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# The Higgs boson and the Higgs field in fractal models of the Universe: supermassive black holes, relativistic jets, solar coronal holes, active microobjects 

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#### Abstract

To describe the masses of black holes, their relationships with the parameters of the Higgs boson, models based on the distribution density functions of the number of quanta in the ground and excited states for relic photons, and on the basis of the density distribution functions of the radiation intensity are proposed. It is proposed to represent the central region of a supermassive black hole near the upper mass boundary as a Bose condensate from black holes. Various states for a black hole with an intermediate mass are introduced. The following estimates have been made: masses for light black holes, binary and supermassive black holes; the speeds of motion of relativistic jets (emissions of matter); widths of active regions of coronal holes on the Sun; a number of parameters of active microobjects. These estimates are consistent with experimental data. Keywords: supermassive black holes, Bose condensate from black holes, Higgs boson, relic photons, relativistic jets, coronal holes on the Sun, active microobjects.


## 1 Introduction

Roger Penrose, Reinhard Henzel, Andrea Gez are the laureates of the 2020 Nobel Prize in Physics. Using the general theory of relativity, R. Penrose theoretically predicted the gravitational collapse of massive stars, space-time singularities, and the birth of black holes [1, 2]. R. Genzel, A. Gez discovered and described a supermassive black hole in the center of our Milky Way galaxy [3, 4]. Earlier K.S. Thorne [5] showed that a star can collapse under the influence of its own gravity: the space around it becomes curved, the star disappears and a black hole appears. It has been experimentally established, that the merger of two black holes [6], two neutron stars [7] is accompanied by the emission of gravitational waves. In [8, 9], a description of the parameters of gravitational waves, relict photons and their relations with the parameters of the Higgs boson was carried out in the framework of the Dicke superradiance model. In this case, supernonradiative states of gravitational fields are possible [10, 11]. However, the mechanisms of transitions from black holes with light masses (of the order of $29 \div 32 M_{s}$ [6, 7], where $M_{s}$ is the mass of the Sun) to supermassive (of the order of $4 \div 5 \cdot 10^{6} M_{s}[3,4]$ ) and relativistic (of the order
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$10^{11} M_{s}$ ) black holes have not yet been described. The creation of such theoretical models requires taking into account stochastic processes, the mass distribution functions of black holes in the Universe, the effect of ordering operators and the presence of qubit states [12, 13] for binary black holes and neutron stars. It also becomes necessary to describe the ejections of matter (relativistic jets) from a supermassive black hole [14]. The use of experimental methods with high angular resolution [15] makes it possible to study the nature of the Higgs field by the example of the behavior of solar active regions (coronal holes). The parameters of active objects are determined by the connections with the Higgs boson and with the different nature of the Higgs field. In [16], experimental evidence was obtained for the decay of the Higgs boson into a lepton pair and a photon, which indicates to the presence of an asymmetry of matter and antimatter [16, 17]. Experimentally in [18] the processes of formation and decay of tetraquarks were investigated. The authors believe that the structure of the new tetraquark contains charmed diquark and antidiquark, which are coupled by gluon interaction. In [19], a target made of gaseous deuterium was irradiated with a proton beam and the cross section for reactions with the formation of a helium isotope was measured. The authors estimated the baryon density for the early Universe during the process of primordial nucleosynthesis. However, the contributions of nonzero rest mass antineutrinos to Higgs fields have not been described.
The aim of this work is to describe the parameters of black holes, relativistic jets, active microobjects, their connections with the Higgs boson and the Higgs field of various nature (taking into account antineutrinos with nonzero rest mass) within the framework of a number of fractal cosmology models.

## 2 Models for describing black hole masses

In [8, 9] the Dicke superradiance model was used to describe gravitational waves and relict photons from binary black holes and neutron stars. For the ratio of the radiation intensities (maximum $I_{m}$ to initial $I(0)$ ) was obtained

$$
I_{m} / I(0)=\left(a_{0}+a_{m}\right)\left(a_{0}-a_{m}+1\right) ; a_{0}^{2}=a_{m}^{2}+z_{\mu}^{\prime}\left(z_{\mu}^{\prime}+2\right) / 4 ; a_{m}^{2}=z_{A 2}^{\prime} ; N_{r a}=z_{A 2}^{\prime}+z_{\mu}^{\prime} .
$$

Here $z_{A 2}^{\prime}=1034.109294$ and $z_{\mu}^{\prime}=7.18418108$ are the usual and cosmological redshifts; the number of relic photons $N_{r a}=1041.293475$; intensity ratio $I_{m} / I(0)=81.06580421$. Supernonradiative states (of which the radiation intensity is equal to zero) were considered within the framework of the models $\mathrm{A}_{0}, \mathrm{~A}_{1}[8,9]$. In the model $\mathrm{A}_{0}$, the characteristic value of the number of bosons in the equilibrium state $N_{0 A}=3.557716045 \cdot 10^{5}$ was obtained. This made it possible to determine the characteristic energy $E_{0 A}=N_{0 A} E_{G}=4.311073329 \mathrm{eV}$, where the rest energy of the graviton $E_{G}=12.11753067 \mu \mathrm{~V}$. In the model $\mathrm{A}_{1}$, a characteristic distribution density
function $n_{z g}^{\prime}=0.114317037$ is obtained, where $n_{z g}^{\prime}+\left|n_{z g}\right|=1$ for Fermi-type particles. This function allows us to determine the characteristic frequencies

$$
\begin{align*}
& v_{z g}^{\prime}, v_{z g}^{*}, v_{D 0} \\
& \quad v_{z g}^{\prime}=n_{z g}^{\prime} v_{G 0} ; \quad v_{z g}^{*}=v_{z g}^{\prime} / \psi_{01} ; \quad \psi_{01}=\varepsilon_{01} / E_{H 0} ; \quad v_{G 0}=N_{0 A} \cdot v_{D 0} \tag{2}
\end{align*}
$$

Here $E_{H 0}=125.03238 \mathrm{GeV}$ and $\varepsilon_{01}=126.9414849 \mathrm{GeV}$ are Higgs boson energies obtained without and taking into account the Higgs field; frequency $v_{G 0}=2.9304515 \mathrm{GHz}, \psi_{01}=1.015268884[8,9]$. Based on (2), we find the numerical values $\quad v_{z g}^{\prime}=335.0005326 \mathrm{MHz}, \quad v_{z g}^{*}=329.9623754 \mathrm{MHz}$, $v_{D 0}=8.236889799 \mathrm{kHz}$. Our calculated frequency $v_{z g}^{*}$ practically coincides with the frequency of 330 MHz , at which dark matter dominates from observations of radio filaments [20].
Model $\mathbf{B}_{\mathbf{0}}$. Black holes with light masses $M_{b h}$ are described on the basis of spectra for occupation numbers $n_{A x}=n_{A 0} S_{0 x}^{\prime}$ and $n_{A x}^{*}=n_{A 0} S_{x u} \quad(x=1,2,3,4$; spectral parameters $S_{0 x}^{\prime}$ and $S_{x u}$ are determined in [8, 9]) within the framework of the anisotropic model, where the main parameter $n_{A 0}=58.04663887$ is determined based on the expressions

$$
\begin{align*}
& n_{A 0}=\left(z_{\mu \lambda}^{\prime}\right)^{2}-1=\left(z_{\mu}^{\prime}+3 / 2\right)\left(z_{\mu}^{\prime}-1 / 2\right) ; \quad z_{\mu \lambda}^{\prime}=z_{\mu}^{\prime}+1 / 2 ; \quad 1 / z_{\mu \lambda}^{\prime}=\sin \varphi_{\mu \lambda}^{\prime} ; \\
& n_{A 0}^{\prime}=\left(z_{\mu \lambda}^{\prime}\right)^{2} ; \quad n_{A 0}^{\prime}-n_{A 0}=1 ; \quad \varphi_{\mu \lambda}^{\prime}=\varphi_{a} Q_{H 2} ; \quad\left(n_{h 1}+n_{h 2}\right)-2 n_{A 4}=n_{G} . \tag{3}
\end{align*}
$$

Using the example of binary black holes in [8,9] and expressions (3), the quanta number of the second black hole $n_{h 2}=M_{h 2} / M_{s}=n_{A 0} / 2=29.02331944$, the first black hole $n_{h 1}=M_{h 1} / M_{s}=35.98093926$ before their merger was obtained. After the merger, a black hole is formed with a number of quanta $2 n_{A 4}=M_{b h} / M_{s}=62.0042587$ and a number of quanta $n_{G}=1 / Q_{H 2}=3$ are carried away by gravitational waves. In the general case, the number of quanta $n_{A 0}, n_{G}$ and the cosmological redshift $z_{\mu}^{\prime}$ determine the number of quanta of the gluon field $n_{g}=2 n_{G} /\left[z_{\mu}^{\prime}\left(z_{\mu}^{\prime}+1\right)-n_{A 0}\right]$. At $n_{G}=3$, constant parameters $z_{\mu}^{\prime}$ from (1), $n_{A 0}$ from (3) we obtain $n_{g}=8$. If $n_{G}, z_{\mu}^{\prime}, n_{A 0}$ are variables, then the number of quanta of the gluon field $n_{g}$ becomes a function of these three arguments, that is typical for bulk fractal structures of the Universe.
Model $\mathbf{B}_{1}$. To estimate the masses of supermassive black holes, we write down the basic relations for the energies

$$
E_{H 0} / E_{G}=v_{H 0}^{*} / v_{G 0}=N_{H G} ; \quad E_{G} / v_{G 0}=E_{H 0} / v_{H 0}^{*}=2 \pi \hbar ;
$$

$$
\begin{equation*}
E_{H 0} / E_{0 A}=N_{0 n} ; \quad E_{H 0} / \varepsilon_{0 n}=N_{0 n}^{*} ; \quad N_{0 n}^{*}=\left(1+n_{z g}^{\prime}\right) N_{0 n} . \tag{4}
\end{equation*}
$$

Here $\hbar$ is Planck's constant. Taking into account (4), we find the parameters $N_{H G}=1.031830522 \cdot 10^{16}, N_{0 n}=2.900261036 \cdot 10^{10}, \quad N_{0 n}^{*}=3.231810284 \cdot 10^{10}$, energy $\varepsilon_{0 n}=3.86880321 \mathrm{eV}$. The parameter $N_{H G}$ is a function of the main parameters $N_{r a}, N_{0 A}, N_{0 n}$

$$
\begin{align*}
& N_{H G}=N_{r a} N_{c v}=N_{0 A} N_{0 n}=N_{D V} n_{r a}=N_{r a} N_{0 A} n_{r a} ; \\
& N_{c v}=N_{0 A} n_{r a} ; \quad N_{0 n}=N_{r a} n_{r a} ; \quad N_{D v}=N_{r a} N_{0 A}, \tag{5}
\end{align*}
$$

where parameters $\quad n_{r a}=2.785248449 \cdot 10^{7}, \quad N_{D v}=3.704626502 \cdot 10^{8}$, $N_{c v}=9.909123093 \cdot 10^{12}$ are additional. For bulk fractal structures of the Universe, the main and additional parameters from (5) can be operators. In the general case, these operators do not commute; when describing light and supermassive black holes, the appearance of stochastic properties is possible. We introduce the distribution density functions in the ground $f_{r a}$ and excited $f_{r a}^{\prime}$ states for relic photons

$$
\begin{equation*}
f_{r a}^{\prime}-f_{r a}=1 ; f_{r a}^{\prime}=\left\langle\hat{c}_{r a} \hat{c}_{r a}^{+}>=N_{r a} /\left(N_{r a}-z_{\mu}^{\prime}\right) ; f_{r a}=\left\langle\hat{c}_{r a}^{+} \hat{c}_{r a}>=z_{\mu}^{\prime} /\left(N_{r a}-z_{\mu}^{\prime}\right),\right.\right. \tag{6}
\end{equation*}
$$

where $\hat{c}_{r a}^{+}, \hat{c}_{r a}$ are creation and annihilation operators of relic photons; $\left.<\ldots\right\rangle$ is averaging symbol. Based on (6), (1), we find the numerical values $f_{r a}=0.006947216, f_{r a}^{\prime}=1.006947216$. Expressions (1) - (6) make it possible to estimate the masses $M_{0 B}, M_{b 0}, M_{b 0}^{\prime}$ black holes by the formulas

$$
\begin{equation*}
M_{0 B}=f_{r a}^{\prime} M_{b 0} ; M_{b 0} / M_{s}=n_{g}\left(1+n_{z g}^{\prime}\right) n_{r a} / n_{A 0} ; M_{b 0}^{\prime}=M_{0 B}-M_{b 0}=f_{r a} M_{b 0} . \tag{7}
\end{equation*}
$$

The numerical values are equal: $M_{0 B} / M_{s}=4.307173111 \cdot 10^{6}$, $M_{b 0} / M_{s}=4.277456693 \cdot 10^{6}, M_{b 0}^{\prime} / M_{s}=0.029716418 \cdot 10^{6}$. Our estimate of the mass $M_{0 B} / M_{S}$ practically coincides with the mass of the central body $4.31 \cdot 10^{6}$ of a supermassive black hole in the center of the Milky Way galaxy [3, 4]. The value $2 M_{b 0}^{\prime} / M_{s}=0.059432836 \cdot 10^{6}$ determines the error $0.06 \cdot 10^{6}$, associated with the error in measuring the parameters of the orbit of the S2 star, rotating around the central body $[3,4]$.
Model $\mathbf{B}_{2}$. The fractal structure of the Universe is characterized by the distribution of masses of black holes, which are found in the center of various galaxies. So for a supermassive black hole in the core of the galaxy M87, using the Event Horizon Telescope [21, 22], a shadow image in the radio range was obtained. Using four Chandra X-ray observations [14] for the MAXI J1820+070 binary black hole relativistic jets were detected. To estimate the upper mass limit $M_{J 0}=N_{0 A} M_{b 0}$, we will represent the central body of a supermassive
black hole as a Bose condensate of black holes with masses $M_{b 0}$. In this case, for the parameter $N_{0 A}=\psi_{1 A} N_{G E}^{*}$, representation is acceptable, where $\psi_{1 A}^{2}=1+\Omega_{m}^{*}, N_{G E}^{*}=M_{s} / M_{E}=R_{G s} / R_{G E}$. Here $M_{E}$ is mass of the Earth; $R_{G s}$ and $R_{G E}$ are Schwarzschild gravitational radii of the Sun and Earth; $N_{G E}^{*}=3.32958 \cdot 10^{5}$. In this model, the density of matter near supermassive black holes $\Omega_{m}^{*}=0.141730642$ is close to our calculated value $\Omega_{m}=0.141145722$ from [23,24] and the value of 0.141 obtained by the Planck observatory, based on the new Hubble constant $H_{0}^{*}$ for the attenuation of $\gamma$-rays against the intergalactic background. As a result, we find $M_{J 0} / M_{s}=15.21797631 \cdot 10^{11}$. For experimentally search of supermassive black holes near the upper mass boundary, brightness distributions, changes in stellar radiation intensity when photographing galaxies with high resolution, adaptive optical spectroscopy to compensate for fluctuations in the atmosphere, and speckle spectroscopy are used [4]. For the maximum radiation intensity $I_{m}$ from (1) near the upper mass boundary, the representation is acceptable

$$
\begin{gather*}
I_{m}=I_{1}^{*}+I_{2}^{*} ; \quad I_{1}^{*}=n_{z g}^{\prime} I_{m}=\mathrm{v}_{1 J}^{2} I_{m} \sin ^{2}\left(\theta_{W}^{*}\right) ; \quad I_{2}^{*}=n_{z g} I_{m}=\left(\mathrm{u}_{1 J}^{2}+\mathrm{v}_{1 J}^{2} \cos ^{2}\left(\theta_{W}^{*}\right)\right) I_{m} ; \\
\mathrm{v}_{1 J}^{2}=k_{1 J}^{2}=0.5\left(1-I(0) / I_{m}\right) ; \quad \mathrm{u}_{1 J}^{2}=\left(k_{1 J}^{\prime}\right)^{2}=0.5\left(1+I(0) / I_{m}\right) ; \quad \mathrm{u}_{1 J}^{2}+\mathrm{v}_{1 J}^{2}=1 ; \\
I_{1}^{*} / I_{m}=k_{1 J}^{2} \mathrm{sn}^{2}\left(u_{1 W} ; k_{1 J}\right)=n_{z g}^{\prime} ; \quad I_{2}^{*} / I_{m}=\mathrm{dn}^{2}\left(u_{1 W} ; k_{1 J}\right)=n_{z g} . \tag{8}
\end{gather*}
$$

Here $k_{1 J}, k_{1 J}^{\prime}$ and $u_{1 W}$ are moduli and effective displacement for elliptic functions $\operatorname{sn}\left(u_{1 W} ; k_{1 J}\right), \operatorname{cn}\left(u_{1 W} ; k_{1 J}\right), \operatorname{dn}\left(u_{1 W} ; k_{1 J}\right) ;$ the angle $\theta_{W}^{*}$ acts as the effective Cabibo angle for supermassive black holes; parameters $\mathrm{u}_{1 J}, \mathrm{v}_{1 J}$ depend on the initial and maximum radiation intensity and are analogous to the N.N. Bogolyubov's transformation parameters in the theory of superconductivity. Numerical values are equal: $k_{1 J}^{2}=0.493832171$, $\left(k_{1 J}^{\prime}\right)^{2}=0.506167829, \quad \sin ^{2}\left(\theta_{W}^{*}\right)=0.231489651, \quad \cos ^{2}\left(\theta_{W}^{*}\right)=0.768510349$, intensity distribution density functions $f_{J 1}=I_{1}^{*} / I_{2}^{*}=0.129072187$, $f_{J 1}^{\prime}=I_{m} / I_{2}^{*}=1.129072187$. Expressions (8) allow us to estimate the masses of black holes $M_{J 1}^{\prime}, M_{J 1}$ near the upper mass boundary by the formulas

$$
\begin{equation*}
M_{J 1}^{\prime}-M_{J 1}=M_{J 0} ; \quad M_{J 1}^{\prime}=f_{J 1}^{\prime} M_{J 0} ; \quad M_{J 1}=f_{J 1} M_{J 0} ; \quad f_{J 1}^{\prime}-f_{J 1}=1 \tag{9}
\end{equation*}
$$

Based on (9), we obtain a numerical value $M_{J 1} / M_{s}=1.964217483 \cdot 10^{11}$, that is close to the experimental value $1.96 \cdot 10^{11} M_{S}$ for the supermassive black hole SDSS J140821.67+025733.2. For intermediate masses of black holes, the maximum radiation intensity $I_{m}^{*}$ can change over a segment $I(0) \leq I_{m}^{*} \leq I_{m}$.

