0 + \sigma) \subseteq (I + \sigma A) X_0 + o_1(\sigma) B(0, 1) \subseteq \tag{11}
\]

\[
E(a^+(t_0 + \sigma), Q^+(t_0 + \sigma)) + o_2(\sigma) B(0, 1),
\]

where \(\sigma^{-1} o_i(\sigma) \to 0\) for \(\sigma \to +0\) \((i = 1, 2)\) and

\[
(I + \sigma A) X_0 = \bigcup_{x \in X_0} \bigcup_{A \in A} \{x + \sigma Ax\}.
\]
Proof. The inclusion (11) follows directly from (10) and presents a special case of the inclusion related to the discrete version of the integral funnel equation for the system (4)-(5) (Kurzhanski and Varaiya\cite{15}, Kurzhanski and Filippova\cite{13}).

The following example illustrates the result of Theorem 1.

Example 1. Consider the following system

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= (c(t) - 1) x_1, \\
0 &\leq t \leq 1, \quad x_0 \in X_0 = B(0, 1)
\end{align*}
\] (12)

where \(c(t)\) is an unknown but bounded measurable function with \(|c(t)| \leq 0.8\) \((0 \leq t \leq 1)\). The trajectory tube \(X(t)\) and its external ellipsoidal estimate \(E(a^+(t), Q^+(t))\) found by Theorem 1 are shown in Figure 1.

![Fig. 1. Trajectory tube \(X(t)\) and its ellipsoidal estimating tube \(E(a^+(t), Q^+(t))\) for the bilinear control system with uncertain initial states.](image)

We see here that the trajectory tube \(X(t)\) of bilinear system (12), issued from the convex set \(X_0 = B(0, 1)\), loses the convexity over time. External ellipsoidal tube \(E(a^+(t), Q^+(t))\) contains the reachable set \(X(t)\) and in some points is enough accurate (it touches the boundary of \(X(t)\)).

3.2 Systems with quadratic nonlinearity

Consider the control system of type (1) but with a known matrix \(A = A^0\)

\[
\dot{x} = A^0 x + f(x) + u(t), \quad x_0 \in X_0 = E(a_0, Q_0), \quad t_0 \leq t \leq T.
\] (13)
We assume here that \( u(t) \in U = E(\hat{a}, \hat{Q}) \), vectors \( d, a_0, \hat{a} \) are given, a scalar function \( f(x) \) has a form \( f(x) = x'Bx \), matrices \( B, Q_0, \hat{Q} \) are symmetric and positive definite.

Denote the maximal eigenvalue of the matrix \( B^{1/2}Q_0B^{1/2} \) by \( k^2 \), it is easy to see this \( k^2 \) is the smallest number for which the inclusion \( X_0 \subseteq E(a_0, k^2B^{-1}) \) is true. The following result describes the external ellipsoidal estimate of the reachable set \( X(t) = X(t; t_0, X_0) \) \((t_0 \leq t \leq T)\).

**Theorem 2 (Filippova[10]).** The inclusion is true for any \( t \in [t_0, T] \)

\[
X(t; t_0, X_0) \subseteq E(a^+(t), r^+(t)B^{-1}),
\]

where functions \( a^+(t), r^+(t) \) are the solutions of the following system of ordinary differential equations

\[
\dot{a}^+(t) = A^0a^+(t) + ((a^+(t))'Ba^+(t) + r^+(t))d + \dot{a}, \quad t_0 \leq t \leq T,
\]

\[
\dot{r}^+(t) = \max_{\|d\|=1} \left\{ l'(2r^+(t)B^{1/2}(A^0 + 2d \cdot (a^+(t))'B)B^{-1/2} + q^{-1}(r^+(t))B^{1/2}Q^1/2)l \right\} + q(r^+(t))r^+(t),
\]

with initial state

\[
a^+(t_0) = a_0, \quad r^+(t_0) = k^2.
\]

**Corollary 2 (Filippova[8]).** The following upper estimate for \( X(t_0 + \sigma) = X(t_0 + \sigma; t_0, X_0) \) \((\sigma > 0)\) holds

\[
X(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0, 1),
\]

where \( \sigma^{-1}o(\sigma) \to 0 \) when \( \sigma \to +0 \) and

\[
a^+(\sigma) = a(\sigma) + \sigma \hat{a}, \quad a(\sigma) = a_0 + \sigma(A^0a_0 + a_0'Ba_0d + k^2d),
\]

\[
Q^+(\sigma) = (p^{-1} + 1)Q(\sigma) + (p + 1)\sigma^2\hat{Q},
\]

\[
Q(\sigma) = k^2(I + \sigma R)B^{-1}(I + \sigma R)', \quad R = A^0 + 2d \cdot a_0'B
\]

and \( p \) is the unique positive root of the equation

\[
\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p + 1)}
\]

with \( \alpha_i \geq 0 \) \((i = 1, \ldots, n)\) being the roots of the following equation \(|Q(\sigma) - \alpha\sigma^2\hat{Q}| = 0\).

Numerical algorithms basing on Theorem 2 and producing the discrete-time external ellipsoidal tube estimating the reachable set of the system (13) \(\text{together with related examples}\) are given in Filippova[10], Filippova and Matviychuk[12].

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4 Main results

Consider the general case
\[ \dot{x} = A(t)x + x'Bx \cdot d + u(t), \quad t_0 \leq t \leq T, \]  
(20)
with initial state
\[ x_0 \in X_0 = E(a_0, Q_0) \]  
(21)
and control constraints
\[ u(t) \in U = E(\hat{a}, \hat{Q}), \]  
(22)
and with the uncertain matrix
\[ A(t) = A^0 + A^1(t), \quad A^1(t) \in \mathcal{A}, \]  
(23)
where the set \( \mathcal{A} \) is defined in (3). As before we assume that matrices \( B, \hat{Q} \) and \( Q_0 \) are symmetric and positive definite.

The next theorem describes discrete external ellipsoidal estimates of reachable sets \( X(t) \) of the uncertain control system (20)–(23), containing both bilinear and quadratic nonlinearities.