These changes are described by a variable number of quanta $n_{m}^{*}$ and an inversion parameter $B_{J m}$

$$
\begin{align*}
& n_{m}^{*}=I_{m}^{*} / I_{m}=\mathrm{u}_{1 J}^{2}+\mathrm{v}_{1 J}^{2} B_{J m} ; B_{1}^{*}=B_{J m}\left(n_{z g}^{\prime}\right)=-\mathrm{u}_{1 J}^{2} / \mathrm{v}_{1 J}^{2}+\sin ^{2}\left(\theta_{W}^{*}\right) ;-1 \leq B_{J m} \leq 1 ; \\
& B_{2}^{*}=B_{J m}\left(n_{z g}\right)=\cos ^{2}\left(\theta_{W}^{*}\right) ; n_{1 J}=\mathrm{v}_{1 J}^{2}+n_{z g}^{\prime} ; \quad n_{1 J}^{\prime}=\mathrm{u}_{1 J}^{2}-n_{z g}^{\prime} ; n_{1 J}+n_{z g}^{\prime}=1 . \tag{10}
\end{align*}
$$

From (10) it follows, that a black hole with an intermediate mass can be in different states, which are determined by a pair of parameters $n_{m}^{*}$ and $B_{J m}$. Let's introduce these states: ground $J 1\left(n_{m}^{*}=1, B_{J m}=1\right)$, supernonradiative $J 2$ $\left(n_{m}^{*}=\mathrm{u}_{1 J}^{2}, B_{J m}=0\right)$, fully inverse state $J 3\left(n_{m}^{*}=\mathrm{u}_{1 J}^{2}-\mathrm{v}_{1 J}^{2}, B_{J m}=-1\right)$, partially inverse state $J 4\left(n_{m}^{*}=n_{z g}^{\prime}, B_{J m}=B_{1}^{*}\right)$, deviated from the ground $J 5\left(n_{m}^{*}=n_{z g}\right.$, $\left.B_{J m}=B_{2}^{*}\right)$. The parameters $B_{1}^{*}=-0.793489803,1+B_{1}^{*}=0.206510197, B_{2}^{*}$, $n_{1 J}=0.391850792, n_{1 J}^{\prime}=0.608149208$ carry information about the characteristic parameters (velocities, energies) of a relativistic jet (ejection of matter from a supermassive black hole) [14].

## 3 Relativistic jets

To describe the parameters of a relativistic jet, we will use the basic model equations

$$
\begin{equation*}
Q_{H 2} R_{A B} / 2 R_{A H}=n_{F}^{\prime}+0.5 ; \quad \Omega_{0 v}=\left(n_{F}^{\prime}\right)^{2} ; \quad \Omega_{\tau L} E_{W 0}=\Omega_{\tau L}^{*} E_{Z 0} \tag{11}
\end{equation*}
$$

Here the parameter $Q_{H 2}=1 / 3$ is determined by the expression from (3) and is related to the angles $\varphi_{\mu \lambda}^{\prime}, \varphi_{a}=22.43261135^{\circ}$, cosmological redshift $z_{\mu}^{\prime}$ from our anisotropic model [8] of the expanding Universe; the number of quanta $n_{F}^{\prime}=0.054219932$ determines the Fermi level and the neutrino density $\Omega_{0 \nu}=0.002939801$; the lepton quantum number $\Omega_{\tau L}=0.002402187$ is related to the quantum number $\Omega_{\tau L}^{*}=0.002116741$ through the rest energies $E_{W 0}=80.35235464 \mathrm{GeV}$ and $E_{Z 0}=91.188 \mathrm{GeV}$ for $W 0$ and $Z 0$ bosons, respectively; Hubble radius $R_{A H}=13.75 \cdot 10^{9} \cdot L_{c 0}$. From (11) we find the characteristic radius (horizon of matter particles) $R_{A B}=45.72314437 \cdot 10^{9} \cdot L_{c 0}$. Note, that parameters are: $L_{c 0}=$ light year $=c_{0} \tau_{c 0}=N_{c 0} L_{E S}=0.306597989 \mathrm{pc}$; limiting speed of light in vacuum $c_{0}=2.99792458 \cdot 10^{5} \mathrm{~km} \mathrm{~s}^{-1}$, $N_{c 0}=6.324043414 \cdot 10^{4}, \quad \tau_{c 0}=365.2503353$ day $=3.155762897 \cdot 10^{7} \mathrm{~s}$, $L_{E S}=1 \mathrm{au}=1.495995288 \cdot 10^{8} \mathrm{~km}$. Based on (11), we introduce the refractive index $n_{A B}$ of the medium of matter particles

$$
\begin{equation*}
n_{A B}=Q_{A B}^{2} ; \quad Q_{A B}=R_{A B} / R_{A H}=2\left(n_{F}^{\prime}+0.5\right) / Q_{H 2} \tag{12}
\end{equation*}
$$

Numerical values are $n_{A B}=11.05775038 ; Q_{A B}=3.325319591$. Next, we find the particle velocities $v_{A H}, v_{0 \xi}, v_{A W}$ and velocities ratios $\xi_{A H}, \xi_{0 J}, \xi_{A W}$

$$
\begin{equation*}
v_{A H}^{2}=c_{0}^{2} / n_{A B} ; \quad v_{A H} S_{1 u}^{2}=v_{0 \xi} S_{2 u}^{2}=v_{A W} ; \quad \xi_{A H}=v_{A H} / c_{0} ; \quad \xi_{0 J}=v_{0 \xi} / c_{0} \tag{13}
\end{equation*}
$$

Values are: $\quad v_{A H}=9.015447983 \cdot 10^{4} \mathrm{~km} \mathrm{~s}^{-1}, \quad v_{0 \xi}=1.803089597 \cdot 10^{5} \mathrm{~km} \mathrm{~s}^{-1}$, $v_{A W}=196.9672387 \mathrm{~km} \mathrm{~s}^{-1} ; \quad \xi_{A H}=0.300722975, \quad \xi_{0 J}=0.60144595$, $\xi_{A W}=v_{A W} / c_{0}=657.0119876 \cdot 10^{-6}$.
From our model $\mathbf{B}_{2}$ it follows, that the density of matter near supermassive black holes $\Omega_{m}^{*}>\Omega_{m}$. This leads to a change in the refractive index of the medium $n_{A B}$, the radius $R_{A B}$ from (12), the neutrino density $\Omega_{0 v}$ from (11), and the particle velocities from (13). Accounting for these changes near supermassive black holes is described by new parameters

$$
\begin{gather*}
\bar{n}_{A B}=\bar{Q}_{A B}^{2} ; \quad \bar{Q}_{A B}=\bar{R}_{A B} / R_{A H}=2\left(\bar{n}_{0 v}+0.5\right) / Q_{H 2} ; \quad \bar{\Omega}_{0 v}=\left(\bar{n}_{0 v}\right)^{2} ; \\
2 \bar{n}_{0 v}=\Omega_{m}^{*}-S_{2 u} ; \quad \bar{v}_{A H}=c_{0} / \bar{Q}_{A B} ; \quad \bar{v}_{A H} S_{1 u}^{2}=\bar{v}_{0 \xi} S_{2 u}^{2}=\bar{v}_{A W} . \tag{14}
\end{gather*}
$$

Numerical values are equal: medium refractive index $\bar{n}_{A B}=11.06252927$, radius $\bar{R}_{A B}=45.73302352 \cdot 10^{9} \cdot L_{c 0}, \quad$ parameters $\quad \bar{Q}_{A B}=3.326038074$, $\bar{n}_{0 v}=0.054339679$, density of the relativistic neutrino $\bar{\Omega}_{0 v}=0.002952801$; velocities $\quad \bar{v}_{A H}=9.013500487 \cdot 10^{4} \mathrm{kms}^{-1}, \quad \bar{v}_{0}=1.802700097 \cdot 10^{5} \mathrm{kms}^{-1}$, $\bar{v}_{A W}=196.9246903 \mathrm{kms}^{-1} ;$ velocities ratios $\quad \bar{\xi}_{A H}=1 / \bar{Q}_{A B}=0.300658013$, $\bar{\xi}_{0 J}=2 \bar{\xi}_{A H}=0.601316027, \bar{\xi}_{A W}=\bar{v}_{A W} / c_{0}=656.8700603 \cdot 10^{-6}$.
Further, taking into account (1), (2), (8), we find the energies of the jet particles $E_{0 J}$ and $E_{1 J}, E_{2 J}$ in the absence and presence of the Higgs field, respectively, by the formulas

$$
\begin{equation*}
E_{0 J} / E_{H 0}=I_{m} / I(0)=I_{1}^{*} / I(0)+I_{2}^{*} / I(0) ; \quad E_{1 J}=\psi_{01} E_{0 J} ; \quad E_{2 J}=\psi_{02} E_{0 J} \tag{15}
\end{equation*}
$$

Numerical values of energies are equal $E_{0 J}=10.13585044 \mathrm{TeV}$, $E_{1 J}=10.29061357 \mathrm{TeV}, E_{2 J}=9.978687329 \mathrm{TeV}$, where $\psi_{02}=0.984494334$.

The effective Cabibo angle $\theta_{W}^{*}$ allows us to estimate the angular width $\varphi_{W}^{*}$ of the jet based on the angular parameters $\varphi_{n 0}, \varphi_{E n}, \varphi_{E 0}$ by the formulas

$$
\begin{equation*}
\varphi_{W}^{*}=\varphi_{n 0} / \sin ^{2}\left(\theta_{W}^{*}\right) ; \varphi_{n 0}=2 n_{G} \varphi_{E n} / n_{g} ; \varphi_{E n}=m_{n} \varphi_{E 0} / m_{p}\left(1+S_{1 u}\right) \tag{16}
\end{equation*}
$$

where the parameters $n_{G}, n_{g}$ are determined in the model $\mathbf{B}_{\mathbf{0}}$ by expressions (3);
$m_{n} / m_{p}=1.008985047$ is the ratio of the neutron mass $\left(m_{n}\right)$ to the proton mass $\left(m_{p}\right)$. Based on (16), we find estimates of the angular parameters $\varphi_{W}^{*}=2.592779092^{\prime \prime}, \varphi_{n 0}=0.600201527^{\prime \prime}, \varphi_{E n}=0.800268702^{\prime \prime}$, where the parameter $\varphi_{E 0}=0.830215001^{\prime \prime}$ describes the behavior of photons near supermassive bodies in Einstein's theory [5]. The obtained estimates of the parameters $\xi_{0 J}, E_{0 J}, \varphi_{W}^{*}$ do not contradict the experimental data [14] for a velocity ratio of 0.6 , an energy of 10 TeV , and an angular width $2.5928^{\prime \prime}$ of jet particles. Based on the effective radii $R_{A B}$ from (12), $\bar{R}_{A B}$ from (14), we obtain estimates of the distance $R_{0}$ from the Sun to the supermassive black hole in the center of our Milky Way galaxy and the errors $\delta R_{0}$ by the formulas

$$
\begin{gather*}
R_{0}=\bar{\delta}_{A B} / n_{R 0} ; \delta R_{0}=\bar{\delta}_{A B} / N_{R 0} ; \quad \bar{\delta}_{A B}=\left(1+\delta_{Q}\right) \delta_{A B} ; \delta_{A B}=\bar{R}_{A B}-R_{A B} ; \\
N_{R 0}=n_{g}\left(N_{r a}+0.5 I_{m} / I(0)\right) ; \quad n_{R 0}=Q_{H 2}\left(N_{r a}+n_{A 0}-n_{g}-\bar{\xi}_{0 J}\right) . \tag{17}
\end{gather*}
$$

The numerical values of the parameters are equal: $N_{R 0}=8654.611017$, $n_{R 0}=363.5795993, \quad \delta_{A B}=9.87915 \cdot 10^{6} \cdot L_{c 0}, \quad \bar{\delta}_{A B}=9.879150543 \cdot 10^{6} \cdot L_{c 0}$. Based on (17), we find estimates of the distance $R_{0}=8.330851608 \mathrm{kpc}$ and error $\delta R_{0}=0.349978489 \mathrm{kpc}$.
Based on the distribution density function $f_{J 1}^{\prime}$ from (9), the number of quanta $\bar{n}_{0 v}$ from (14), we find the radius $r_{J B}$ of the central body by the formulas

$$
\begin{gather*}
N_{G 0} r_{J B}=\delta_{J B}^{\prime}+l_{A B} ; \quad \delta_{J B}^{\prime}=\bar{\delta}_{A B} f_{J 1}^{\prime} ; \quad l_{A B}=\bar{\delta}_{A B} \sin \left(\theta_{0 v}\right) ; \quad N_{G 0}=N_{a} / N_{H G} ; \\
N_{G 0} E_{H 0}=N_{a} E_{G} ; \sin \left(\theta_{0 v}\right)=\bar{n}_{0 v}\left(1-\bar{n}_{0 v}\right)=\bar{n}_{0 v}-\bar{\Omega}_{0 v} . \tag{18}
\end{gather*}
$$

Values of the parameters are equal: $N_{G 0}=5.839561703 \cdot 10^{7}, \theta_{0 v}=2.945548561^{\circ}$, $\sin \left(\theta_{0 v}\right)=0.051386878, l_{A B}=5.076587037 \cdot 10^{5} \cdot L_{c 0}, \delta_{J B}^{\prime}=11.15427411 \cdot 10^{6} \cdot L_{c 0}$. From (18) we obtain $r_{J B}=0.199705618 \cdot L_{c 0}=1.262947001 \cdot 10^{4} \mathrm{au}$.

Next, we find estimates for the semi-axes $x_{0 S}, y_{0 S}$ the elliptical orbit of the star S 2 , rotating around the central body by the formulas

$$
\begin{align*}
& y_{0 S}=r_{J B} / \bar{n}_{A B}\left(1+\Omega_{m}^{*}\right) ; \quad x_{0 S}^{2} / y_{0 S}^{2}=S_{1 u}^{2} \sin \left(\varphi_{0 g}\right) / S_{2 u}^{2} ; \\
& \sin ^{2}\left(\varphi_{0 g}\right)=\left(n_{A 0}-n_{g}\right)\left(E_{e}+E_{e h}\right) / E_{0 g} ; \quad E_{0 g}=n_{g} E_{H 0} . \tag{19}
\end{align*}
$$

Here the rest energies of the gluon $E_{0 g}=1.00025904 \mathrm{TeV}$, electron $E_{e}$ and electron hole $E_{e h}$ are assumed to be equal $E_{e}=E_{e h}=0.51099907 \mathrm{MeV}$; $\sin \left(\varphi_{0 g}\right)=0.007150827$, the angle of polarization of the radiation
$\varphi_{0 g}=0.409715696^{\circ}$; semi-axes $y_{0 S}=999.9241011 \mathrm{au}, x_{0 S}=119.5804463 \mathrm{au}$.
Our estimates of the parameters $R_{0}, \delta R_{0}, r_{J B}, x_{0 S}, y_{0 S}$ agree with the experimental data [3, 4] for the distance 8.33 kpc from the Sun to the supermassive black hole in the center of the Milky Way galaxy, the error 0.35 kpc , the radius of the central body $0.2 \cdot L_{c 0}$, for the semi-axes $120 \mathrm{au}, 1000 \mathrm{au}$ the elliptical orbit of the S2 star, rotating around the central body, respectively.

## 4 Asymmetry of matter, antimatter and the Higgs field

The presence of a Higgs field of various nature (gluon, lepton, neutrino, hadronic based on the parameter $\Omega_{\tau L}^{*}$ from (11), gravitational, etc.) leads to changes in the rest energy of the Higgs boson $E_{H 0}$ in (18); energies of holes (antiparticles) $E_{e h}$ in (19), $E_{\mu h}, E_{\tau h}$ for $e, \mu, \tau$-leptons, respectively; the appearance of asymmetry of matter and antimatter. We introduce the energy $E_{0 L}$ based on the total energy $\varepsilon_{0 L}$ of paired leptons, the number of quanta of gluons $n_{g}$

$$
\begin{equation*}
E_{0 L}=n_{g} \varepsilon_{0 L} ; \quad \varepsilon_{0 L}=\left(E_{e}+E_{e h}\right)+\left(E_{\mu}+E_{\mu h}\right)+\left(E_{\tau}+E_{\tau h}\right) \tag{20}
\end{equation*}
$$

Here $E_{\mu}=E_{\mu h}=105.658389 \mathrm{MeV}, E_{\tau}=E_{\tau h}=1777.00 \mathrm{MeV}$ are rest energies for $\mu, \quad \tau$-leptons, respectively. From (20) we find the energies $\varepsilon_{0 L}=3.766338776 \mathrm{GeV}, E_{0 L}=30.13071021 \mathrm{GeV}$ (close to the data from [16]).
Next, we introduce the distribution density functions of the Bose type $f_{g A}$ (ground state), $f_{g A}^{\prime}$ (excited state) based on the number of quanta of black holes $\left(n_{A 0}\right)$, gluons ( $n_{g}$ ). Based on $E_{H 0}$ we find the energy $E_{g A}, E_{g A}^{\prime}$

$$
\begin{align*}
& f_{g A}^{\prime}-f_{g A}=1 ; \quad f_{g A}=n_{g} /\left(n_{A 0}-n_{g}\right) ; \quad f_{g A}^{\prime}=n_{A 0} /\left(n_{A 0}-n_{g}\right) ; \\
& E_{g A}=E_{H 0} f_{g A} / 2 ; \quad E_{g A}^{\prime}=E_{H 0} f_{g A}^{\prime} / 2 ; \quad E_{g A}^{\prime}-E_{g A}=E_{H 0} / 2 . \tag{21}
\end{align*}
$$

The numerical values are equal: $f_{g A}=0.159850895, E_{g A}=9.993268924 \mathrm{GeV}$, $E_{g A}^{\prime}=72.50945893 \mathrm{GeV}$. Taking into account the energy $E_{g A}$ from (21), the expressions for the rest energies of leptons have the form

$$
\begin{equation*}
E_{e}=E_{g A} \sin ^{2}\left(\varphi_{e g}\right) ; \quad E_{\mu}=E_{g A} \sin ^{2}\left(\varphi_{\mu g}\right) ; \quad E_{\tau}=E_{g A} \sin ^{2}\left(\varphi_{\tau g}\right) \tag{22}
\end{equation*}
$$

Here angles are equal: $\varphi_{e g}=\varphi_{0 g}, \varphi_{\mu g}=5.901862921^{\circ}, \varphi_{\tau g}=24.94112323^{\circ}$. To describe the interaction of $\mu$ and $e$-leptons, we find the energies $E_{\mu}^{\prime}, E_{\mu}^{*}$ from the expressions

$$
E_{\mu}^{\prime}=E_{g A} \sin ^{2}\left(\varphi_{\mu g}+\varphi_{e g}\right)=\left(E_{\mu}^{2}+4 \Delta_{\mu}^{2}\right)^{1 / 2} ; \quad 2 \Delta_{\mu}=n_{A 0} E_{e x} ; \quad E_{e x}=E_{e}+E_{h}^{\prime} ;
$$

$$
\begin{align*}
& E_{\mu}^{*}=E_{g A} \sin ^{2}\left(\varphi_{\mu g}-\varphi_{e g}\right)=\left(E_{\mu}^{2}-4\left(\Delta_{\mu}^{*}\right)^{2}\right)^{1 / 2} ; 2 \Delta_{\mu}^{*}=n_{A 0} E_{e x}^{*} ; E_{e x}^{*}=E_{e}+E_{h}^{*} \\
& E_{e} / E_{e x}=0.5+\sin \left(\varphi_{e x}\right) ; E_{h}^{\prime} / E_{e x}=0.5-\sin \left(\varphi_{e x}\right) ; E_{e} / E_{e x}^{*}=0.5+\sin \left(\varphi_{e x}^{*}\right) \tag{23}
\end{align*}
$$

For variant I (sum of angles), the parameter values are: $\varphi_{\mu g}+\varphi_{e g}=6.311578617^{\circ}$, $E_{\mu}^{\prime}=120.7760733 \mathrm{MeV}, \quad E_{\mu}^{\prime}-E_{\mu}=15.11768432 \mathrm{MeV}, \quad$ energy gap $\Delta_{\mu}=29.25390878 \mathrm{MeV}$, energy $E_{e x}=1.007944968 \mathrm{MeV}$, hole energy $E_{h}^{\prime}=0.496945898 \mathrm{MeV}, \quad \sin \left(\varphi_{e x}\right)=0.0069712$, characteristic angle $\varphi_{e x}=0.399423573^{\circ}$. For variant II (angle difference), the parameter values are: $\varphi_{\mu g}-\varphi_{e g}=5.492147225^{\circ}, \quad E_{\mu}^{*}=91.54109182 \mathrm{MeV}$, energy gap $\Delta_{\mu}^{*}=26.38145028 \mathrm{MeV}$, energy $E_{e x}^{*}=0.908974259 \mathrm{MeV}$, hole energy $E_{h}^{*}=0.397975189 \mathrm{MeV}, \quad \sin \left(\varphi_{e x}^{*}\right)=0.062171112, \quad$ characteristic angle $\varphi_{e x}^{*}=3.564441086^{\circ}, E_{h}^{*} / E_{e x}^{*}=0.5-\sin \left(\varphi_{e x}^{*}\right)$. Note, that the values of the angle differences $\quad\left(\varphi_{e g}-\varphi_{e x}\right) / 2=18.52582072^{\prime \prime}, \quad\left(\varphi_{e g}-\varphi_{e x}\right) / 4=9.26291036^{\prime \prime} \quad$ are characteristic of the angular widths of coronal holes on the Sun [15]. From (23) we find expressions easy for analyzing the asymmetry of individual contributions from $E_{e}, E_{\mu}$, different angles, in energy $E_{\mu}^{\prime}, E_{\mu}^{*}$ in the form

$$
\begin{equation*}
\left(E_{\mu}^{\prime}+E_{\mu}^{*}\right) / 2=E_{e} \cos ^{2}\left(\varphi_{\mu g}\right)+E_{\mu} \cos ^{2}\left(\varphi_{e g}\right) ; E_{\mu}^{\prime}-E_{\mu}^{*}=E_{g A} \sin \left(2 \varphi_{\mu g}\right) \sin \left(2 \varphi_{e g}\right) . \tag{24}
\end{equation*}
$$

Based on the energy $E_{0 L}$ from (20) we found characteristic energies $\varepsilon_{d L}, \varepsilon_{d 0}$, $\varepsilon_{d z}^{\prime}$ and the Higgs boson energies $E_{H d}, E_{H d}^{\prime}, E_{H g}, E_{H g}^{\prime}, E_{H L}, E_{H L}^{\prime}$

$$
\begin{gather*}
E_{0 L}=n_{g} \varepsilon_{0 L}=n_{G} \varepsilon_{d L} ; \quad \varepsilon_{d 0}=n_{A 0} \varepsilon_{d L} ; \quad \varepsilon_{d z}^{\prime}=z_{\mu}^{\prime}\left(z_{\mu}^{\prime}+1\right) \varepsilon_{d L} ; \quad \varepsilon_{d z}^{\prime}=\varepsilon_{d 0}+2 \varepsilon_{0 L} ; \\
E_{H d}^{2}=E_{H 0}^{2}+\varepsilon_{d L}^{2} ;\left(E_{H d}^{\prime}\right)^{2}=E_{H 0}^{2}-\varepsilon_{d L}^{2} ; \quad E_{H g}^{2}=E_{H 0}^{2}+E_{g A}^{2} ;\left(E_{H g}^{\prime}\right)^{2}=E_{H 0}^{2}-E_{g A}^{2} ; \\
E_{H L}^{2}=E_{H 0}^{2}+\varepsilon_{0 L}^{2} ; \quad\left(E_{H L}^{\prime}\right)^{2}=E_{H 0}^{2}-\varepsilon_{0 L}^{2} \tag{25}
\end{gather*}
$$

Characteristic energies are $\varepsilon_{d L}=10.04357007 \mathrm{GeV}$ (close to the energy for dark matter from [20]), $\varepsilon_{d 0}=582.9954848 \mathrm{GeV}, \varepsilon_{d z}^{\prime}=590.5281624 \mathrm{GeV}$. Energies $\varepsilon_{d L}, E_{g A}, \varepsilon_{0 L}$ describe the different nature of the Higgs field. The presence of the Higgs field leads to the appearance of active particles with energies $E_{H d}=125.4351201 \mathrm{GeV}, \quad E_{H d}^{\prime}=124.6283385 \mathrm{GeV}, \quad E_{H g}=125.4311025 \mathrm{GeV}$, $E_{H g}^{\prime}=124.6323819 \mathrm{GeV}, E_{H L}=125.0890937 \mathrm{GeV}$ (corresponds to the peak for the Higgs boson decay process from [16]), $E_{H L}^{\prime}=124.9756406 \mathrm{GeV}$. Energy differences $\delta E_{H g}=E_{H d}-E_{H g}=4.0176 \mathrm{MeV}, \delta E_{H g}^{\prime}=E_{H g}^{\prime}-E_{H d}^{\prime}=4.04343 \mathrm{MeV}$ describe the line width in the energy spectrum for the Higgs boson [16].

## 5 Active microobjects

Based on the energies $\varepsilon_{d L}, E_{g A}, \varepsilon_{0 L}$ from (25) we find the radii $R_{d L}, R_{g A}, R_{0 L}$ of active microobjects associated with the different nature of the Higgs field

$$
\begin{align*}
& R_{d L}=A_{G} \varepsilon_{d L} ; \quad R_{g A}=A_{G} E_{g A} ; \quad R_{G L}=n_{G} R_{d L}=n_{g} R_{0 L} \\
& R_{d z}^{\prime}=z_{\mu}^{\prime}\left(z_{\mu}^{\prime}+1\right) R_{d L} ; \quad R_{d 0}=n_{A 0} R_{d L} ; \quad R_{d z}^{\prime}-R_{d 0}=2 R_{0 L} . \tag{26}
\end{align*}
$$

Here $A_{G}=0.960836162 \mathrm{fm}(\mathrm{eV})^{-1}$ is the constant from [23, 24]. The gravitational radii are: $R_{d L}=9.6502253 \mu \mathrm{~m}, R_{g A}=9.6018942 \mu \mathrm{~m}, R_{0 L}=3.6188345 \mu \mathrm{~m}$. For characteristic radii we obtain: $R_{d 0}=560.1631441 \mu \mathrm{~m}$ (coupled to the number of quanta of the black hole $n_{A 0}$ ); $R_{d z}^{\prime}=567.4008131 \mu \mathrm{~m}$ (coupled to cosmological redshift $z_{\mu}^{\prime}$ ); $R_{G L}=28.95067596 \mu \mathrm{~m}$ (coupled with the number of quanta of the gravitational field in an excited state $n_{G}$, or with the number of quanta of the gluon field $n_{g}$ ). Next, we find the characteristic lengths $l_{d 0}, l_{d z}^{\prime}, l_{0 L}$ of active objects

$$
\begin{gather*}
l_{d 0} / R_{d 0}=l_{d z}^{\prime} / R_{d z}^{\prime}=l_{0 L} / 2 R_{0 L}=E_{\alpha u} / E_{H 0}=S_{12 u} ; S_{12 u}=S_{1 u}-S_{2 u} ; E_{\alpha u}-E_{\alpha S}=E_{c} ; \\
E_{\alpha S}=S_{012} E_{H 0}=\xi_{g S} E_{0 g} ; \quad S_{012}=S_{01}^{\prime}-S_{02}^{\prime} ; \quad \xi_{g S}=S_{012} / n_{g} . \tag{27}
\end{gather*}
$$

Here the parameters are: $S_{12 u}=0.013690291, \quad S_{012}=0.005451282$, $\xi_{g S}=0.00068141$; the rest energy $E_{c}=1.030142904 \mathrm{GeV}$, c-quark gravitational radius $R_{c}=A_{G} E_{c}=0.989798554 \mu \mathrm{~m}$; energy $E_{\alpha S}=0.681586763 \mathrm{GeV}$ is determined either through the rest energy of the Higgs boson, or through the energy of the gluon, energy $E_{\alpha u}=1.711729667 \mathrm{GeV}$; gravitational radii $R_{\alpha S}=A_{G} E_{\alpha S}=654.8932091 \mathrm{~nm}, R_{\alpha u}=A_{G} E_{\alpha u}=1.644691763 \mu \mathrm{~m}$. From (27) we obtain values of characteristic lengths $l_{d 0}=7.6687965 \mu \mathrm{~m}, l_{d z}^{\prime}=7.7678822 \mu \mathrm{~m}$, $l_{0 L}=99.085795 \mathrm{~nm}$. From (27) it follows that it is possible to describe particles and antiparticles, compound particles (hadrons), which are experimentally observed at the LHC [16], on the basis of energies $E_{\alpha u}, E_{\alpha S}, E_{c}$. As an example, consider the possibility of describing the energies $E_{T Q}, E_{T Q}^{\prime}$ of a tetraquark, a hadron by

$$
\begin{gather*}
E_{T Q}=2 E_{c}+2 \bar{E}_{c} ; \quad \bar{E}_{c}=E_{c}+E_{\alpha S}+\Delta_{\mu}^{*}=E_{c}+\xi_{g S} E_{0 g}+\Delta_{\mu}^{*}=E_{\alpha u}+\Delta_{\mu}^{*} \\
E_{T Q}-E_{T Q}^{\prime}=2\left(E_{\mu}+E_{\mu}^{\prime}\right) ; \quad E_{T 1}=E_{T Q}-2 E_{\mu}^{\prime}-\Delta_{\mu} ; \quad E_{T 2}=E_{T Q}-2 E_{\mu}^{*}+\Delta_{\mu}^{*} \tag{28}
\end{gather*}
$$