**Theorem 3.** The following external ellipsoidal estimate holds
\[ X(t_0 + \sigma) \subseteq E(a^*(t_0 + \sigma), Q^*(t_0 + \sigma)) + o(\sigma)B(0, 1) \]  
(24)
where \( \sigma^{-1}o(\sigma) \to 0 \) for \( \sigma \to +0 \) and where
\[ a^*(t_0 + \sigma) = \hat{a}(t_0 + \sigma) + \sigma(\hat{a} + a'_0Ba_0 \cdot d + k^2d), \]  
(25)
\[ Q^*(t_0 + \sigma) = (p^{-1} + 1)\hat{Q}(t_0 + \sigma) + (p + 1)\sigma^2\hat{Q}, \]  
(26)
with functions \( \hat{a}(t), \hat{Q}(t) \) calculated as \( a^+(t), Q^+(t) \) in Theorem 1 but when we replace matrices \( Q_0 \) and \( A^0 \) in (6)–(9) by
\[ \tilde{Q}_0 = k^2B^{-1}, \quad \tilde{A}^0 = A^0 + 2d \cdot a'_0B \]  
(27)
respectively, and \( p \) is the unique positive root of the equation
\[ \sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p + 1)} \]  
(28)
with \( \alpha_i \geq 0 \) \( (i = 1, ..., n) \) being the roots of the following equation \[ |Q(t_0 + \sigma) - \alpha_0B| = 0. \]

**Proof.** Analyzing both results of Theorem 1 and Theorem 2 and of their corollaries and using the general scheme of the proof of Theorem 2 in Filippova[8] (see also techniques in Filippova[9]) we obtain the formulas (24)-(28) of the Theorem.
The following iterative algorithm basing on Theorem 3 may be used to produce the external ellipsoidal tube estimating the reachable set \( X(t) \) on the whole time interval \( t \in [t_0, T] \).

**Algorithm.** Subdivide the time segment \([t_0, T]\) into subsegments \([t_i, t_{i+1}]\) where \( t_i = t_0 + ih \) (\( i = 1, \ldots, m \)), \( h = (T - t_0)/m \), \( t_m = T \).

- Given \( X_0 = E(a_0, Q_0) \), find the smallest \( k = k_0 > 0 \) such that
  \[
  E(a_0, Q_0) \subseteq E(a_0, k^2 B^{-1})
  \]
  \((k^2)\) is the maximal eigenvalue of the matrix \( B^{1/2} Q_0 B^{1/2} \).

- Take \( \sigma = h \) and define by Theorem 3 the external ellipsoid \( E(a_1, Q_1) \) such that
  \[
  X(t_1) \subseteq E(a_1, Q_1) = E(a^*(t_0 + \sigma), Q^*(t_0 + \sigma)).
  \]

- Consider the system on the next subsegment \([t_1, t_2]\) with \( E(a_1, Q_1) \) as the initial ellipsoid at instant \( t_1 \).

- Next steps continue iterations 1-3. At the end of the process we will get the external estimate \( E(a(t), Q(t)) \) of the tube \( X(t) \) with accuracy tending to zero when \( m \to \infty \).

**Example 2.** Consider the following control system

\[
\begin{align*}
\dot{x}_1 &= x_2 + u_1, \\
\dot{x}_2 &= -x_1 + c(t)x_1 + x_1^2 + x_2^2 + u_2, \quad x_0 \in X_0, \quad t_0 \leq t \leq T.
\end{align*}
\]  

(29)

Here we take \( t_0 = 0, T = 0.35 \), \( X_0 = B(0, 1) \) and \( U = B(0, 0.1) \), the uncertain but bounded measurable function \( c(t) \) satisfies the inequality \(|c(t)| \leq 0.8 \) \((t_0 \leq t \leq T)\). The trajectory tube \( X(t) \) and its external ellipsoidal estimating tube \( E(a^*(t), Q^*(t)) \) calculated by the Algorithm are given in Figure 2.

## 5 Conclusions

The paper deals with the problems of state estimation for uncertain dynamical control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded.

We study here the case when the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. The problem may be reformulated as the problem of describing the motion of set-valued states in the state space under nonlinear dynamics with state velocities having bilinear-quadratic type.

Basing on results of ellipsoidal calculus developed earlier for some classes of uncertain systems we present the modified state estimation approach which uses the special structure of nonlinearity and uncertainty in the control system and allows constructing the external ellipsoidal estimates of reachable sets.
Fig. 2. Trajectory tube $X(t)$ and its ellipsoidal estimating tube $E(a^*(t), Q^*(t))$ for the system with bilinear and quadratic nonlinearities.

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References


Implications of Chaos Theory in Management Science

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Abstract. We live in a dynamic world that is most often described as being “chaotic” and unpredictable. From our human perspective, we do not see the greater framework of the system that we live in, and can only try to approximate its boundaries. However, with technological advances and continued adaptability, this does not limit our progression, because humans are complex creatures that seek to control chaos. It follows that we function in organizations that become complex systems, or systems that provide a balance between rigid order and random chaos. This realization defines a new paradigm for “emergent” leadership and management based on chaos theory, where emergent leaders become “strange attractors”; this means they are leaders that are flexible and have the skill set to accept unpredictability to enable the organization to adapt accordingly.

Keywords: Control, Non-linear systems, Uncertainty, Unpredictability, Attractors, Leadership, Emergent leader, Positive motivators.

1 Introduction

When we think of the word “chaos”, the prominent meanings that come to mind are confusion, disorder, and lack of control. However, these definitions represent the modern English meaning of the word. Chaos was first conceptualized and defined through mythology, which described the origins (or birth) of humankind. “Myth is as logical as philosophy and science, although the logic of myth is that of unconscious thought” (Caldwell[3]). The word itself is rooted in Greek origins, its authentic form being \(Χάος\) (Khaos). In Greek mythology, Chaos is “the embodiment of the primeval Void which existed before Order had been imposed on the universe” (Grimal and Kershaw[4]). In this definition it is evident how humankind had tried to contain a vastness that was (and is) difficult to comprehend in its natural form. Hesiod’s *Theogony* agrees with the undefinable origin concept, as “first of all, the Void came into being, next broad-bosomed Earth, the solid and eternal home of all... Out of Void came Darkness and black Night” (Brown [1]).
The synonymy of “void” and “chaos”, and the birth of darkness from the “embodiment of the primeval void”, implies that chaos is an “impenetrable darkness and unmeasurable totality, of an immense opacity in which order is nonexistent or at least unperceived” (Caldwell[3]); that chaos describes the collection of everything that humankind cannot grasp and cannot control. Now, there is the duality of the controllable and uncontrollable; an unspoken demarcation of what the human mind is capable of elucidating.

However, the characterization of chaos is incomplete without the following line, as “Earth, the solid and eternal home of all” is formed, born as a separate entity and representative as “the primordial maternal symbol” (Caldwell[3]). The key to this is to note the synonymy of the “maternal symbol” and the “solid and eternal home of all”, the implication being that “solidity” and order is established so that the lineage of humankind can be traced back to something tangible. Again, there is a polarity of order and non-order, which can be seen in the chronologic succession of chaos, then the formation of earth. This sequence becomes significant in implying “that Chaos, a state prior to perception, represents the situation of the child in the symbiotic state” and “may be regarded as a representation of the symbiotic phase as un-differentiation and imperception, as a formless totality” (Caldwell[3]). Through this implication, chaos is better defined as being everything before perception, rather than confusion or disorder; chaos is what is unknown or intangible.