Here $\bar{E}_{c}=1.738111117 \mathrm{GeV}, E_{\mu}+E_{\mu}^{\prime}=226.4344623 \mathrm{MeV}$ are the c-antiquark, muon pair energies, respectively. Energies $E_{T 1}=6628.875515 \mathrm{MeV}$, $E_{T 2}=6742.980837 \mathrm{MeV}$ determine the features of the type of local maximum, minimum on the experimental dependence of the number of events on the state of
the tetraquark [18]. The base narrow peak corresponds to the tetraquark energy $E_{T Q}=6899.681571 \mathrm{MeV}$, and the broadened peak corresponds to the hadron energy $\quad E_{T Q}^{\prime}=6446.812646 \mathrm{MeV}$. Note, that the energy difference $E_{c}-E_{\mu}^{*}=938.6018122 \mathrm{MeV}$ (for the c-quark and antimuon) is close to the sum of the energies $E_{p}+E_{e}=938.7833217 \mathrm{MeV}$ (for the proton and the electron). This indicates us to the need and possibility of describing additional contributions from the neutrinos, hadron Higgs fields to the energies of active objects.
The classical decay of a neutron into a pair of proton-electron and antineutrino is described by the expressions

$$
\begin{gather*}
E_{n}=\left(E_{p}+E_{e}\right)+n_{r a} \varepsilon_{v n} ; \quad \varepsilon_{v n}=\left(\varepsilon_{H G}^{2}+\Delta_{v n}^{2}\right)^{1 / 2} ; \quad \Delta_{v n}^{2}=z_{v n}\left(z_{v n}+2\right) \varepsilon_{H G}^{2} ; \\
\varepsilon_{v n}=\varepsilon_{H G}+z_{v n} \varepsilon_{H G}=\psi_{v n} \varepsilon_{H G} ; \psi_{v n}=1+z_{v n} . \tag{29}
\end{gather*}
$$

Here the rest energies are for neutrino $\varepsilon_{H G}=280.0460475 \mathrm{meV}$ [23, 24], neutron $E_{n}=946.7027435 \mathrm{MeV}$, proton $E_{p}=938.2723226 \mathrm{MeV}$. From (29) we find the antineutrino energy $\varepsilon_{v n}=284.3344848 \mathrm{meV}$, energy gap $\Delta_{v n}=49.1966514 \mathrm{meV}$, parameters of the neutrino field $z_{v n}=0.015313329, \psi_{v n}=1.015313329$.
We take into account the contribution from the hadronic Higgs field by replacing the energy of the pair $\left(E_{p}+E_{e}\right)$ in (29) with the energy of the difference $E_{c}-E_{\mu}^{*}$ for the c-quark and antimuon. In this case, the antineutrino energy $\varepsilon_{v n}$ is replaced by the renormalized antineutrino energy $\bar{\varepsilon}_{v n}$ and is determined from the expressions

$$
\begin{gather*}
E_{n}=\left(E_{c}-E_{\mu}^{*}\right)+n_{r a} \bar{\varepsilon}_{v n} ; \quad \bar{\varepsilon}_{v n}=\left(\varepsilon_{H G}^{2}+\bar{\Delta}_{v n}^{2}\right)^{1 / 2} ; \quad \bar{\Delta}_{v n}^{2}=\bar{z}_{v n}\left(\bar{z}_{v n}+2\right) \varepsilon_{H G}^{2} ; \\
\bar{\varepsilon}_{v n}=\varepsilon_{H G}+\bar{z}_{v n} \varepsilon_{H G}=\bar{\psi}_{v n} \varepsilon_{H G} ; \quad \bar{\psi}_{v n}=1+\bar{z}_{v n}=\psi_{v n}^{*}+0.5 n_{v n} ; n_{v n}^{2}=\Omega_{\tau L}^{*} ; \\
\bar{\varepsilon}_{v n}=\varepsilon_{h v}+\varepsilon_{v n}^{*} ; \quad \varepsilon_{h v}=0.5 n v_{v n} \varepsilon_{H G} ; \quad \varepsilon_{v n}^{*}=\left(\varepsilon_{H G}^{2}+\left(\Delta_{v n}^{*}\right)^{2}\right)^{1 / 2}=\psi_{v n}^{*} \varepsilon_{H G} . \tag{30}
\end{gather*}
$$

Here, the parameter $n_{v n}=0.046008054$ from (11) describes the contribution from the hadron Higgs field to the energy $\varepsilon_{h v}=6.442186838 \mathrm{meV}$. Based on (30), we find the renormalized antineutrino energy $\bar{\varepsilon}_{v n}=290.8512992 \mathrm{meV}$, energy gap $\bar{\Delta}_{v n}=78.54100538 \mathrm{meV}$, parameters $\bar{z}_{v n}=0.038583839, \bar{\psi}_{v n}=1.038583839$. The energy $\varepsilon_{v n}^{*}=284.4091123 \mathrm{meV}$, energy gap $\Delta_{v n}^{*}=49.62614656 \mathrm{meV}$, field parameters $z_{v n}^{*}=\bar{z}_{v n}-0.5 n_{v n}=0.015579812, \psi_{v n}^{*}=1+z_{v n}^{*}$ describe a different state of the antineutrino, compared to the state from (29).
Taking into account (29), we find the baryon densities of the Universe $\Omega_{b 1}$ (ground state of matter), $\Omega_{b 2}$ (hole state of matter) from the expressions

$$
\begin{equation*}
\Omega_{b 1}=\left(0.5-\mathrm{z}_{v n}\right) n_{v n} ; \quad \Omega_{b 2}=\left(0.5+\mathrm{z}_{v n}\right) n_{v n} ; \quad \Omega_{b 1}+\Omega_{b 2}=n_{v n} \tag{31}
\end{equation*}
$$

Numerical values are equal: $\Omega_{b 1}=0.022299491, \Omega_{b 2}=0.023708563$. At the same time $\Omega_{b 1}<\Omega_{b 2}$, that confirms the presence of two states of baryonic matter due to the presence of the Higgs antineutrino field $z_{v n}$. Replacing in (31) $z_{v n}$ by $z_{v n}^{*}$ from (30) leads to other values of the baryon density $\Omega_{b 1}^{*}=0.02228723$, $\Omega_{b 2}^{*}=0.023720824$. Hence it follows, that the baryon density of the Universe depends on the states of the antineutrino field. On the other hand, within the framework of our anisotropic model (taking into account the polarization of the CMB), the base parameter $n_{v n}$ can be independently determined from

$$
\begin{equation*}
n_{v n}=\left|\chi_{e f}\right| \sin \left(\varphi_{0 g}\right)+\psi_{r c}+2 \Omega_{0 G} ; \Omega_{b 1}=0.5 n_{v n}-2 n_{\tau L} \sin \left(\varphi_{0 g}\right) ; n_{\tau L}^{2}=\Omega_{\tau L} \tag{32}
\end{equation*}
$$

Here $\left|\chi_{e f}\right|=0.2504252, \psi_{r c}=0.04420725, \Omega_{0 G}=4.99501253 \cdot 10^{-6}$ from [23, 24]. The values $\Omega_{b 1}$ from (32) and (31) coincide and agree with the baryon density of the Universe 0.0223 from the experimental data [19]. Note, that expressions (31) allow us to describe the inverse (at $\left.z_{v n}<0\right)$ states, states with shifts $\Omega_{b 1}^{\prime}, \Omega_{b 2}^{\prime}$ or $\bar{\Omega}_{b 1}, \bar{\Omega}_{b 2}$ of the baryon density of the Universe

$$
\begin{equation*}
\Omega_{b 1}^{\prime}=\Omega_{b 1}-\Omega_{\tau L}^{*} ; \quad \Omega_{b 2}^{\prime}=\Omega_{b 2}+\Omega_{\tau L}^{*} ; \quad \bar{\Omega}_{b 1}=\Omega_{b 1}+\Omega_{\tau L}^{*} ; \quad \bar{\Omega}_{b 2}=\Omega_{b 2}-\Omega_{\tau L}^{*} \tag{33}
\end{equation*}
$$

due to the presence of a contribution from $\Omega_{\tau L}^{*}$ while preserving the quantum number $n_{v n}$. Numerical values are equal: $\Omega_{b 1}^{\prime}=0.02018275, \bar{\Omega}_{b 1}=0.024416232$. Expressions (31) - (33) can be used to describe the effective susceptibilities $\chi_{\nu x}$ of active regions ( $x=A, B, C, D, E$ ) of coronal holes on the Sun. In [15] the parameters $N_{2 x}, N_{1 x}$ for these regions were measured. Based on the formulas

$$
\begin{align*}
& \chi_{v x}=N_{2 x} / N_{1 x} ; \quad \chi_{b x}=\chi_{v x}+\Omega_{b 1}^{\prime} ; \quad z_{b x}=\left(1+\chi_{b x}^{2}\right)^{1 / 2}-1=1-\left(1-\bar{\chi}_{b x}^{2}\right)^{1 / 2} \\
& \chi_{N A}^{2}=\left|\chi_{21} \cdot \chi_{12}\right|+2 \Omega_{0 G} ; \quad \varepsilon_{b A}=\left(\varepsilon_{H G}^{2}+\Delta_{b A}^{2}\right)^{1 / 2} ; \quad \bar{\varepsilon}_{b A}=\left(\varepsilon_{H G}^{2}+\bar{\Delta}_{b A}^{2}\right)^{1 / 2} \tag{34}
\end{align*}
$$

estimates for each of the regions can be obtained: $\chi_{V A}=0.1721131$, $\chi_{V B}=0.1689743, \chi_{V C}=0.1744639, \chi_{V D}=0.1789925, \chi_{V E}=0.1608336$. From (34) independently (based on the susceptibility components $\chi_{21}, \chi_{12}$, constants $a_{T}, a_{\lambda}$ from our anisotropic model $[23,24]$ ), we find $\chi_{N A}$ exactly coinciding with $\chi_{v A}$. For the susceptibility with a shift, we find: $\chi_{b A}=0.1922958$, $z_{b A}=0.0183210, \bar{\chi}_{b A}=0.1905423$. These susceptibilities determine the gaps $\Delta_{b A}=\chi_{b A} \varepsilon_{H G}=53.851680 \mathrm{meV}, \quad \bar{\Delta}_{b A}=\bar{\chi}_{b A} \varepsilon_{H G}=53.360611 \mathrm{meV} \quad$ (which correspond to the effective temperatures $T_{b A}=a_{T} \Delta_{b A}=39.489369^{\circ} \mathrm{C}$, $\bar{T}_{b A}=a_{T} \bar{\Delta}_{b A}=36.6398^{\circ} \mathrm{C}$ ), energies $\varepsilon_{b A}=285.17677 \mathrm{meV}, \bar{\varepsilon}_{b A}=285.08445 \mathrm{meV}$ (which correspond to the wavelengths $\lambda_{b A}=a_{\lambda} / 2 \varepsilon_{b A}=2.1734659 \mu \mathrm{~m}$,
$\left.\bar{\lambda}_{b A}=a_{\lambda} / 2 \bar{\varepsilon}_{b A}=2.1741698 \mu \mathrm{~m}\right)$ in the spectra of neutrinos with nonzero rest mass. These active microobjects can be part of the solar and intergalactic winds and affect to various physical, chemical, biological processes on Earth and in Universe.

## Conclusions

The relationships between the base parameters of the Higgs boson and the parameters of black holes are established. Based on the distribution density functions of the number of quanta in the ground and excited states for relic photons, a lower mass estimate for a supermassive black hole is obtained. Based on the density distribution functions of the radiation intensity, an estimate of the mass near the upper boundary is obtained. The description of the central region of a supermassive black hole is made in terms of Bose condensate from black holes. Various states for a black hole with intermediate mass are introduced. Estimates for the mass and radius of the central body, the distance from the Sun to the supermassive black hole in the center of the Milky Way galaxy, the semiaxes of the elliptical orbit of S2 (rotating around the central body) are obtained. The model equations are used to describe the base parameters of a relativistic jet: velocities, energy, angular width of jet particles.
It is shown, that the presence of a Higgs field of different nature leads to changes in the rest energy of the Higgs boson and the energies of holes (antiparticles) for paired leptons; the appearance of active microobjects with different energies and sizes; the appearance of asymmetry of matter and antimatter. A model for the classical decay of a neutron into a proton-electron pair and an antineutrino with a nonzero rest mass is proposed. The possibility of using this model to describe tetraquarks, the baryon density of the Universe, which depends on the states of antineutrinos, is shown.
Parameter estimates are consistent with experimental data.

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# Memory cell based on qubit states and its control in a model fractal coupled structure 

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#### Abstract

Model fractal coupled structures are considered, for which a characteristic feature of the behavior of the deformation field is the presence of such superposition qubit states where there is no damping. Such states can be memory cell. The possibility of internal and external control of the structure of the memory cell, the possibility of performing the operations of write and delete information has been established. It is shown, that changes in deformation fields in a memory cell are anisotropic. External control of a memory cell in a coupled structure is performed using different fractal indices of separate structures. In this case, fractal indices do not depend on iterative processes. It is shown, that there is a critical value of the fractal index of separate structures, when passing through which effective damping occurs. This effect can be used to control the storage of information in a memory cell. When fractal indices depend on an iterative process, self-organization (internal control) occurs. By the example of the sinusoidal law of change in the fractal index of separate structures, it is shown, that structures of the following type arise: vertical, horizontal, inclined stripes; lattice structures of various orientations.


Keywords: coupled fractal structures, memory cell, superposition of qubit states, deformation field, control of memory cell structure.

## 1 Introduction

In [1], the description of the complex deformation field of model fractal coupled structures was carried out on the basis of different qubit states of separate structures such as circular and elliptical cylinders. A distinctive feature of the behavior of the deformation field of such coupled structures is the presence of qubit states, for which there is no damping (the imaginary part of the deformation field is zero). Such states can be memory cells.
The relevance of the work is associated with the problem of creating quantum computers [2,3], that encode information in qubits; with quantum cryptography, where information is recorded in a memory cell below the noise level. Physical systems, that realize qubits, can be any objects, that have two quantum states. Modern nanotechnology makes it possible to create such active objects. Various nanostructures, metamaterials [4-6], superconductors [7] can act as active objects. These active objects can be in superposition qubit states, exhibit stochastic properties, quantum entanglement, which is the basis for the creation of quantum computers. Control, storage of quantum information, the possibility of its extraction are important steps for quantum communication. Modern

[^0]nanotechnologies use various periodic structures and metamaterials [4], where the amplitude and phase of the deformation field is carried out by external control. In the presence of an iterative process, quantum chaos and the phenomenon of self-organization (internal control) appear in a memory cell. Therefore, the question of preserving a memory cell for fractal coupled structures requires additional research. Random matrices are used to describe quantum chaos [8]. Elements of random matrices are formed as a result of an iterative process. In this case, it becomes necessary to describe and take into account the effect of ordering of separate operators of deformation fields in a coupled structure [9, 10], which based on various qubit states.
The aim of this work is to describe the deformation field of a memory cell in a fractal coupled structure with elements of cylindrical type, internal and external control of its structure.

## 2 Memory cell of a model fractal coupled structure

To describe the deformation field of memory cells, let us consider a model coupled structure, which consists of two fractal cylinders of elliptic type $(i=1,2)$, located in a bulk discrete lattice $N_{1} \times N_{2} \times N_{3}$, whose nodes are given by integers $n, m, j$. Nonlinear equations for the dimensionless displacement function $u$ of the lattice node are $[1,9,10]$

$$
\begin{align*}
& u=\sum_{i=1}^{2} u_{R i} ; \quad u_{R i}=R_{i} k_{u i}^{2}\left(1-2 \mathrm{sn}^{2}\left(u-u_{0 i}, k_{u i}^{\prime}\right)\right) ; \quad i=1,2 ;  \tag{1}\\
& k_{u i}^{2}=\left(1-\alpha_{i}\right) / Q_{i} ; \quad k_{u i}^{\prime}=\left(1-k_{u i}^{2}\right)^{1 / 2} ; \quad p_{0 i}=p_{0 i}+p_{1 i} n+p_{2 i} m+p_{3 i} j ;  \tag{2}\\
& Q_{i}=p_{0 i}-b_{1 i}\left(n-n_{0 i}\right)^{2} / n_{c i}^{2}-b_{2 i}\left(m-m_{0 i}\right)^{2} / m_{c i}^{2}-b_{3 i}\left(j-j_{0 i}\right)^{2} / j_{c i}^{2} . \tag{3}
\end{align*}
$$

Here $\alpha_{i}$ are the fractal dimensions of the deformation field $u$ along the axis $O z ; u_{0 i}$ are the constant (critical) displacements; variable modules $k_{u i}, k_{u i}^{\prime}$ are functions of indices $n, m, j$ nodes of the bulk discrete lattice. Different structures are characterized by parameters: $p_{0 i}, p_{1 i}, p_{2 i}, p_{3 i}, b_{1 i}, b_{2 i}, b_{3 i}, n_{0 i}, n_{c i}, m_{0 i}$, $m_{c i}, j_{0 i}, j_{c i}, R_{i}$. In our model, the choice of different states of qubits in the plane $n O m$ is determined by the nonzero coefficients of the linear terms in the functions $p_{0 i}, Q_{i}$ from (2), (3). The initial state $(0,0)$ of an separate structure is determined by zero coefficients $p_{1 i}=0, p_{2 i}=0$. Various basic and superposition states of qubits were considered in [1]. In this work, we will focus only on the superposition state $(-1,-1)$, in which the parameters $p_{1 i}, p_{2 i}$ have the form $p_{1 i}=-0.00423, p_{2 i}=-0.00572$.
Consider a superposition state $(-1,-1)$ of two fractal coupled structures $(A),(B)$. In structure (A), the operation of scalar multiplication of the complex deformation fields of separate structures (I) and (II) is realized, while the
deformation field of this structure is described by a function $u_{A}=u_{R 1} f_{A}\left(u_{R 2}\right)$ with a corresponding matrix $\mathbf{M}_{A}$. The elements of the matrix $\mathbf{M}_{A}$ are obtained by solving equations (1) - (3) by the method of iteration over the index $n$. This procedure simulates coupled (dependent) stochastic processes of the original independent stochastic processes for structures (I) and (II), which are described by the functions $u=u_{R 1}$ and $u=u_{R 2}$. Structure (I) is a circular cylinder with constant semi-axes $n_{c 1}=m_{c 1}$, and structure (II) is an elliptical cylinder with variable semi-axes $n_{c 2}, m_{c 2}$.
To take into account the ordering of separate operators of deformation fields in a coupled structure, structure $(B)$ is considered, where the operation of scalar multiplication of complex deformation fields of separate structures (II) and (I) is realized. The deformation field of this structure is described by a function $u_{B}=u_{R 2} f_{B}\left(u_{R 1}\right)$ with a corresponding matrix $\mathbf{M}_{B}$. In the numerical modeling, it was assumed that $N_{1}=240, N_{2}=240, u_{0}=29.537, p_{0}=1.0423$, $b_{1 i}=b_{2 i}=1, \quad n_{0 i}=121.1471, \quad m_{0 i}=120.3267, \quad j_{0 i}=31.5279, \quad j_{c i}=11.8247$, $b_{3 i}=0$. Values of the semi-axes of a circular cylinder (I) are $n_{c 1}=m_{c 1}=57.4327$ with $R_{1}=1$. For elliptical cylinder (II) with $R_{2}=1$, we have the following dimensions of the semi-axes: variant 1 are $n_{c 2}=43.0746, m_{c 2}=19.1443$; variant 2 are $n_{c 2}=55.2537, m_{c 2}=14.9245$; variant 3 are $n_{c 2}=119.9327$, $m_{c 2}=6.8758$. Further consider only coupled structures $(A),(B)$, in which separate structures (I) and (II), (II) and (I) have the same fractal dimensions $\alpha_{i}$ and the same superposition qubit states $(-1,-1)$, but differ in the order of the deformation field operators $u_{A}=u_{R 1} f_{A}\left(u_{R 2}\right), u_{B}=u_{R 2} f_{B}\left(u_{R 1}\right)$.
As an example, Fig. 1 shows the behavior of the deformation field $u_{A}=\operatorname{Re} u_{A}$ of the structure (A) with the same fractal dimension $\alpha_{i}=0.5$ of separate structures (I) and (II). The variable semi-axes of the elliptical cylinder of structure (II) correspond to variants $1,2,3$. A change in the semi-axes of an elliptical cylinder of structure (II) (internal control of the structure parameters) does not lead to the appearance of an imaginary part of the displacement function, which is a characteristic feature of the behavior of the deformation field. For $u_{A}=\operatorname{Re} u_{A}$ the presence of a stochastic peak is characterized, for which the structure and region of localization in the plane nOm changes with depending on the semi-axes of the elliptical cylinder (II) (Fig. $1 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ). to a decrease in the semi-axis $m_{c 2}$ of the elliptical cylinder (II) (Fig. $1 \mathrm{~g}, \mathrm{~h}, \mathrm{i}$ ). The cross sections $\operatorname{Re} u_{A}$ (Fig. $1 \mathrm{~g}, \mathrm{k}, \mathrm{l}$ ) confirm the anisotropic nature of the alteration of the structure of the inner region of the stochastic peak: there is a change in the shape and structure of separate elliptical rings, the effect of mixing of separate trajectories.


Fig. 1. Dependences of the deformation field of the structure $(A)$ at $\alpha_{i}=0.5$ on the variable semi-axes of the structure (II): $u=u_{A}(\mathrm{a}, \mathrm{b}, \mathrm{c})$ - general view; projections on the planes $n O u(\mathrm{~d}, \mathrm{e}, \mathrm{f}), m O u(\mathrm{~g}, \mathrm{~h}, \mathrm{i}) ;(\mathrm{j}, \mathrm{k}, \mathrm{l})-$ cross sections (top view).

These changes are anisotropic. In this case, along the axis On, the peak broadens due to an increase in the semi-axis $n_{c 2}$ of the elliptical cylinder (II), the amplitudes of the peaks are of the order of 10 (Fig. 1d), 12 (Fig. 1e), 11 (Fig. 1f) dimensionless units. A narrowing of the peak occurs along the axis Om due
to a decrease in the semi-axis $m_{c 2}$ of the elliptical cylinder (II) (Fig. $1 \mathrm{~g}, \mathrm{~h}, \mathrm{i}$ ).
The cross sections $\operatorname{Re} u_{A}$ (Fig. $1 \mathrm{~g}, \mathrm{k}, \mathrm{l}$ ) confirm the anisotropic nature of the alteration of the structure of the inner region of the stochastic peak: there is a change in the shape and structure of separate elliptical rings, the effect of mixing of separate trajectories.
For both the structure ( $A$ ) and the coupled structure $(B)$ with the same superposition states $(-1,-1)$ of separate structures feature of the behavior of the deformation field is the absence of effective damping in all over region ( $\left.\operatorname{Im} u_{B}=0\right)$. For $\operatorname{Re} u_{B}$ it is also characterized by the presence of a expanded stochastic peak with a structure close to the peak $\operatorname{Re} u_{A}$ (Fig. 1), but $\operatorname{Re} u_{B}-\operatorname{Re} u_{A} \neq 0$. In this case, the conditions are carried out

$$
\begin{equation*}
u_{B}-u_{A}=u_{R 2} f_{B}\left(u_{R 1}\right)-u_{R 1} f_{A}\left(u_{R 2}\right) \neq 0, \quad \mathbf{M}_{B}-\mathbf{M}_{A} \neq 0, \tag{4}
\end{equation*}
$$

which is related to the dependence of the considered stochastic processes. This indicates that the operators of the displacement fields of separate structures (II), (I) and (I), (II) do not commute in coupled structures (B) and (A). The results of numerical modeling for structure $(B)$ are not presented in this work.

## 3 External control of a memory cell

External control of the structure of a memory cell will be carried out due to a different choice of constant fractal dimensions $\alpha_{i}$ of separate structures (I), (II).
In this case, fractal indices $\alpha_{i}$ do not depend on iterative processes. On Fig. 2 shows the behavior of the deformation fields $u_{A}=\operatorname{Re} u_{A}$ of the structure (A) for the same fractal dimensions of structures (I), (II): $\alpha_{1}=\alpha_{2}=0.0$ (Fig. $2 \mathrm{a}, \mathrm{d}$ ), $\alpha_{1}=\alpha_{2}=0.9$ (Fig. $2 \mathrm{~b}, \mathrm{e}$ ), $\alpha_{1}=\alpha_{2}=0.99$ (Fig. $2 \mathrm{c}, \mathrm{f}$ ); the semi-axes of structure (II) correspond to variant 1 . When fractal dimensions of structures (I), (II) increase, then a change in the shape and structure of stochastic peaks is observed, which is accompanied by a sharp decrease in amplitudes from 46 (Fig. 2a), 0.33 (Fig. 2 b) to 0.0036 (Fig. 2 c) of dimensionless units. The cross sections (Fig. 2 d , e, f) confirm a significant alteration of the structure of the inner region from a wave-like state (Fig. 2 d) to an almost regular behavior (Fig. 2 f). When $\alpha_{1}=\alpha_{2}=1.0$ the deformation field becomes zero $u_{A}=\operatorname{Re} u_{A}=0$. This makes it possible to interpret such a change in fractal dimensions as an operation of delete information in a memory cell. Next, consider the structure (A), where the fractal dimensions $\alpha_{i}$ of separate structures (I) and (II) are chosen to be different. In this case, for an elliptical cylinder (II) with $R_{2}=1$, we have the parameters of variant 1 . Fig. 3 shows the dependences of the deformation field of the structure $(A)$ on various joint changes in the fractal dimensions $0 \leq \alpha_{i}<1$ of structures (I), (II): an increase $\alpha_{1}$ for structure
(I) and a decrease $\alpha_{2}$ for structure (II) in the segment [ $0 ; 0.99$ ]. For projections on the plane $n O u$ (Fig. $3 \mathrm{a}, \mathrm{d}, \mathrm{g}, \mathrm{j}$ ), $m O u$ (Fig. 3 b , e, h, k), the following characteristic features of the behavior of the deformation field are observed. The amplitudes and shapes of stochastic peaks change. Cross sections (Fig. 3 c, f, i, l) $u \in\left[-10^{-4} ; 10^{-4}\right]$ allow more detailed information to be extracted.

a) $\alpha_{1}=\alpha_{2}=0$

$$
\begin{array}{llll}
50 & 100 & 150 & 200 \mathrm{n}
\end{array}
$$

$$
\begin{aligned}
& \text { d) } \alpha_{1}=\alpha_{2}=0, \\
& u \in[-0.1 ; 0.1]
\end{aligned}
$$


b) $\alpha_{1}=\alpha_{2}=0.9$

e) $\alpha_{1}=\alpha_{2}=0.9$,
$u \in[-0.01 ; 0.01]$

c) $\alpha_{1}=\alpha_{2}=0.99$

f) $\alpha_{1}=\alpha_{2}=0.99$ $u \in[-0.0001 ; 0.0001]$

Fig. 2. Dependences of the deformation field of the structure $(A)$ on the same fractal dimensions of the structures (I), (II): $u=u_{A}=\operatorname{Re} u_{A}(\mathrm{a}, \mathrm{b}, \mathrm{c})-$ projections on the plane $n O u$, (d, e, f) - cross sections (top view); $0 \leq \alpha_{i}<1$.

The circular cylinder of structure (I) with $\alpha_{1}=0$ defines the external regular wavelike behavior of the deformation field, and the elliptical cylinder of structure (II) with $\alpha_{2}=0.99$ defines the internal stochastic behavior of the deformation field (core) (Fig. 3 c ). With a further joint change in fractal dimensions, a significant change in the structure of both the core and the outer region occurs: there is an intersection (Fig. 3 f), breaks (Fig. 3 i) of regular and stochastic rings; the appearance of rings with superposition (Fig. 31 ) of regular and stochastic behavior. When $\alpha_{1}=\alpha_{2}=1.0$ the deformation field becomes zero $u_{A}=\operatorname{Re} u_{A}=0$, which follows from the basic equations (1) - (3). This allows for the possibility of interpretation as an operation of delete information in a memory cell. With a further increase in the values of fractal dimensions $\alpha_{i}>1$ of separate structures (I), (II) (Fig. 4, Fig. 5), one should expect a significant alteration of the
deformation field in the coupled structure ( $A$ ).


Fig. 3. Dependences of the deformation field of the structure $(A)$ on various fractal dimensions of the structures (I), (II): projections $u=u_{A}=\operatorname{Re} u_{A}$ on the planes
$n O u(\mathrm{a}, \mathrm{d}, \mathrm{g}, \mathrm{j}), m O u(\mathrm{~b}, \mathrm{e}, \mathrm{h}, \mathrm{k}) ;(\mathrm{c}, \mathrm{f}, \mathrm{i}, \mathrm{l})-u \in\left[-10^{-4} ; 10^{-4}\right]$ cross sections (top view); $0 \leq \alpha_{i}<1$.

As an example, Fig. 4 shows the dependences of the deformation field of the structure (A) on the same fractal dimensions of structures (I), (II) for $\alpha_{1}=\alpha_{2}=1.01$ and $\alpha_{1}=\alpha_{2}=1.1$. Wherein, the amplitudes of the peaks increase from $4 \cdot 10^{-3}$ (Fig. $4 \mathrm{a}, \mathrm{b}$ ) to 0.4 (Fig. 4 d , e), there are features like an inflow near the stochastic core of the peaks. The cross section (Fig. 4 c) confirms the formation of an almost regular convex region (inflow) around the stochastic core. With an increase in fractal dimension, the cross section (Fig. 4 f ) is characterized by the formation of broadened rings of complex shape (in contrast to circular and elliptical rings from Fig. 3). Stochastic rings are present within the core (Fig. 4 f ).
Note, that the imaginary part of the displacement function is as before equal to zero.


Fig. 4. Dependences of the deformation field of the structure $(A)$ on the same fractal dimensions of structures (I), (II): projections $u=u_{A}=\operatorname{Re} u_{A}$ on the planes $n O u$ (a, d), $m O u$ (b, e); (c, f) - cross sections $u \in\left[-10^{-4} ; 10^{-4}\right]$ (top view); $1<\alpha_{i} \leq 1.1$.

A further increase in the fractal dimension leads to the appearance of the imaginary part of the displacement function $\operatorname{Im} u_{A} \neq 0$ (Fig. 5 d , e, f). The amplitude of the peaks $\operatorname{Re} u_{A}$ (Fig. $5 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ) increases compared to (Fig. 4 d , e, f), and the shape of the peaks becomes asymmetric. Wherein, the area of the stochastic core expands with the formation of intersecting circular and elliptical rings (Fig. 5 c). The appearance of the imaginary part $\operatorname{Im} u_{A} \neq 0$ can be interpreted as a possible mechanism for the loss of a part of information from a memory cell. Comparison of the behavior of the deformation fields of the structure $(A)$ shows, that
there is a critical value of the fractal dimension $\alpha=\alpha_{c r}=1.12$ (where $1.1<\alpha<1.2$ ), when passing through which, damping occurs.


Fig. 5. Dependences of the deformation field of the structure $(A)$ on the same fractal dimensions of the structures (I), (II): projection $\operatorname{Re} u_{A}$ and $\operatorname{Im} u_{A}$ on the planes $n O u(\mathrm{a}, \mathrm{d}), m O u(\mathrm{~b}, \mathrm{e}) ;$ cross sections ( $\mathrm{c}, \mathrm{f}$ ) (top view); $\alpha_{1}=\alpha_{2}=1.2$.

## 4 Internal control of a memory cell

Modern nanotechnology uses various periodic structures and metamaterials [4], where the amplitude and phase of the deformation field is performed by external control. The question about preservation the memory cell at the presence of an iterative process for fractal coupled periodic structures requires additional research. In this work, using examples of various sinusoidal laws of change in the fractal dimensions of separate structures (I), (II) of the coupled structure (A), we investigate the behavior of the deformation field, the change in the structure of the memory cell depending on the iterative process. In this case, selforganization (internal control) occurs. We realize various sinusoidal laws of change in fractal dimensions $\alpha_{1}$ and $\alpha_{2}$ separate structures (I) and (II) of the coupled structure (A) in expressions (2) as different functions of lattice indices $n, m$ (Fig. 6, Fig. 7). Note, that for Fig. 6, Fig. 7 the imaginary part of the deformation field are equal $\operatorname{Im} u_{A}=0$.
Fig. 6 shows the dependences of the deformation field of the structure ( $A$ ) for
the same sinusoidal laws of change in the fractal dimensions of separate structures (I), (II) on the lattice indices $n, m: \quad \alpha_{1}=\alpha_{2}=\sin \varphi_{1}$, $\varphi_{1}=6 \pi(n-1) / 39$ (Fig. $6 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ); $\alpha_{1}=\alpha_{2}=\sin \varphi_{2}, \varphi_{2}=6 \pi(m-1) / 39$ (Fig. 6 d); $\alpha_{1}=\alpha_{2}=\sin \varphi_{3}, \varphi_{3}=\varphi_{1}+\varphi_{2}$ (Fig. 6 e); $\alpha_{1}=\alpha_{2}=\sin \varphi_{4}, \varphi_{4}=\varphi_{1}-\varphi_{2}$ (Fig. 6 f). Deformation field dependence $u=u_{A}=\operatorname{Re} u_{A}$ for projection on the plane $n O u$ (Fig. 6 a) is a zug (a sequence of peaks of different amplitude along the axis On ). This is due to the presence of a sinusoidal law $\sin (6 \pi(n-1) / 39)$ for the fractal dimensions of separate structures (I), (II) from the lattice index $n$, according to which the iterative process is performed.