2 Control in Chaos

For decades we have repressed this unknown through systems and controls of mathematical equations and patterns. The reduction of chaos began with Sir Isaac Newton, in his attempt to mechanize reality through linearization of (what was later accepted as) a nonlinear, dynamic system. The theory had been based on the idea that with a linear reality, predictions could be made and phenomena could be controlled simply by deconstruction of the universe into its most “basic parts” and “logically” putting them back together (Burns[2]). In truth, this type of linearized reality has helped to advance humankind not only technologically, but also socially. “The social sciences have always attempted to model physical science paradigms” (Burns[2]). This is evident in the early formation of the field of Psychology, where Freud’s developmental stages build upon each other. It was assumed that if one stage is (or becomes) dysfunctional, problems in the human psyche occur. Other human systems, such as the development and function of political parties, economic systems, and the development of children’s concept learning strategies, are also built under the assumption of Newton’s linear reality (Burns[2]). With the help of quantum physicists, and theoretical meteorologist, Edward Lorenz, the way actual reality functions became easier to accept: the reality that the universe is chaotic and
cannot be linearized and deconstructed into simpler mathematics. The realization that social systems could no longer be defined as linear.

Finally, the study of reality no longer models the constrained limitations of a linear way of thinking, and instead begins to model non-linear, dynamic chaos. By extension, because organizations exist in reality, it can be assumed that social systems develop within a chaotic system. Therefore, “organizations are nonlinear, dynamic systems” (Otten and Chen[10]) that make it imperative for leadership, and leadership practices, to be constructed through chaotic-system thinking. “In chaos theory leadership is not reduced to the ‘leadership’ behavior of a key position holder or team of ‘top’ people. Leadership is conducted throughout the organization, through all agents… Leadership is broadly conducted precisely because in chaotic systems, all agents have potential access to vital information from the environment” (Burns[2]).

The very definition of an organization is a body of people who share a purpose, vision, or mission. The primary functions of leadership within the organization are to: a) ensure that the agents of the organization keep the purpose and core values in mind, and b) ensure that the primary mission and values adapt (continuously) with environmental demands. By empowering all levels of the organization, the environment is monitored constantly and the overall mission is clarified because it is continuously evaluated and defined from different perspectives (Burns[2]). The acceptance of chaos in social systems is the basis that leaders must begin with. The assumption that outcomes are predictable is parallel to the assumption that chaos can be predicted. However, if chaos is defined as the unknown, the assumption that chaos is predictable is illogical. Therefore, it is the prerogative of leaders to influence the perspectives of the agents to accept unpredictability, so as to allow them to develop the capability to receive information and adapt accordingly. Leaders must have the skill set to shift thought processes in order to focus on the possibilities of outcomes and choose which ones are “desirable” to the organization, rather than fixate on a single possibility and try to control and direct chaos to produce this outcome.

3 Chaos Theory and Complex Systems Defined

Chaos theory states that the behavior of complex systems are highly sensitive to the slightest changes in conditions, which results in small changes to giving rise to more unpredictable, prominent effects on the system. With the introduction of quantum mechanics came a better understanding of how chaos theory applies to the real world. “Chaos theory, in essence, is an attempt to remove some of the darkness and mystery which permeates the classical concept of chaos by explaining, at least in some dynamic systems, how the system exhibits chaotic behavior” (Hite[7]). Chaos theory emphasizes that the conditions and state of change are no longer simple linear
cause-and-effect relations; instead it assumes that both the cause and effect can originate and result from a multitude of variables that could come from various directions. This implies that a chaotic system is a flexible macro-structure that is vulnerable to the slightest disturbances on the micro level, although these changes are bounded by a pre-established framework.

Within the framework of an organization, chaos theory implies (but is not limited to) six critical points: 1) organizational life is predictable and unpredictable; 2) it is virtually impossible to define a single cause for any reactions; 3) diversity provides a more productive base; 4) self-organization will reduce concern for anarchy prevailing over chaos; 5) individual action in combination with a multiplier effect will focus responsibility on the individual; and 6) “scale-invariant properties and irreversibility are components of all chaotic organizations” (Grint[5]).

Organizational life may be predictable on the macro-level, as there will appear to be repetitive behaviors or patterns that appear aperiodically. On the other hand, at the micro-level of an organization, it will seem unpredictable because humans, as individuals, will appear to be random and to express unconnected, chaotic tendencies. One example found in nature is seen in the actions of ants; the activities of a single ant will appear random and disconnected, but the greater picture shows that it is a part of a larger social organization that has a single value. Because of this type of reasoning, the second critical point holds true: to define a single cause to explain an effect is impossible, as there could be many causes that occur simultaneously to produce an outcome. Every individual agent of the organization will establish multiple links, or connections, with other agents and various sources of information from the environment. Therefore, multiple reasons behind following directives or strategies will develop over time or simultaneously. Each unique link and motive must be taken into account when trying to align the goals of the organization with that of the individual. The strategies that are established should be aimed towards the acceptance of unpredictability and uncertainty, so as to give the impression “that they have control over something which is inherently uncontrollable” (Grint[5]).

The acceptance of uncertainty and unpredictability will help agents to recognize the value of dissenting voices and contrary cultures. The idea behind this is to shift the organization from a hierarchical top-down structure to a self-organizing structure, where the environment is defined by fundamental, interactive guidelines that allow for the flexibility in handling each situation uniquely. This idea is akin to giving an organization a set of standards and regulations that suggest how to handle general issues, instead of stating rigorous rules on how things should and should not be. It would be ideal to just hint at the overall culture and let each experimental, self-organizing group within the structure contribute to the definition of organizational life by facilitating their own resolutions (because it would be unique to each group) instead of following orders. The allowance of this kind of problem solving will enable the agents to voice their opinions and implement
actions without reprimand, unlike positive- and negative-reinforcement managerial styles that may dissolve the organization into anarchy. Agents who do not feel constrained by rules and regulations feel that they are contributing to the overall system, and are less likely to cause destructive disorder. From this point, it is up to the leader or manager to be able to allow the loss of total control, and to allow for the birth and decay of motivational schemes in order to become effectively adaptive.

With the loss of control, it usually follows that there is a loss of responsibility placed solely on the leader of an organization. This happens because the agents create and form the culture, and therefore have the obligation to uphold the culture. The leader or manager, and even the individual agents, must also understand the irreversibility of individual actions; the multiple connections that form between various agents will contribute greatly to the multiplier effect, and propel smaller-scale decisions and strategies into larger arrangements. A component of chaotic structures that this is commonly compared to is called a fractal, where similar ordering properties can be seen at different levels of the organization, and be recognizable to all levels. And, like a fractal, these similar patterns will build upon each other to create a complex structure.