Fig. 6. Dependences of the deformation field of the structure $(A)$ for the same sinusoidal laws of change in the fractal dimensions of separate structures (I), (II): projections $u=u_{A}=\operatorname{Re} u_{A}$ on the plane $n O u$ (a), $m O u$ (b); (c, d, e, f) - cross

$$
\text { sections } u \in[-0.1 ; 0.1] \text { (top view). }
$$

There is no iterative process along the axis $O m$, therefore a broadened stochastic peak for projection $u=u_{A}=\operatorname{Re} u_{A}$ on the plane $m O u$ is observed (Fig. 6 b ). The cross section (Fig. 6 c ) shows, that the core of the coupled structure $(A)$ is a sequence of broadened stochastic stripes parallel to the axis Om . Stochastic elliptical rings with internal periodicity (outer region of the coupled structure $(A)$ ) around the core are observed.
Note, that when choosing an iterative process along the axis $O m$, a zug will be observed for projection on the plane $m O u$, and a broadened stochastic peak will be
observed for projection on the plane $n O u$. In this case, the core of the coupled structure $(A)$ will be a sequence of broadened stochastic stripes parallel to the axis On. Thus, the choice of the iterative process makes it possible to additionally control the deformation field of the coupled structure ( $A$ ).
When the same fractal dimensions of separate structures (I), (II) $\alpha_{1}=\alpha_{2}=\sin \varphi_{2}$, $\varphi_{2}=6 \pi(m-1) / 39$ depend on the lattice index $m$ (Fig. 6 d ), then the core of the coupled structure $(A)$ is a sequence of broadened stochastic stripes parallel to the axis $O n$. Other stochastic elliptic rings with internal periodicity and discontinuous trajectories around the core are observed.
When choosing fractal dimensions of separate structures (I), (II), depending on the superposition of lattice indices $n, m \quad\left(\alpha_{1}=\alpha_{2}=\sin \varphi_{3}, \varphi_{3}=\varphi_{1}+\varphi_{2}\right.$ (Fig. 6 e); $\alpha_{1}=\alpha_{2}=\sin \varphi_{4}, \varphi_{4}=\varphi_{1}-\varphi_{2}($ Fig. 6 f$)$ ) inclined periodic structures appear in the core of the coupled structure (A). Note, that the angle of rotation for inclined structures of the core is anticlockwise (Fig. 6 e) and clockwise (Fig. 6 f ). In this case, a significant difference for the deformation field of the coupled structure $(A)$ is observed.


Fig. 7. Dependences of the deformation field of the structure ( $A$ ) for various sinusoidal laws of change in the fractal dimensions of separate structures (I), (II): projections $u=u_{A}=\operatorname{Re} u_{A}$ on the plane $n O u(\mathrm{a}, \mathrm{d}), m O u$ (b, e); (c, f) - cross sections $u \in[-0.1 ; 0.1]$ (top view).

Fig. 7 shows the dependences of the deformation field of the structure $(A)$ for various ( $\alpha_{1} \neq \alpha_{2}$ ) sinusoidal laws of change in the fractal dimensions of separate
structures (I), (II) on the lattice indices $n, m: \alpha_{1}=\sin \varphi_{1}, \alpha_{2}=\sin \varphi_{2}$, $\varphi_{1}=6 \pi(n-1) / 39, \varphi_{2}=6 \pi(m-1) / 39($ Fig. $7 \mathrm{a}, \mathrm{b}, \mathrm{c}) ; \alpha_{1}=\sin \varphi_{3}, \alpha_{2}=\sin \varphi_{4}$, $\varphi_{3}=\varphi_{1}+\varphi_{2}, \varphi_{4}=\varphi_{1}-\varphi_{2}($ Fig. $7 \mathrm{~d}, \mathrm{e}, \mathrm{f})$.
Compared to Fig. 6 a, b here the dependences of the deformation field $u=u_{A}=\operatorname{Re} u_{A}$ for projections on the plane $n O u$ (Fig. 7 a) and (Fig. 7 b) are zugs (sequences of peaks of different amplitudes) both along the axis $O n$ and along the axis $O m$, respectively. This is due to the fact that the fractal dimension $\alpha_{1}$ is a function of the lattice index $n$, and the fractal dimension $\alpha_{2}$ is a function of the lattice index $m$. The section (Fig. 7 c ) shows, that the core of the coupled structure $(A)$ is a lattice of square-shaped sub-elements. For the variant, when fractal dimensions $\alpha_{1}, \alpha_{2}$ are functions of two lattice indices $n, m$, instead of pronounced zugs (Fig. $7 \mathrm{a}, \mathrm{b}$ ), stochastic peaks with a thin structure are observed (Fig. 7 d , e). The cross section (Fig. 7 f ) shows, that the core of the coupled structure $(A)$ is now a lattice of rhombic-shaped sub-elements. This behavior of the deformation field (the appearance of rhombic-shaped sub-elements) is associated with the presence of joint rotations of inclined structures both anticlockwise and clockwise, in comparison with Fig. 6 e and Fig. 6 f (where rotations are done separately).
Choosing other functions for fractal dimensions $\alpha_{1}, \alpha_{2}$ one can expect, that the core of the coupled structure $(A)$ will be an irregular lattice of sub-elements of various shapes (such as quantum dots, curved stripes, hexagonal cells).
Modern nanotechnology makes it possible to create such structures, for example, on the surface of thin membrane.
The operators of the displacement fields of separate structures (II), (I) and (I), (II) do not commute in the coupled structures $(B)$ and $(A)$ in an iterative process.
Taking into account the ordering of separate operators of deformation fields in a coupled structure $(B)$ leads to a different behavior of the deformation field depending on the functions of fractal dimensions $\alpha_{2}, \alpha_{1}$ separate structures (II), (I). The results of numerical modeling for structure ( $B$ ) are not presented in this work.

## Conclusions

It is shown, that a pronounced feature of the behavior of the deformation field of coupled structures $(A),(B)$ with the same superposition qubit states $(-1,-1)$ of separate structures is the absence of the imaginary part of the displacement function in the all region $\left(\operatorname{Im} u_{A}=\operatorname{Im} u_{B}=0\right)$ at $0<\alpha_{1}<1.12,0<\alpha_{2}<1.12$, which indicates to the absence effective damping. This makes it possible to interpret coupled structures $(A),(B)$ with the same superposition states $(-1,-1)$ of separate structures (I), (II) as memory cells. It is shown, that there is a critical value of the fractal dimension $\alpha=\alpha_{c r}=1.12$, when passing through which, damping occurs.

The possibility of internal and external control of the parameters of the structure of the memory cell by changing the semi-axes of the elliptical cylinder of the structure (II) and the fractal dimensions $\alpha_{i}$ of separate structures (I), (II) has been established. The possibility performing operations of write, delete information in a memory cell has been established. Changes in the deformation fields of coupled structures are anisotropic.
The behavior of the deformation field of the structure $(A)$ from constant (same and different) fractal dimensions of separate structures (I), (II) (external control of the memory cell) is investigated. It is shown, that a change in the fractal dimensions leads to alteration of the shape and structure of stochastic peaks, the core of the coupled structure (A).
Internal control of the memory cell is performed by realization various sinusoidal laws of change in fractal dimensions $\alpha_{1}$ and $\alpha_{2}$ separate structures (I) and (II) of the coupled structure (A), as different functions of lattice indices $n, m$. It is shown, that substructures of the type of vertical, horizontal, inclined strips, lattice structures with sub-elements of various shapes appear in the core of a coupled structure (A).

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## To stochastic resonance in homopolar dynamo

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## Headings content

[^1]
#### Abstract

This chapter is devoted to discussion of the behavior of one-disk dynamo under the action of harmonic and random signals. Evaluations of separated effects of harmonic and random external voltages in the framework of the linearized Bullard equations have been presented. As random signals with zero average the Gaussian delta-correlated noise and the Langevin stochastic process have been considered. In particular, as physical values characterizing these influences both autocorrelation functions of observables and their spectral densities have been calculated. This information is important for design and testing of homopolar dynamo layout to perform analog research of stochastic resonance in this device in nonlinear regime.


## Abbreviations

GSSP - the Gaussian stationary stochastic process.
ACF - the autocorrelation function.
SD - the spectral density.

### 2.1 Introduction

Stochastic resonance is known to be a cooperative effect in nonlinear systems manifesting itself in increasing of the output signal-to-noise ratio under addition of the optimal portion of noise [1].

At present great attention is paid to studying of stochastic resonance in multidimensional systems arising from physics through chemistry to biology and neuroscience [2-5]. However, in our opinion, the most correct path in investigation of stochastic resonance leading to a real understanding of the essence of this phenomenon is choosing of a fairly simple dynamic system with a relatively small dimension and a detailed study of this one. As a rule there are no analytical solutions both the nonstationary Fokker-Plank-Kolmogorov equation for such system and stochastic differential equations describing its behaviour. Numerical solution of these problems is quite hard too [6, 7]. Therefore this system ought to allow experimental investigation.

On the one hand, from the point of view of clarity, preference should be given to mechanical systems. Such systems are easily perceived and interpreted due to our daily experience. On the other hand, electrical systems are characterized by the ease of controlling of external influences. Hence it is convenient to take an electromechanical system as a model system for experimental and theoretical research of stochastic resonance.

In the framework of this approach we study one-disk dynamo (the so-called Bullard dynamo). At first this electromechanical system was suggested in article [8] in order to illustrate a number of astrophysical and geophysical effects concerning motion of electrically conducting fluid in a magnetic field (see [9] and references therein). Contrary to original article [8] we take into consideration both electrical load in parallel with the field coil and friction at the axis of the dynamo. But we restrict ourselves by investigation of the linear response of the Bullard dynamo because of our final aim is design of functioning homopolar dynamo for analog modeling of stochastic resonance in this system. We stress that in our research there is no any magnetohydrodynamic background - compare for instance with work [10].

The rest of the chapter is organized as follows: in section 2, we discuss equations of motion for the Bullard dynamo and their linearization. Section 3 is devoted to calculations of influence of harmonic external voltage on the linearized Bullard system. Section 4 deals with linear responses of the system on random signals with zero average, namely, on the Gaussian delta-correlated noise and the Langevin stochastic process. Final section is devoted to discussion of results elaborated and conclusions.

### 2.2 Main equations

Mathematical model of the homopolar dynamo is given by the following system of stochastic ordinary differential equations:

$$
\left\{\begin{array}{l}
L \cdot \frac{\mathrm{~d} J}{\mathrm{~d} t}+R \cdot J=M \cdot J \cdot \Omega+U(t)  \tag{2.1}\\
I \cdot \frac{\mathrm{~d} \Omega}{\mathrm{~d} t}=K-M \cdot J^{2}-2 \cdot \gamma \cdot \Omega
\end{array}\right.
$$

where $J(t)$ is electric current via the inductance $L$ on Fig. 2.1;
$\Omega(t)$ is angular speed of rotation of the disk of dynamo;
$R$ is value of resistance in the electrical circuit on Fig. 2.1.;
$M$ is coefficient of mutual inductance:
$U(t)$ is an external voltage;
$I$ is moment of inertia for the dynamo;
$K$ is constant mechanical torque on the axis of the dynamo;
$2 \cdot \gamma$ is coefficient of mechanical friction on the dynamo axis.


Fig. 2.1. Structural scheme of the homopolar dynamo
To study stochastic resonance in the system on Fig. 2.1 one ought to choose external voltage in (2.1) as follows:

$$
\begin{equation*}
U(t)=U_{0} \cdot \cos (v \cdot t)+V(t) \tag{2.2}
\end{equation*}
$$

where $U_{0}$ is amplitude of harmonic signal;
$v$ is circular frequency of harmonic signal;
$V(t)$ is the Gaussian stationary stochastic process (GSSP) with zero average:

$$
\begin{equation*}
<V(t)>=0 \tag{2.3}
\end{equation*}
$$

and fixed autocorrelation function (ACF):

$$
\begin{equation*}
<V(t) \cdot V\left(t^{\prime}\right)>=B\left(t^{\prime}-t\right) \tag{2.4}
\end{equation*}
$$

We underline that our approach in (2.1) differs sharply from one in paper [11] because of authors of this paper apply separation of the magnetic flux on magnetic flux across disk of the dynamo and magnetic flux across the loops of inductance. This separation of magnetic flux on two parts leads to increasing of dimension of phase space of the system.


Fig. 2.2. Phase plane of the homopolar dynamo in the absence of external load
For further analysis of system (2.1) it is convenient to introduce the next dimensionless variables and parameters:

$$
\begin{array}{rll}
x_{1}=\sqrt{\frac{M}{K}} \cdot J, & x_{2}=\sqrt{\frac{M \cdot I}{L \cdot K}} \cdot \Omega, & v_{0}=\sqrt{\frac{M \cdot K}{L \cdot I}}, \\
\mu=R \cdot \sqrt{\frac{I}{M \cdot L \cdot K}}, & \delta=\gamma \cdot \sqrt{\frac{L}{I \cdot K \cdot M}}, & U_{m}=K \cdot \sqrt{\frac{L}{I}} . \tag{2.5}
\end{array}
$$

After that one can rewrite system (2.1) in the following form:

$$
\left\{\begin{array}{c}
\dot{x}_{1}=-\mu \cdot x_{1}+x_{1} \cdot x_{2}+u(\tau)  \tag{2.6}\\
\dot{x}_{2}=1-x_{1}^{2}-2 \cdot \delta \cdot x_{2}
\end{array}\right.
$$

where $u(\tau)=U(t) / U_{m}$ is dimensionless external voltage;
$\dot{x}_{1,2}$ are derivatives of dimensionless variables $x_{1,2}$ with respect to dimensionless time $\tau=v_{0} \cdot t$.

The system (2.6) in the absence of external voltage is defined as:

$$
\left\{\begin{array}{c}
\dot{x}_{1}=-\mu \cdot x_{1}+x_{1} \cdot x_{2}  \tag{2.7}\\
\dot{x}_{2}=1-x_{1}^{2}-2 \cdot \delta \cdot x_{2}
\end{array}\right.
$$

It is easy to see that if $0<\delta<1 / 2 \mu$ then system (2.7) possesses by three equilibrium states: $O^{s}(0,1 /(2 \cdot \delta))$ and $O^{ \pm}( \pm \sqrt{1-2 \cdot \delta \cdot \mu}, \mu)$. It is not difficult to check that if $0<\delta<\sqrt{2+4 \mu^{2}}-2 \mu$ then points $O^{ \pm}$are stable focuses and if $\sqrt{2+4 \mu^{2}}-2 \mu<\delta<1 / 2 \mu$ then points $O^{ \pm}$are stable nodes. Point $O^{s}$ is saddle point in both cases.

We shall suppose that dimensionless damping factor $\delta$ is quite small therefore we shall deal with situation when points $O^{ \pm}$are stable focuses. Phase plane of system (2.7) at $\mu=1.0$ and $\delta=0.1$ corresponding to the case under consideration is shown on Fig. 2.2.

It is obvious that system (2.7) is invariant under transformation of variables $\left(x_{1}, x_{2}\right) \rightarrow\left(-x_{1}, x_{2}\right)$ therefore to calculate linear response of the system (2.6) it is enough to take into account only vicinity of the point $O^{+}$.