4 The “Strange” Attractor

The development of Lorenz’s mathematical model of a chaotic system emphasized the idea that dynamic, complex systems are highly dependent on initial conditions; his model of the system demonstrated that a slight change in the input values produced very different outputs. However, no matter what changes were made, the visual pattern that computers generated based on Lorenz’s model reflected that of butterfly wings. “The resulting figure displays a curve that weaves itself into a circular pattern, but never repeats itself exactly. Because it never returns to the initial state, though it may come arbitrarily close, the system is aperiodic” (Singh and Singh[12]). An embedded circular shape within the “wings” forms as the model continues; however, it is almost like a void space – the pathways never cross through this space. This void space is an “attractor” that will draw “point trajectories into its orbit, yet two arbitrarily close points may diverge away from each other and still remain within the attractor” (Burns[2]).

“Conventional theory asserts that the world is predictable and stable, and able to be explained by causal links that can be measured and monitored. Chaos theory implies that in the short term anything can happen, but that in the long term patterns, or ‘strange attractors’, are discernible” (Grint[5]). These strange attractors represent a key concept in this definition of chaos theory. “A system attractor, in essence, operates like a magnet in a system. It is the point or locus around which dynamical system activity coalesces... It is the attractor that provides the system with some sense of unity, if not uniformity. The attractor may be strong and
definite, as with a fixed point, or it may be weak and indefinite, as with strange attractors” (Hite[7]). The strange attractor is not “weak” as in the classical sense of the definition. It is weak in the sense that it is flexible in its structure and has the ability to adapt infinitely. The strange attractor is better conceptualized as the pinpoint where the basis of the new or current dynamic system begins; this is similar to agents and how they interact within an organization. The difference between the agent and the attractor is that the attractor is an individual who possesses innate qualities that other agents may eventually gravitate towards.

In essence, the strange attractors of the organization are the values and vision that is shared, and “attractor” agents will exemplify these values and vision; but it is unlikely that individual agents will “orbit” the vision and values in the same way. This will result in the creation of multiple pathways to achieve the same overall mission of the organization. The “Butterfly Effect” theory was named after complexity science “where a butterfly flapping its wings in one location gives rise to a tornado or similar event occurring in another remote part of the world... the butterfly effect is nonlinear and amplifies the condition upon each iteration” (Osborn et al. [9]). And, as the butterfly effect explains, because these paths differ, these small changes in trajectories will result in larger changes to the overall system, though it will still be within the same framework. However, the timeframe of these changes, and to what extent the changes will have an effect, will be unknown; something small can begin a chain of events that will cause something relatively larger or smaller, in another part of the world or in close vicinity; but how quickly or slowly that happens will be unpredictable. At this point the difference between a complex system and chaotic system becomes difficult to define.

5 The Line between Chaotic and Complex Systems

“Where chaos theory addresses systems that appear to have high degrees of randomness and are sensitive to initial conditions, complexity theory has to do with systems that operate just at the line of separating coherence from chaos” (Hite[7]). Returning to the definition that chaos is everything unknown to humankind, it was also seen that the state of chaos thrives within the condition of symbiosis, by undifferentiating or non-delineating the self from the total. Now, instead of chaos being the unknown, as in uncertainty or ambiguity, it is transformed into being the unknown, as in the unawareness of individuality; there is no self or other, there is only totality; there is only interdependence in oneness (Singh [11]). Complex systems operate between order and chaos, where the state of symbiosis exists, but the conditions surrounding the symbiotic relationship are defined.

By extension of this thought, the theory of the “Butterfly Effect” is emphasized. The initial conditions put into the system are known, which is representative of imposing a type of order into the system. However, the outcome
will always be unpredictable in the short-term. Nevertheless, in the long-term, there will be aperiodic behaviors that a complex system will adapt to. Thus, if new initial conditions based upon these behaviors are inputted into the system, no matter how unpredictable the outcomes, the system will iterate and adapt to try to return to a flexible state of equilibrium, even if the speed of this change is unknown. It must also be accepted that this state of equilibrium is fleeting, as there will be another change in the system occurring somewhere else at any given point in time, giving credence to the idea that complex systems are dynamic in nature. And, because the system will always be in flux and dynamic, it is logical to say that how leadership is defined and how management is applied also need to be continuously dynamic.

6 Leadership Actualized

There is no universal explanation for what leadership is, or how to define it -- only contextual examples of what leadership accomplishes. Through the understanding of chaos and complexity, it becomes easier to digest that a solid definition for leadership may never be found; the essence of leadership is continuously adapted and remolded to fit what the organization needs. There are a few reasons behind why leadership is so difficult to define. Like the Butterfly Effect, the extent, speed, and actual dimensions of the response(s) to leadership will never be clearly known, and so cannot be clearly defined. However, the connotations of leadership are known to be adaptable to the culture of the organization.

Therefore, defining the culture would mean determining the style of leadership that is needed. Because culture varies from organization to organization, what defines a leader will also differ, as they will need to adapt to specific and unique organizational needs. And, as a leader, it is important to note that leadership is not delivered by a single individual, but rather, is dependent on the interaction between an agent and its organization and is constructed from social recognition (Osborn et al. [9]). “The point of leadership is to initiate change and make it feel like progress… Leadership is what takes us and other people into a better world. Leadership insists that things must be done differently. Leadership rides the forces that are pulling individuals, groups, organizations, markets, economies, and societies in different directions, and lends a coherence that will enable us to benefit from the change around us. Leadership says, ‘We cannot just carry on doing what we have done before. See all these forces of change around us; they are not just threats, they are also opportunities. But we must do this rather than that’” (Yudelowitz et al. [13]). Leadership seems to represent the “space between” what a leader does and how the organization responds; leadership manifests itself in the interaction, and what makes someone a leader is the leader’s awareness of this fact and to what extent his or her influence can be recognized.
7 Organizations are Complex Adaptive Systems

In an adaptive organization, leaders monitor the overall well-being of the system, both internally and externally. Attractors influence the organization’s culture and dynamics, while agents drive the system. A relatively new understanding of an organization is that it follows a “complex adaptive system” theory [CAST] -- a framework for explaining the emergence of system-level order that arises through the interactions of the system’s interdependent components (agents)” (Lichtenstein and Plowman[8]). Because these interactions and influences can begin from anywhere within an organization, the model of an organization that seems to emerge is a decentralized structure that allows change to originate from anywhere, at any time. However, this does not mean that the unity and cohesiveness of the structure will become affected. What a complex adaptive system offers is a flexible structure that allows for the input of all the variables from the environment to influence the system, then adapts accordingly by beginning with individual agents. This is very reflective of the Butterfly Effect; “when an agent adjusts to new information, the agent expands his/her own behavioral repertoire, which, in effect, expands the behavioral repertoire of the system itself” (Lichtenstein and Plowman[8]).

In an empirical study, B.B. Lichtenstein and D.A. Plowman found that there are four sequential conditions that form an element termed “emergence”. Multiple cases were examined, where each case exemplified an organization undergoing the process of adaptation and how they “emerged” to be able to survive within the present environmental conditions. The four prevailing, sequential conditions found in each case are: dis-equilibrium, amplification of actions, recombination or self-organization, and stabilizing feedback.

Dis-equilibrium describes the system when it is in a state of dynamism and is usually initiated by the occurrence of an incongruity or change. This disruption can be caused by external or internal influences, such as, competition or new opportunities, and can be volatile enough to push the system beyond the existing perceptions of the norm. The study found that this state must be sustained for a long period of time in order to be considered a precursor to an emergent ordered system.

The second condition, amplifying actions, is when the dis-equilibrium caused by small actions and events begins to fluctuate and amplify throughout the system, seemingly to move toward a “new attractor”, and grows until a threshold is reached. And, as learned from chaos theory, these actions and alterations will not follow a linear path throughout the organization; the change will easily “jump channels” (because all the agents are interconnected in some way) and can escalate in unpredictable, and unexpected ways (Lichtenstein and Plowman[8]).

The recombination, or self-organization, is the third (and most defining) condition that must be reached, as this is where a new order is established that increases the efficiency and capacity of the entire system. Once the organization has
crossed the aforementioned threshold, it “emerges” as a “new entity with qualities that are not [yet] reflected in the interactions of each agent within the system” (Lichtenstein and Plowman[8]). The hope of this self-organization is that the system will recombine in such a way that new patterns of interaction between agents will improve the functions and capacity of the organization. In truth, this critical step will determine the survivability of the organization because, instead of restructuring progressively, the system could collapse or self-disorganize. This could be due to a) the lack of innovative ideas, b) poor assessment of the environment (because the reconstruction is dependent on reform), c) an inadequate “strategic fit” or core competency to handle the changes made, or d) a resistance to change (which is characteristic of a stable system) (Yukl and Lepsinger [14]).

The final condition of this emergent ordered system is the stabilizing feedback (“damping feedback”), or the anchors that keep the change in place and slow the amplification that produced the emergence in the initial stages. This anchoring is important, as it is reflective of how the interactions between agents sustain the change successfully and solidify legitimacy to the new paradigm. The new emergent order will dramatically increase “the capacity of the system to achieve its goals” (Lichtenstein and Plowman[8]). The study also surmised that leaders with certain characteristics will enable this emergence in an organization.

8 Characteristics of Leaders of Emergence

Leaders of emergence will generate or “enable” circumstances that will purposefully create the conditions needed to bring about the new emergent order. Lichtenstein and Plowmen noticed that certain characteristics were prominent and recognizable within each case used in the study. To achieve the dis-equilibrium condition, a leader will need to disrupt existing patterns and rally support for the uncertainty in the disturbance. Most importantly, a leader will need to acknowledge these conflicts and controversies with the intention that the farther the “ripple” spreads, the more perspective and diverse solutions will be generated. In this case, it is not the “people at the top” of the formal hierarchy that will brainstorm and decide what solution to take. Instead, the role of the leader becomes distributed through all branches of the organizations, where conflict and diversity are acknowledged, and can be accepted, equally. Next, it becomes the role of these emerging leaders to “amplify” the perspectives and conflict through the rest of the organization by encouraging innovative ideas and solutions, in order to instigate the second condition. By allowing experimental procedures, for example, to be enacted in a certain part of the organization, new ideas can be tested instead of just talked about; the belief or disbelief in the success of an experiment is only truly forged when the results are attainable. And, by encouraging the expression of innovation, “new attractors” may be birthed, and a type of “relational space” can be created, where “a certain high
quality of interactions, reflecting a shared context of mutual respect, trust, and psychological safety in the relationship” is created (Lichtenstein and Plowman[8]). And, “as predicted by complexity theory (and managerial psychology), these rich interactions strengthened interpersonal networks, which helped to amplify the changes as they emerged” (Lichtenstein and Plowman[8]).

A leader who seeks the creation of a new emergent structure will assess the feasibility of the new structure that this attractor presents and not blindly following the new internal trend. Some points that a leader may ask about the proposed system are a) if it is attainable, b) if it will fit within the environment, and c) if it is progressive or retrogressive to the organization’s values and vision. If the leader is fairly sure that the new regime is “better” for the organization, he or she will need to begin to rally other agents to support it, so that collective action can contribute to a solidified installment of the changes made.

The final condition of this complex adaptive system depends on the ability of the leaders to re-stabilize the structure. To do this, the leaders must remind the organization of the values and vision of the organization, and promote awareness of the cultural and environmental constraints that will affect the new emergent structure. It is the leader’s job to keep the structure grounded in reality while allowing it to thrive at the increased capacity that was achieved. And, while it is true that these four sequential conditions and characteristics were founded upon a limited number of case studies, this model for understanding the functions and reactions of a complex adaptive system are relevant and supported by aspects grounded in chaos theory, presenting an “underlying order in chaos” (Otten and Chen[10]).
9 Possible Motivators for an Emergent System

Both models have only scratched the surface of the new order of leadership and management in an organization. They very clearly express that leaders are no longer the apex of the organization, but, instead, are more effective when they are “orbidted” and “in-plane” with the agents. However, in order for the agents to begin to collect around a supported attractor, they must be motivated to do so. The leader will need to give purpose and meaning to the new attractor that will make sense to the emerging paradigm. The empirical study of the emergent system found that the creation of correlated language and symbols helped to initiate recombination or “self-organization”. These symbols resonated the most when performed through symbolic actions that legitimize the change, while the language used helps to relate emotionally on a personal level with each agent. Another way to inspire meaning and connection to the new structure is to consolidate or recombine important resources, such as, capital, space, or skills, so as to give the impression that the system is expanding towards a “better” paradigm. The idea is that self-organization will be supported, and, thus, gain favor throughout the system. And, because there is not only a centralized leader within the structure of this complex system, the multiple leaders who emerge become symbols (Lichtenstein and Plowman[8]).

Hamel [6], discusses a management style called “Management 2.0” that humanizes the structure of an organization, acknowledges the autonomy of the individual, and sets a complex system motivated by humanistic, not materialistic, ideals: it redefines the language of the system, supporting ideals such as justice, community, and collaboration, as opposed to corruption, profit, and rivalry. The motivators behind this foundation are unique and requires a distinct leadership style to achieve it. One technique to increase motivation to uphold these ideals is to “reduce fear and increase trust”. To reduce fear means to eliminate positive-negative reinforcement of actions, and encourage risk-taking innovations. With autonomy, now, comes an inherent trust between the leader and agents, where a leader trusts the agents of the organization to function within the values and boundaries established, and the agents trust the leader to provide stability and dynamism, without erasure of the individuality of the agent. And, democratization of information allows agents to act independently, thus preserving autonomy.

Summary and Conclusions

Empowering the agents allows them to have the capability to drive the system. However, without the presence of the attractors to influence the culture, the system
may not emerge according to values of the organization. Ultimately, the obligation of the leader is to bridge the values with the vision and mission of the entity, and give purpose to the organization. Leaders will also need to monitor the internal and external influences to the system. The use of complex adaptive systems theory will enable the leaders to guide the adaptation of a system by creating an emergent structure that reconfigures the organization into new patterns that improve the function and capacity of the system, while still aligning with its core competency. Although it is fundamentally impossible to control chaos, it is possible to increase the survivability of an organization to adapt to the chaotic environment through complex adaptive systems theory.