Introducing for system (2.6) new variables $y_{1,2}$ as follows:

$$
\begin{equation*}
x_{1}=+\sqrt{1-2 \cdot \delta \cdot \mu}+y_{1}, \quad x_{2}=\mu+y_{2} \tag{2.8}
\end{equation*}
$$

and rejecting terms with powers of $y_{1,2}$ greater than one we find that system (2.6) is reduced to this one:

$$
\left\{\begin{array}{c}
\dot{y}_{1}=\sqrt{1-2 \cdot \delta \cdot \mu} \cdot y_{2}+u(\tau)  \tag{2.9}\\
\dot{y}_{2}=-2 \cdot \sqrt{1-2 \cdot \delta \cdot \mu} \cdot y_{1}-2 \cdot \delta \cdot y_{2}
\end{array}\right.
$$

From system (2.9) it is easy to observe that variable $y_{2}$ obeys to the equation of motion for harmonic oscillator with damping factor $\delta$ and fundamental frequency $\omega_{0}=\sqrt{2 \cdot(1-2 \cdot \delta \cdot \mu)}$ under the action of external force:

$$
\begin{equation*}
\ddot{y}_{2}+2 \cdot \delta \cdot \dot{y}_{2}+\omega_{0}^{2} \cdot y_{2}=-\sqrt{2} \cdot \omega_{0} \cdot u(\tau) \tag{2.10}
\end{equation*}
$$

and that the behaviour of variable $y_{1}$ is governed by the behaviour of variable $y_{2}$ as follows:

$$
\begin{equation*}
y_{1}=-\frac{\dot{y}_{2}+2 \cdot \delta \cdot y_{2}}{\sqrt{2} \cdot \omega_{0}} \tag{2.11}
\end{equation*}
$$

At last for self-consistency of above presented linearization external dimensionless voltage ought to be weak: $|u(\tau)| \ll 1$.

### 2.3 Action of harmonic signal on the linearized Bullard dynamo

At first let us consider behaviour of the system (2.9) under the influence of external voltage:

$$
\begin{equation*}
u(\tau)=A_{0} \cos (\omega \tau) \tag{2.12}
\end{equation*}
$$

where in accordance with formulae (2.5) $A_{0}=\frac{U_{0}}{U_{m}}$;
$\omega=v / v_{0}$.


Fig. 2.3. Amplitude responses of the homopolar dynamo
Looking at equation (2.10) with right hand side (2.12) one can see that in this case it describes harmonically excited linear oscillator with damping therefore we may solve it in the framework of the well-known complex amplitude method.

Seeking solution of equation (2.10) in the following form:

$$
\begin{equation*}
y_{2}(\tau)=\operatorname{Re}\left[A_{2}(\omega) \cdot \exp (i \omega \tau)\right], \tag{2.13}
\end{equation*}
$$

one can easily find that complex amplitude $A_{2}(\omega)$ is equal to:

$$
\begin{equation*}
A_{2}(\omega)=-\frac{\sqrt{2} \cdot \omega_{0}}{\omega_{0}^{2}-\omega^{2}+2 i \delta \omega} \cdot A_{0} \tag{2.14}
\end{equation*}
$$

Further substituting expression (2.13) into equation (2.11) and using formula (2.14) it is not difficult to establish that

$$
\begin{equation*}
y_{1}(\tau)=\operatorname{Re}\left[A_{1}(\omega) \cdot \exp (i \omega \tau)\right], \tag{2.15}
\end{equation*}
$$

complex amplitude $A_{1}(\omega)$ in formula (2.15) being equal to:

$$
\begin{equation*}
A_{1}(\omega)=\frac{i \omega+2 \delta}{\omega_{0}^{2}-\omega^{2}+2 i \delta \omega} \cdot A_{0} \tag{2.16}
\end{equation*}
$$

Thus from formulas (2.14) and (2.16) it is easy to obtain that amplitude responses of dynamical variables of system (2.9) on voltage (2.12) are equal to:

$$
\begin{align*}
& \frac{\left|A_{1}(\omega)\right|}{A_{0}}=\sqrt{\frac{\omega^{2}+4 \cdot \delta^{2}}{\left(\omega^{2}-\omega_{0}^{2}\right)^{2}+4 \cdot \delta^{2} \omega^{2}}} .  \tag{2.17}\\
& \frac{\left|A_{2}(\omega)\right|}{A_{0}}=\frac{\sqrt{2} \cdot \omega_{0}}{\sqrt{\left(\omega^{2}-\omega_{0}^{2}\right)^{2}+4 \cdot \delta^{2} \omega^{2}}} .
\end{align*}
$$

Graphs of dependences (2.17) on dimensionless frequency $\omega$ for $\mu=1.0$ and $\delta=0.1$ are presented on Fig. 2.3. On this Figure continuous line corresponds to function $A_{1}(\omega)$ and dashed line corresponds to function $A_{2}(\omega)$. Both of them demonstrate typical resonance behavior.

### 2.4 Action of the Gaussian delta-correlated noise and the Langevin stochastic process on the linearized Bullard dynamo

Let us now suppose that external voltage is GSSP purely. In this case it is interesting to determine the following ACF:

$$
\begin{equation*}
B_{J}\left(t, t^{\prime}\right)=<\left(J(t)-J_{a}\right)\left(J\left(t^{\prime}\right)-J_{a}\right)>, \tag{2.18}
\end{equation*}
$$

where

$$
\begin{equation*}
J_{a}=\langle J(t)\rangle \tag{2.19}
\end{equation*}
$$

is average value of electric current in the circuit on Fig. 2.1.
Reducing in accordance with formulas (2.5) input GSSP voltage to dimensionless form:

$$
\begin{equation*}
u(\tau)=\frac{V(t)}{U_{m}} \tag{2.20}
\end{equation*}
$$

and substituting expression (2.20) into formula (2.3) we establish that:

$$
\begin{equation*}
<u(\tau)>=0 \tag{2.21}
\end{equation*}
$$

therefore from formulas (2.10) and (2.11) one can immediately obtain that:

$$
\begin{equation*}
\left\langle y_{1}(\tau)\right\rangle=\left\langle y_{2}(\tau)\right\rangle=0 . \tag{2.22}
\end{equation*}
$$

Thus combining formulas (2.5), (2.8) and (2.22) it is easy to find that:

$$
\begin{equation*}
J_{a}=\sqrt{\frac{K}{M}} \cdot \sqrt{1-\frac{2 \gamma \cdot R}{K \cdot M}}, \tag{2.23}
\end{equation*}
$$

hence

$$
\begin{equation*}
B_{J}\left(t, t^{\prime}\right)=\frac{K}{M} \cdot B_{1}\left(\tau, \tau^{\prime}\right) \tag{2.24}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{1}\left(\tau, \tau^{\prime}\right)=<y_{1}(\tau) y_{1}\left(\tau^{\prime}\right)> \tag{2.25}
\end{equation*}
$$

On the other side in correspondence with formula (2.11) behavior of value $y_{1}(\tau)$ is controlled by value $y_{2}(\tau)$ therefore ACF (2.25) is expressed via the next ACF:

$$
\begin{equation*}
B_{2}\left(\tau, \tau^{\prime}\right)=<y_{2}(\tau) y_{2}\left(\tau^{\prime}\right)>. \tag{2.26}
\end{equation*}
$$

Inserting expression (2.11) into definition (2.25) and using the simplest properties of ACF [12] it is not hard to prove that:

$$
\begin{equation*}
B_{1}\left(\tau, \tau^{\prime}\right)=\frac{1}{2 \omega_{0}^{2}}\left[\frac{\partial^{2} B_{2}\left(\tau, \tau^{\prime}\right)}{\partial \tau \partial \tau^{\prime}}+2 \delta\left(\frac{\partial B_{2}\left(\tau, \tau^{\prime}\right)}{\partial \tau}+\frac{\partial B_{2}\left(\tau, \tau^{\prime}\right)}{\partial \tau^{\prime}}\right)+4 \delta^{2} B_{2}\left(\tau, \tau^{\prime}\right)\right] . \tag{2.27}
\end{equation*}
$$

Further after looking at formula (2.4) and comparing it with formula (2.20) it is obvious that:

$$
\begin{equation*}
<u(\tau) \cdot u\left(\tau^{\prime}\right)>=b_{u}\left(\tau^{\prime}-\tau\right) \tag{2.28}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{u}\left(\tau^{\prime}-\tau\right)=\frac{1}{U_{m}^{2}} B\left(t^{\prime}-t\right) . \tag{2.29}
\end{equation*}
$$

It is clear that formulas (2.28) and (2.29) demonstrates stationary state of dimensionless input voltage therefore value $y_{2}(\tau)$ is GSSP too because of it obeys to linear differential equation with constant coefficients (2.10) [12]. It means that ACF (2.26) in fact depends only on variable $\theta=\tau^{\prime}-\tau$ :

$$
\begin{equation*}
B_{2}\left(\tau, \tau^{\prime}\right) \equiv B_{2}(\theta) \tag{2.30}
\end{equation*}
$$

Substituting representation (2.30) into formula (2.27) one can easily derive that:

$$
\begin{equation*}
B_{1}(\theta)=\frac{1}{2 \omega_{0}^{2}}\left[-\frac{d^{2} B_{2}(\theta)}{d \theta^{2}}+4 \delta^{2} B_{2}(\theta)\right], \tag{2.31}
\end{equation*}
$$

hence $y_{1}(\tau)$ is also GSSP.
For further advance it is convenient in accordance with the WienerKhinchin theorem [12] to introduce spectral densities (SD) of ACF (2.30) and (2.31) as follows:

$$
\begin{equation*}
S_{1,2}(\omega)=\int_{-\infty}^{+\infty} B_{1,2}(\theta) \cdot \exp (-i \cdot \omega \cdot \theta) \cdot d \theta . \tag{2.32}
\end{equation*}
$$

After the Fourier transform relation (2.31) is reduced to the next one between SD $S_{1}(\omega)$ and $S_{2}(\omega)$ :

$$
\begin{equation*}
S_{1}(\omega)=\frac{\omega^{2}+4 \delta^{2}}{2 \omega_{0}^{2}} \cdot S_{2}(\omega) . \tag{2.33}
\end{equation*}
$$

At last it is well-known that for linear homogeneous system (2.10) connection between input and output SD is expressed via its amplitude response (2.17) [12] namely:

$$
\begin{equation*}
S_{2}(\omega)=\left|\frac{A_{2}(\omega)}{A_{0}}\right|^{2} \cdot S_{u}(\omega) \tag{2.34}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{u}(\omega)=\int_{-\infty}^{+\infty} b_{u}(\theta) \cdot \exp (-i \cdot \omega \cdot \theta) \cdot d \theta \tag{2.35}
\end{equation*}
$$

is SD for ACF (2.28).
Thus combining formulas (2.17), (2.33) and (2.34) one can obtain that:

$$
\begin{equation*}
S_{1}(\omega)=\frac{\omega^{2}+4 \cdot \delta^{2}}{\left(\omega^{2}-\omega_{0}^{2}\right)^{2}+4 \cdot \delta^{2} \cdot \omega^{2}} \cdot S_{u}(\omega) \tag{2.36}
\end{equation*}
$$

Inverse Fourier transform of expression (2.36) is known to represent ACF (2.25):

$$
\begin{equation*}
B_{1}(\theta)=\int_{-\infty}^{+\infty} \frac{\omega^{2}+4 \cdot \delta^{2}}{\left(\omega^{2}-\omega_{0}^{2}\right)^{2}+4 \cdot \delta^{2} \omega^{2}} \cdot S_{u}(\omega) \cdot \exp (i \cdot \omega \cdot \theta) \cdot \frac{d \omega}{2 \pi} \tag{2.37}
\end{equation*}
$$

If input voltage is the Gaussian delta-correlated noise (the white noise) then ACF (2.4) is equal to:

$$
\begin{equation*}
B\left(t^{\prime}-t\right)=2 \cdot D_{V} \cdot \delta\left(t^{\prime}-t\right) \tag{2.38}
\end{equation*}
$$

therefore

$$
\begin{equation*}
b_{u}\left(\tau^{\prime}-\tau\right)=2 \cdot D \cdot \delta\left(\tau-\tau^{\prime}\right) \tag{2.39}
\end{equation*}
$$

where intensity of stochastic process is renormalized in accordance with formula (2.29) as $D=D_{V} \cdot v_{0} / U_{m}^{2}$.


Fig. 2.4. Reaction of the homopolar dynamo on the Gaussian delta-correlated noise

Further expression (2.35) gives us that SD of GSSP with ACF (2.39) is equal to $S_{u}(\omega)=2 \cdot D$. Thus integrand in formula (2.37) possesses by four simple poles $\pm \sqrt{\omega_{0}^{2}-\delta^{2}} \pm \mathrm{i} \cdot \delta$ hence using the well-known Jordan's lemma one can calculate explicit representation of ACF (2.25) in this case:

$$
\begin{equation*}
B_{1}(\theta)=\frac{D \cdot \exp (-\delta|\theta|)}{2 \delta \sqrt{\omega_{0}^{2}-\delta^{2}}} \cdot \operatorname{Re}\left[\frac{\omega_{*}^{2}+4 \delta^{2}}{\omega_{*}} \cdot \exp \left(\sqrt{\omega_{0}^{2}-\delta^{2}}|\theta|\right)\right] \tag{2.40}
\end{equation*}
$$

where $\omega_{*}=\sqrt{\omega_{0}^{2}-\delta^{2}}+i \cdot \delta$.
Graph of the ACF (2.40) for $\mu=1.0, \delta=0.1$ and $D=0.002$ is shown on Fig. 2.4.

If input voltage is the Langevin sochasic process then dimensionless ACF (2.29) may be chosen in the following form [13]:

$$
\begin{equation*}
b_{u}\left(\tau^{\prime}-\tau\right)=\sigma^{2} \cdot \exp \left(-\gamma\left|\tau^{\prime}-\tau\right|\right), \quad \gamma>0 \tag{2.41}
\end{equation*}
$$

where $\sigma^{2}$ is dispersion of input GSSP $u(\tau)$.
SD corresponding to ACF (2.41) is equal to [13]:

$$
\begin{equation*}
S_{u}(\omega)=\frac{2 \cdot \gamma \cdot \sigma^{2}}{\omega^{2}+\gamma^{2}} \tag{2.42}
\end{equation*}
$$

It means that in this case two additional simple poles $\pm i \cdot \gamma$ arise in integrand in formula (2.37).


Fig. 2.5. Reaction of the homopolar dynamo on the Langevin stochastic process

In the same manner one can derive that for SD (2.42) ACF (2.25) is equal to the next sum:

$$
\begin{equation*}
B_{1}(\theta)=B_{1}^{1}(\theta)+B_{1}^{2}(\theta), \tag{2.43}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{1}^{1}(\theta)=\frac{\gamma \cdot \sigma^{2} \cdot \exp (-\delta|\theta|)}{2 \cdot \delta \cdot \sqrt{\omega_{0}^{2}-\delta^{2}}} \cdot \operatorname{Re}\left[\frac{\omega_{*}^{2}+4 \delta^{2}}{\omega_{*} \cdot\left(\omega_{*}^{2}+\gamma^{2}\right)} \cdot \exp \left(\sqrt{\omega_{0}^{2}-\delta^{2}}|\theta|\right)\right] \tag{2.44}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{1}^{2}(\theta)=\sigma^{2} \cdot \frac{4 \cdot \delta^{2}-\gamma^{2}}{\left(\omega_{0}^{2}+\gamma^{2}\right)^{2}-4 \cdot \delta^{2} \cdot \gamma^{2}} \cdot \exp (-\gamma|\theta|) \tag{2.45}
\end{equation*}
$$

Graph of the ACF (2.43) for $\mu=1.0, \delta=0.1, \gamma=0.02$ and $\sigma=0.2$ is presented on Fig. 2.5. Comparing Fig. 2.5 with Fig. 2.4 one can observe that this graph also has oscillatory character stipulated by function (2.44). But moreover this graph possesses by variable vertical shift caused by contribution of function (2.45) into expression (2.43).

### 2.5 Conclusion

In the chapter linear responses of the homopolar dynamo both on weak harmonic input voltage and weak GSSP input voltage have been calculated. This preliminary research gives one a possibility of investigation of stochastic resonance in the Bullard dynamo by means of analog modeling.

To realize this research program one ought to evaluate physical parameters of the system on Fig. 2.1 and then use them to make its layout. After that one can perform a number of tests of the operation of the layout.

The first test is an action of weak ( $U_{0} \ll U_{m}$ ) harmonic signal with very slowly varying circular frequency on the homopolar dynamo layout. If dimensionless circular frequency $\omega$ of this input signal gets closer to $\omega_{0}$ then a sharp increase in amplitude of electric current in the circuit should be observed in accordance with formula (2.16) (see also Fig. 2.3).