References

Lyapunov spectrum analysis of natural convection in a vertical, highly confined, differentially heated fluid layer

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Abstract. We use Lyapunov spectrum analysis to characterize the dynamics of a single convection roll between two differentially heated plates. 3D numerical simulation is carried out in a highly confined periodic domain. As the Rayleigh number increases, the intensity of the convection roll displays chaotic features while the roll remains stationary. For still higher values of the Rayleigh number, the roll intermittently moves between two positions separated by half a wavelength. We use Lyapunov spectrum analysis to help determine the characteristics of the flow in both regimes. We show that although the largest Lyapunov exponent is positive on average, the most probable value of the short-time Lyapunov exponent is negative. We compute the flow eigenvectors associated with the strongest variations in the exponent in the chaotic and the intermittent case and identify the corresponding hydrodynamic modes of instability.

Keywords: Natural convection, Period-doubling bifurcations, crisis-induced intermittency, Lyapunov spectrum.

1 Introduction

Natural convection between two vertical plates maintained at different temperatures is an important prototype to model heat transfer in industrial applications, such as plate heat exchangers or solar panels. The properties of heat transfer are deeply influenced by the nature of the flow, which is typically turbulent. It is therefore of interest to study the onset of chaotic dynamics in these flows. The development of instabilities in a differentially heated cavities with adiabatic walls has been studied numerically for a few decades [1,2]. Earlier studies are mostly limited to 2D geometries and relatively low Rayleigh numbers regimes (steady, periodic, quasi-periodic) with a focus on primary instabilities. Recent studies focus on the fully turbulent nature of the natural convection.
convection flow at high Rayleigh numbers [3], which remains a challenge owing to the double kinetic and thermal origin of the fluctuations.

Our studies attempt to bridge the gap between the relatively ordered flow observed at low Rayleigh numbers and the fully turbulent flow at high Rayleigh numbers. To this end, we carried out the three-dimensional direct numerical simulation (DNS) of a fluid layer between two vertical, infinite, differentially heated plates and determined the different stages leading to chaos [4]. The flow is characterized by co-rotating convection rolls which grow and shrink over time and interact with each other in a complex fashion. Similar rolls have also been observed in tall cavities of high aspect ratio [5]. A useful model of the problem can be obtained by limiting the dimensions of the plates in order to study the dynamics of a single convection roll. A cascade of period-doubling bifurcations and a crisis-induced intermittency have been observed in the vertically confined domain [6]. The goal of this paper is examine how Lyapunov exponent analysis can help characterize the chaotic dynamics of the flow in such a configuration.

2 Configuration

We consider the flow of air between two infinite vertical plates maintained at different temperatures. The configuration is represented in Figure 1. The distance between the two plates is \( D \), and the periodic height and depth of the plates are \( L_z \) and \( L_y \) respectively. The temperature difference between the two plates is \( \Delta T \). The direction \( x \) is normal to the plates, the transverse direction is \( y \), and the gravity \( g \) is opposite to the vertical direction \( z \).

![Fig. 1](image)

The simulation domain is constituted by two vertical plates, separated by a distance \( D \) and maintained at different temperatures. Periodic boundary conditions for the plates are enforced in both transverse and vertical directions (\( y \) and \( z \)). The aspect ratios of the periodic dimensions are \( A_y = L_y/D = 1 \) and \( A_z = L_z/D = 2.5 \). The temperature of the back plate at \( x = 0 \) (in red) is \( \Delta T/2 \), while that of the front plate at \( x = 1 \) (in blue) is \(-\Delta T/2\). The distance between the plates is \( D \).

The fluid properties of air, such as the kinetic viscosity \( \nu \), thermal diffusivity \( \kappa \), and thermal expansion coefficient \( \beta \), are supposed to be constant. Four nondimensional parameters characterizing the flow are chosen in the following
way: the Prandtl number $Pr = \frac{\nu}{\kappa}$, the Rayleigh number based on the width of the gap between the two plates $Ra = \frac{\varphi \Delta T D^3}{\nu \kappa}$, and the transverse and vertical aspect ratio $A_y = L_y/D$ and $A_z = L_z/D$, respectively. Only the Rayleigh number is varied in the present study. The Prandtl number of air is taken equal to 0.71. The transverse aspect ratio is set to be $A_y = 1$, the vertical aspect ratio is set to $A_z = 2.5$, which corresponds to the critical wavelength $\lambda_{zc} = 2.513$ obtained by the stability analysis [4].

The flow is governed by the Navier-Stokes equations within the Boussinesq approximation. The nondimensional equations are:

$$\nabla \cdot \vec{u} = 0 \quad (1)$$
$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{Pr}{\sqrt{Ra}} \Delta \vec{u} + Pr \theta \vec{z} \quad (2)$$
$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = \frac{1}{\sqrt{Ra}} \Delta \theta \quad (3)$$

with Dirichlet boundary conditions at the plates

$$\vec{u}(0, y, z, t) = \vec{u}(1, y, z, t) = 0, \quad \theta(0, y, z, t) = 0.5, \quad \theta(1, y, z, t) = -0.5 \quad (4)$$

and periodic conditions in the $y$ and $z$ directions. The equations (1)-(4) admit an $O(2) \times O(2)$ symmetry. One $O(2)$ symmetry corresponds to the translation in the transverse direction $y$ and the reflection $y \rightarrow -y$, while the other corresponds to the translations in the vertical direction $z$ and a reflection that combines centrosymmetry and Boussinesq symmetry: $(x, z, \theta) \rightarrow (1 - x, -z, -\theta)$.

A spectral code [7] developed at LIMSI is used to carry out the simulations. The spatial domain is discretized by the Chebyshev-Fourier collocation method. The projection-correction method is used to enforce the incompressibility of the flow. The equations are integrated in time with a second-order mixed explicit-implicit scheme. A Chebyshev discretization with 40 modes is applied in the direction $x$, while the Fourier discretization is used in the transverse and vertical directions. 30 Fourier modes are used in the transverse direction $y$ for $A_y = 1$, while 60 Fourier modes are used in the vertical direction $z$ for $A_z = 2.5$.

### 2.1 Description

For low Rayleigh numbers, the flow solution is laminar. A cubic velocity and linear temperature profile, which depend only on the normal distance from the plates are observed. The flow presents similar features to those of a confined mixing layer [9,4]. As the Rayleigh number $Ra$ is increased, steady two-dimensional convection rolls appear at $Ra = 5708$, which then at $Ra = 9980$ become steady three-dimensional convection rolls linked together through braids of vorticity (see Figure 2). For still higher Rayleigh numbers, the flow becomes periodic at $Ra = 11500$. The convection roll essentially grows and shrinks with a characteristic period of $T = 28$ convective time units, which is in good agreement with the natural excitation frequency of a mixing layer [9].
As the Rayleigh number increases, a series of period-doubling bifurcations appears, as illustrated in Figure 3. More details can be found in [6]. The onset of chaos was predicted to occur at $Ra \sim 12320$, in agreement with numerical observations. The variations of the roll size become more disorganized and intense, but the position of the roll remains quasi-stationary. When $Ra = 12546$, the variations in the intensity of the roll become so large that the roll breaks down and reforms at another location, separated by half a vertical wavelength from the original one. In terms of dynamics this corresponds to crisis-induced intermittency, which can be seen in Figure 3(b). The difference between the chaotic and the intermittent regimes in terms of phase portraits is illustrated in Figure 4 for two Rayleigh numbers taken in each regime.

![Fig. 2. (Color online) Q-criterion visualization of flow structures colored by the vertical vorticity $\Omega_z$. Bi-periodic domain at $Ra = 12380$, $Q = 0.25$ in the present numerical configuration from Figure 1, i.e. with periodic boundary conditions in both $y$ and $z$ directions ($A_y = A_z = 1$);](image)

3 Lyapunov spectrum

3.1 Definition

Several methods exist to distinguish between regular and chaotic dynamics in a deterministic system. The largest Lyapunov exponent, which measures the divergence rate of two nearby trajectories, is considered as a useful indicator to answer this question. Similarly, the $n$ first Lyapunov exponents $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_n$ characterize the deformation rates of a $n$-sphere of nearby initial conditions. We applied the numerical technique proposed by Benettin et al. [8] to compute the Lyapunov spectrum of our fluid system, by parallelizing the DNS code described above with MPI library. On each processor, the flows are advanced independently in time. The flow on the processor-0 is the reference solution, which is obtained by numerical integration of the nonlinear equations. On the other processors, the randomly initiated perturbations $\delta X$ are integrated in time by solving the linearized DNS code. The modified
Fig. 3. (Color online) Bifurcation diagram obtained by using the local maxima $\theta_n$ of the temperature time series at the point $(0.038, 0.097, 0.983)$. Note: the vertical line in the figure corresponds the largest Rayleigh number in Figure 3 (a) $12000 < Ra < 12500$ (b) $12400 < Ra < 12600$.

Fig. 4. (Color online) Phase portraits. Abscissa: real part of the Fourier transform (in $y$ and $z$) of the vertical velocity $\hat{w}_{01}(x)$ calculated on vertical plane $x = 0.0381$; ordinate: real part of the Fourier transform of the vertical velocity $\hat{w}_{10}$. (a) $Ra = 12380$, (b) $Ra = 12600$.

Gram-Schmidt procedure is applied every 20 time-steps of $dt$ to renormalize the perturbations. At each renormalisation step, the instantaneous Lyapunov exponents were computed as

$$\lambda^\text{inst}_i = \frac{1}{\Delta t} \ln \frac{\|\delta X(j\Delta t)\|_i}{\|\delta X(0)\|_i}$$

Their asymptotic mean values form the long-time Lyapunov spectrum:

$$\lambda_i = \lim_{N \to +\infty} \frac{1}{N\Delta t} \sum_{j \in N} \ln \frac{\|\delta X(j\Delta t)\|_i}{\|\delta X(0)\|_i}$$

where $\lambda_i$ is the $i$-th Lyapunov exponent and the norm measuring the distance between two nearby trajectories is chosen as $\|\delta X(t)\| = \sqrt{\int_V [\delta \hat{u}(t)^2 + \delta \theta(t)^2]dV}$. 

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3.2 Long-time Lyapunov exponents

The computation of Lyapunov spectrum for our fluid system was carried out at different Rayleigh numbers between $Ra = 12360$ and $Ra = 12900$. Errorbars for the Lyapunov exponent are estimated from the standard error of the mean assuming a Gaussian distribution and a 95% confidence interval. We note that the error on the exponent may be somewhat underestimated, as we do not take into account other sources of error, such as the distance to the attractor.

In all that follows, we focus on two Rayleigh numbers: one corresponds to the chaotic, non-intermittent system $Ra = 12380$. The other $Ra = 12600$ corresponds to a chaotic, intermittent case. Convergence tests were run for these two Rayleigh numbers $Ra = 12380$ and $Ra = 12600$ and two different time-discretizations $dt = 1 \times 10^{-3}$ and $dt = 1 \times 10^{-2}$. The 15 leading Lyapunov exponents are computed, among which the first 8 ones are listed in Table 1.

![Fig. 5. (Color online) (a) The largest Lyapunov exponent $\lambda_1$ for different Rayleigh numbers; Error bars are 1.96 times the standard error. (b) Fractal dimension obtained by application of the Kaplan-Yorke formula as a function of the Rayleigh number. The position of the solid line spanning each figure represents the value of the Rayleigh number at the onset of the crisis.](image)

As shown in Figure 3.2, the largest asymptotic Lyapunov exponent is positive for $Ra \geq 12360$, and increases quasi-linearly for $12400 < Ra < 12546$. This suggests that temporal chaos has been reached. For all Rayleigh numbers considered, only one single positive Lyapunov exponent is found and is on the order of 0.01. The test $0-1$ for chaos proposed by Gottwald and Melbourne [12,13] was applied to an appropriately sampled temperature time series, and returned a value close to 1, which confirms that our flow is chaotic. The Lyapunov exponent is considerably larger for the intermittent case $Ra = 12600$ than for the chaotic case $Ra = 12380$.

We find that the asymptotic value of exponents 2 to 4 is close to zero. We observe that the temporal oscillations of the short-time exponents 2 to 4 decrease with the time step, as can be expected. As shown by Sirovich and Deane [10] for Rayleigh-Bénard convection, three exponents should be zero:
one comes from the fact that the time derivative \( \frac{\partial X}{\partial t} \) of the reference solution \( X \) satisfies the linearized equation, since the system is autonomous. The other two zero exponents reflect the fact that \( \frac{\partial X}{\partial y}, \frac{\partial X}{\partial z} \) also satisfy the linearized equation on account of the homogeneous boundary conditions.

All exponents of order \( n \geq 5 \) were found to be negative. Convergence was more difficult to reach for these higher-order exponents. However even if some uncertainty is present, this does not affect significantly the value of the fractal dimension.

The Lyapunov dimension was estimated using the Kaplan-Yorke formula [11]:

\[
D_L = K + \frac{S_K}{|\lambda_{K+1}|}
\]

where \( K \) is the largest \( n \) for which \( S_n = \sum_{i=1}^{n} \lambda_i > 0 \). It was found to be between 4.2 and 4.6, as can be seen in Figure 3.2 (b). An inflection point, corresponding to a sharp increase in the largest exponent, is observed at the onset of intermittency for both the largest exponent and the Lyapunov dimension.

### 4 Short-time Lyapunov exponent

As pointed out by Vastano and Moser [15], examination of the short-time Lyapunov exponent provides additional information about the flow. Figure 6 and 7 shows the distribution of the first Lyapunov exponent for the two Rayleigh numbers and the two time resolutions. We can see that the distributions are very similar for both time intervals, which shows the convergence of the computations. Corresponding time series of the largest Lyapunov exponent and their Fourier spectrum are represented in Figure 8. The fundamental excitation frequency \( f = 0.22 \) is dominant in the chaotic case. Lower frequencies become important in the chaotic case.

A striking fact is that for both Rayleigh numbers, although the mean value of the exponent is positive, the maximum value of probability distribution function (p.d.f.) is actually negative. This is markedly different from the

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**Table 1.** First 8 Lyapunov exponents at two different Rayleigh numbers for two different time steps.

<table>
<thead>
<tr>
<th>( \lambda_i dt )</th>
<th>( \text{Ra} = 12380 )</th>
<th>( \text{Ra} = 12600 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 0.0094 \pm 0.0004 )</td>
<td>( 0.0199 \pm 0.0005 )</td>
</tr>
<tr>
<td>2</td>
<td>( -0.00047 \pm 0.000067 )</td>
<td>( -0.00001 \pm 0.0008 )</td>
</tr>
<tr>
<td>3</td>
<td>( 0.00075 \pm 0.000048 )</td>
<td>( 0.00036 \pm 0.00006 )</td>
</tr>
<tr>
<td>4</td>
<td>( 0.0010 \pm 0.00053 )</td>
<td>( 0.00011 \pm 0.00062 )</td>
</tr>
<tr>
<td>5</td>
<td>( -0.0579 \pm 0.00020 )</td>
<td>( -0.0594 \pm 0.00017 )</td>
</tr>
<tr>
<td>6</td>
<td>( -0.0726 \pm 0.0006 )</td>
<td>( -0.0696 \pm 0.0006 )</td>
</tr>
<tr>
<td>7</td>
<td>( -0.0709 \pm 0.0006 )</td>
<td>( -0.0732 \pm 0.0006 )</td>
</tr>
<tr>
<td>8</td>
<td>( -0.0843 \pm 0.0006 )</td>
<td>( -0.0919 \pm 0.0006 )</td>
</tr>
</tbody>
</table>
results reported by Kapitaniak [14] for quasi-periodically forced systems, where the mean value of the exponent appeared to correspond to the maximum of the distribution. We note that no external forcing is imposed in our configuration, which is characterized by self-sustained oscillations. The distributions at $Ra = 12380$ and $Ra = 12600$ present many similarities. The main difference is that in the intermittent case the local maximum of the distribution for small positive values in Figure 6 disappears, while a band of significantly higher positive values (larger than 0.2) is created in Figure 7.

We computed the vector associated with local extrema of the short-time Lyapunov exponent which were identified in the time series. This gives us insight into the perturbations most likely to disorganize the flow. We checked that observations made at a particular time held for other times.

Results are presented in Figure 9 for the chaotic case. For the chaotic case, we have identified two types of relative extrema: (i) relatively small excursions, associated with the local maximum and the local minimum in the histogram from Figure 6 corresponding to positions marked with filled circles in Figure 8 (a). We find that the perturbation associated with a local maximum consists of almost 2D rolls (Figure 9 (a)), while the minimum corresponds to a strongly 3D flow and a relatively weaker convection roll (Figure 9 (b)). (ii) stronger excursions, where both extrema are associated with an essentially 2D flow (positions marked with filled squares in Figure 9 (c)(d)). 2D convection rolls correspond to the most unstable linear modes. However the convection rolls associated with maxima seem to be stronger than those associated with minima.

In the intermittent case, we focus exclusively on largest extrema. Figure 10 (a) shows that the maxima in time corresponds to a flow which is in fact almost 1-D (note the much lower value for the criterion $Q = 0.05$), while the minima in time corresponds to a 2D flow (see Figure 10 (b)). These two states can be associated with the break-up and formation of the roll.

![Fig. 6.](image) (Color online) Probability distribution function (p.d.f.) of instantaneous 1st Lyapunov exponent $\lambda_1^{inst}$ at $Ra = 12380$. 

Fig. 7. (Color online) Probability distribution function (p.d.f.) of instantaneous 1st Lyapunov exponent $\lambda_{\text{inst}}^1$ at $Ra = 12600$.

Fig. 8. (Color online) (a) (b) Evolution of the largest short-time Lyapunov exponent $\lambda_{\text{inst}}^1$ at (a) $Ra = 12380$ (b) $Ra = 12600$; (c) (d) Temporal Fourier spectrum of the largest short-time exponent $\lambda_{\text{inst}}^1$ at (c) $Ra = 12380$ (d) $Ra = 12600$. 

(a) $dt = 1 \times 10^{-2}$
(b) $dt = 1 \times 10^{-3}$
Fig. 9. (Color online) Eigenvector associated with a local extremum of the short-time exponent at $Ra = 12380$ at the positions indicated in Figure 8 (a). Value of the $Q$ isosurface $Q = 0.3$ (a) $t=469$ (maximum) (b) $t=479$ (minimum) (c) $t=552$ (maximum) (d) $t=726$ (minimum)

5 Conclusion

We have considered the numerical simulation of a convection roll between two differentially heated plates of small periodic dimensions. As the Rayleigh number increases, the convection roll shrinks and grows in a periodic, then quasi-periodic, then chaotic. For still higher values, the convection roll breaks down and reforms intermittently at another location. Lyapunov spectrum analysis was used to characterize the dynamical features of the flow. Two cases in the purely chaotic and intermittent regime were examined in detail. We found that although the asymptotic value of the largest exponent is positive, its most probable value is negative. We showed that intermittency corresponds to the occurrence of higher positive values in the Lyapunov exponent corresponding
Fig. 10. (Color online) Eigenvector associated with a local extremum of the short-time exponent at Ra = 12600 at the positions indicated in Figure 8 (b). Value of the Q isosurface (a) t=954 (maximum) $Q = 0.05$ (b) t=968 (minimum) $Q = 0.3$

to the break-up and reformation of the convection roll. The perturbations associated with the extremal values of the short-time largest exponent were identified. In the chaotic case, the perturbations associated with the largest extrema are 2D convection rolls. Maxima are associated with larger rolls, while minima are associated with less intense rolls. In the intermittent case, maxima were associated with a quasi 1-D flow, which corresponds to the break-up of the roll, while minima corresponded to 2D convection rolls and therefore the roll formation stage. These results confirm that the analysis of short-time Lyapunov exponents provides insight into the physics of the flow and suggests that it could be useful for low-order modelling of its complex dynamics.

References

6. Z. Gao, B. Podvin, A. Sergent, S. Xin, Chaotic dynamics of a convection roll in a highly confined, vertical, differentially heated fluid layer, submitted to Physical Review E.