The second test is an application to the layout of the weak Gaussian deltacorrelated noise as an input voltage. In this case measured ACF (2.18) must correspond to the calculated dependence (2.40) (see also Fig. 2.4).

Moreover nonlinearity of a system is known to transform GSSP into nonGaussian stochastic process [12], therefore, in order to control the role of nonlinearity of system (2.1) one should measure the following triple ACF [14]:

$$
\begin{equation*}
T\left(t_{1}, t_{2}\right)=<\left(J(t)-J_{a}\right)\left(J\left(t+t_{1}\right)-J_{a}\right)\left(J\left(t+t_{2}\right)-J_{a}\right)> \tag{2.46}
\end{equation*}
$$

and calculate its bispectrum [14]:

$$
\begin{equation*}
Q\left(\omega_{1}, \omega_{2}\right)=\int_{-\infty}^{+\infty+\infty} \int_{-\infty} T\left(t_{1}, t_{2}\right) \cdot \exp \left(-i \omega_{1} t_{1}-i \omega_{2} t_{2}\right) \cdot d t_{1} d t_{2} \tag{2.47}
\end{equation*}
$$

If the influence of nonlinearity is small then both value (2.46) and value (2.47) must be close to zero due to the Gaussian nature of the input signal.

The third test is an action of the weak Langevin stochastic process as an input voltage. This kind of input voltage can be obtained by means of transferring of the Gaussian delta-correlated noise via four-terminal network with resistance and capacitance [12]. In this case measured ACF (2.18) must correspond to the calculated dependence (2.43) (see also Fig. 2.5). And it is necessary to oversee closeness to zero of values (2.46) and (2.47) too.

At last if the layout overcomes these checks successfully then one can proceed to the experimental study of stochastic resonance in the homopolar dynamo under the action of input voltage (2.2) in nonlinear regime.

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# MODEL OF AI INVOLVING DYNAMICS SYMMETRY BREAKING IN MULTIFRACTAL MEDIUM 

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#### Abstract

In this paper, we show that the existence of a multifractal medium implies a series of characteristics that are specific to artificial intelligence such as memory, multivalent logic, etc. This medium contains the implicit information, the explicit information being evidenced only by a spontaneous breaking symmetry mechanism. In this context, we propose a universal holographic mechanism, mathematically based on group invariances of $S L(2 R)$ type, through which the transition of implicit-explicit information is realized. The principles discussed are found in the most complex structure from an informational point of view, that is, the human brain. Keywords: Multifractal medium, Implicit information, Explicit information, Spontaneous symmetry breaking, Brain.


## 1 Introduction

To describe complex system dynamics in the fractal paradigm, but remaining faithful to the differentiable mathematical procedures, it is necessary to explicitly introduce scale resolutions, both in the expression of the physical variables and in the fundamental equations which govern complex system dynamics [1]. This means that, instead of "working" with a single physical variable described by a strict non-differentiable function, it is possible to "work"
only with approximations of these mathematical functions obtained by averaging them on different scale resolutions. As a consequence, any physical variable purposed to describe complex system dynamics will perform as the limit of a family of mathematical functions, this being non-differentiable for null scale resolutions and differentiable otherwise [2-5]. This non-differentiable function exhibits the property of self-similarity in every of its points, which can be translated into a property of ''holography', type (every part reflects the whole). In the present paper, considering the fractal paradigm as being functional, a non-differentiable model describing the complex system dynamics is proposed. Precisely, we prove how implicit information becomes explicit information based on the group invariance. This means that the mathematical procedure reduces to obtaining joint invariant functions under the simultaneous action of two isomorphic groups of $S L(2 R)$ type. To this end, starting from the idea that any complex system can be modeled by a multifractal, we have shown that its dynamics can be explained by Schrödinger-type regimes at different scale resolutions. Particularly, analyzing the group invariance in the case of multifractal Schrödinger stationary dynamics, we showed that the implicit information becomes explicit information through specific behaviors of period double type, damped oscillations, quasi-periodicity, chaos. All these manifestations of matter are contained in the mechanism of spontaneous breaking of the symmetry of the complex system. The implicit-explicit transition of information is associated with this breaking.

## 2 Results and discussion

### 2.1. Mathematical Model

The complex system is a set of entities (or structured units) that, through their interactions, relationships or dependencies form a unified whole [1]. In the following, the complex system will be assimilated with a multifractal. Such an assumption is sustained by the following example, related to the collision processes in a complex system: between two successive collisions, the trajectory of the complex system particle is a straight line that becomes non-differentiable in the impact point. Considering that all the collision impact points form an uncountable set of points, it results that the trajectories of the complex system particles become continuous and non-differentiable curves, i.e. fractal. In such a context, the Fractal Theory of Motion in the form of Scale Relativity becomes operational through the scale covariant derivative [4, 5]:

$$
\frac{\hat{d}}{d t}=\partial_{t}+\hat{V}^{l} \partial_{l}+\frac{1}{4}(d t)^{\left(\frac{2}{D_{f}}\right)-1} D^{l p} \partial_{l} \partial_{p}
$$

where

$$
\begin{gather*}
\hat{V}^{l}=V_{D}^{l}-V_{F}^{l} \\
D^{l p}=d^{l p}-i \hat{d}^{l p} \\
d^{l p}=\lambda_{+}^{l} \lambda_{+}^{p}-\lambda_{-}^{l} \lambda_{\underline{p}}  \tag{2}\\
\hat{d}^{l p}=\lambda_{+}^{l} \lambda_{+}^{p}+\lambda_{-}^{l} \lambda_{-}^{p} \\
\partial_{t}=\frac{\partial}{\partial t}, \partial_{l}=\frac{\partial}{\partial x^{l}}, \partial_{l} \partial_{p}=\frac{\partial}{\partial x^{l}} \frac{\partial}{\partial x^{p}}, i=\sqrt{-1}, l, p=1,2,3
\end{gather*}
$$

In the above-written relations, $x^{l}$ is the fractal spatial coordinate, $t$ is the non-fractal time having the role of an affine parameter of the motion curves, $\hat{V}^{l}$ is the complex velocity, $V_{D}^{l}$ is the differential velocity independent on the scale resolution $d t, V_{F}^{l}$ is the non-differentiable velocity dependent on the scale resolution, $D_{F}$ is the fractal dimension of the movement curve, $D^{l p}$ is the constant tensor associated with the differentiable-non-differentiable transition, $\lambda_{+}^{l}\left(\lambda_{+}^{p}\right)$ is the constant vector associated with the backward differentiable-nondifferentiable physical processes and $\lambda_{-}^{l}\left(\lambda_{-}^{p}\right)$ is the constant vector associated with the forward differentiable-non-differentiable physical processes. There are many modes, and thus a varied selection of definitions of fractal dimensions: more precisely, the fractal dimension in the sense of Kolmogorov, the fractal dimension in the sense of Hausdorff-Besikovitch etc. [4, 6]. Selecting one of these definitions and operating it in the complex system dynamics, the value of the fractal dimension must be constant and arbitrary for the entirety of the dynamical analysis: for example, it is regularly found $D_{F}<2$ for correlative processes, $D_{F}>$ 2 for non-correlative processes etc. [2, 6]. Now, accepting the functionality of the scale covariance principle i.e. applying the operator (1) to the complex velocity field from (2), in the absence of any external constraint, the motion equations (i.e. the geodesics equation on a multifractal space) takes the following form [4, 5]:

$$
\frac{\hat{d} \widehat{V}^{i}}{d t}=\partial_{t} \hat{V}^{i}+\hat{V}^{l} \partial_{l} \hat{V}^{i}+\frac{1}{4}(d t)^{\left(\frac{2}{D_{f}}\right)-1} D^{l k} \partial_{l} \partial_{k} \hat{V}^{i}=0
$$

This means that the fractal acceleration $\partial_{t} \widehat{V}^{i}$, the fractal convection $\widehat{V}^{l} \partial_{l} \widehat{V}^{i}$ and the fractal dissipation $D^{l k} \partial_{l} \partial_{k} \hat{V}^{i}$, make their balance in any point of the fractal curve.

If the fractalization is achieved by Markov-type stochastic processes [2, 3], then:

| $\lambda_{+}^{i} \lambda_{+}^{l}=\lambda_{-}^{i} \lambda_{-}^{l}=2 \lambda \delta^{i l}$, | (4) |
| :--- | :--- |

where $\lambda$ is a coefficient associated to the differentiable-non-differentiable transition and $\delta^{i l}$ is Kronecker's pseudo-tensor. Under these conditions, the geodesics equation (3) takes the form:

$$
\begin{equation*}
\frac{\hat{d} \widehat{V}^{i}}{d t}=\partial_{t} \hat{V}^{i}+\hat{V}^{l} \partial_{l} \hat{V}^{i}-i \lambda(d t)^{\left(\frac{2}{D_{f}}\right)-1} \partial^{l} \partial_{l} \hat{V}^{i}=0 \tag{5}
\end{equation*}
$$

### 2.2. Dynamics of complex systems in the form of Schrödingertype "regimes"

For irrotational motions of the complex system, the complex velocity field $\widehat{V}^{i}$ from (2) takes the form:

$$
\begin{equation*}
\widehat{V}^{i}=-2 i \lambda(d t)^{\left(\frac{2}{D_{f}}\right)-1} \partial^{i} \ln \Psi \tag{6}
\end{equation*}
$$

where $\ln \Psi$ is the fractal scalar potential of the velocity fields.
Then, substituting (6) in (5), the geodesics equation (5) becomes (for details, see $[4,5])$ :

$$
\lambda\left[\lambda \partial^{l} \partial_{l} /(d t)^{2\left(1-\frac{2}{D_{f}}\right)}+i \partial_{t} /(d t)^{\left(1-\frac{2}{D_{f}}\right)}\right] \Psi=0
$$

This is a Schrödinger equation of fractal type. Therefore, various dynamics of any complex system can be implemented as Schrödinger-type fractal "regimes" (i.e. at various scale resolutions). In the one-dimensional stationary case, the Schrödinger equation of multifractal type takes the form ( $[4,5]$ ):

| $\frac{d^{2} \Psi}{d x^{2}}+k_{0}^{2} \Psi=0$ | (8) |
| :---: | :--- |
| with | $k_{0}^{2}=\frac{E}{2 m_{0} x^{2}(d t)^{\left(\frac{4}{D_{f}}\right)-2}}$ |

In (9) $x$ is the fractal spatial coordinate, $E$ is the fractal energy of the complex system entity and $m_{0}$ is the rest mass of the complex system entity. In the general case, $\Psi(x)$ is a complex function. Considering that $\Psi(x)$ can be written in the form:

(8) in real variables becomes:

| $\frac{d^{2} X}{d x^{2}}+k_{0}^{2} X=0$ | $(11)$ |
| :--- | :--- |
| $\frac{d^{2} Y}{d x^{2}}+k_{0}^{2} Y=0$ | $(12)$ |

Relations (11) and (12) are invariant to the group of $S L(2 R)$-type (for details see [7]):

$$
\begin{gather*}
X^{\prime}=\alpha X+\beta Y \\
Y^{\prime}=\gamma X+\delta Y  \tag{13}\\
\alpha \delta-\beta \gamma=1
\end{gather*}
$$

The basis of this Lie algebra is given by the infinitesimal generators:

$$
X_{1}=Y \frac{\partial}{\partial X}, \quad X_{2}=\frac{1}{2}\left(X \frac{\partial}{\partial X}-Y \frac{\partial}{\partial Y}\right), \quad X_{3}=-X \frac{\partial}{\partial Y},
$$

the generators satisfying the commutation relations:

$$
\left[X_{1}, X_{2}\right]=X_{1}, \quad\left[X_{2}, X_{3}\right]=X_{3}, \quad\left[X_{3}, X_{1}\right]=-2 X_{2}
$$

The solution of equations ((11) and (12)) is written in the form:

$$
\begin{equation*}
[X(x) \mid Y(x)]=z e^{i\left(k_{0} x+\theta\right)}+\bar{z} e^{-i\left(k_{0} x+\theta\right)} \tag{16}
\end{equation*}
$$

where $z$ is a complex amplitude, $\bar{z}$ is the complex conjugate of $z$ and $\theta$ is the specific phase. Thus, $z, \bar{z}$ and $\theta$ label each entity from an eventual complex system that has, as a general characteristic, the same $k_{0}$. Equation (10) has a "hidden" symmetry in the form of a homographic group: the ratio of two independent linear solutions of equation (10), $\tau$, is a solution of Schwartz's differential equation [8]:

$$
\begin{gather*}
\{\tau, x\}=\frac{d}{d x}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)-\frac{1}{2}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)^{2}=2 k_{0}^{2}  \tag{17}\\
\dot{\tau}=\frac{d \tau}{d x}, \ddot{\tau}=\frac{d^{2} \tau}{d x^{2}}
\end{gather*}
$$

The left part of (17) is invariant with respect to the homographic transformation:

$$
\begin{equation*}
\tau \leftrightarrow \tau^{\prime}=\frac{a_{1} \tau+b_{1}}{c_{1} \tau+d_{1}}, \tag{18}
\end{equation*}
$$

with $a_{1}, b_{1}, c_{1}$ and $d_{1}$ real parameters. The relation (18) corresponding to all possible values of these parameters defines the group $S L(2 R)$. Thus, all the entities of the complex systems having the same $k_{0}$ are in biunivocal
correspondence with the transformations of the group $S L(2 R)$. This allows the construction of a "personal" parameter $\tau$ for each entity of the complex system, separately. Indeed, as a "guide" it is chosen the general form of the solution of (17), which is written as:

$$
\begin{equation*}
\tau^{\prime}=u+v \tan \left(k_{0} x+\theta\right) \tag{19}
\end{equation*}
$$

So, through $u, v$ and $\theta$, it is possible to characterize any entity of the complex system. In such a context, identifying the phase from (19) with the one from (16), the "personal" parameter of the entity becomes:

$$
\begin{equation*}
\tau^{\prime}=\frac{z+\bar{z} \tau}{1+\tau}, \quad z=u+i v, \quad \bar{z}=u-i v, \quad \tau \equiv e^{2 i\left(k_{0} x+\theta\right)} \tag{20}
\end{equation*}
$$

The fact that (20) is also a solution of (17) implies the group of $S L(2 R)-$ type ([4, 5, 7]):

$$
\begin{equation*}
z^{\prime}=\frac{a_{1} z+b_{1}}{c_{1} z+d_{1}}, k^{\prime}=\frac{c_{1} \bar{z}+d_{1}}{c_{1} z+d_{1}} k \tag{21}
\end{equation*}
$$

The infinitesimal generators of the group (21) are:

$$
\begin{align*}
& B_{1}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}, \quad B_{2}=z \frac{\partial}{\partial z}+\bar{z} \frac{\partial}{\partial \bar{z}}, \\
& B_{3}=z^{2} \frac{\partial}{\partial z}+\bar{z}^{2} \frac{\partial}{\partial \bar{z}}+(z-\bar{z}) k \frac{\partial}{\partial k}
\end{align*}
$$

with commutation relations:

$$
\begin{equation*}
\left[B_{1}, B_{2}\right]=B_{1}, \quad\left[B_{2}, B_{3}\right]=B_{3}, \quad\left[B_{3}, B_{1}\right]=-2 B_{2} \tag{23}
\end{equation*}
$$

The group (21) admits the differential 1-forms (absolutely invariant through the group) [7]:

$$
\begin{equation*}
\Omega_{0}=-i\left(\frac{d k}{k}-\frac{d z+d \bar{z}}{z-\bar{z}}\right), \quad \Omega_{1}=\frac{d z}{(z-\bar{z}) k}, \quad \Omega_{2}=-\frac{k d \bar{z}}{z-\bar{z}^{\prime}} \tag{24}
\end{equation*}
$$

and the invariant metric:

$$
\begin{equation*}
\frac{d s^{2}}{f}=\Omega_{0}^{2}-4 \Omega_{1} \Omega_{2} \tag{25}
\end{equation*}
$$

with $f$ an arbitrary constant factor. An interesting case is the one induced by means of the parallel transport of direction in the Levi-Civita sense [7, 8]. Then, in the space of variables $(z, \bar{z}, k)$ the differential 1 -form $\Omega_{0}$ is null:

$$
\begin{equation*}
\Omega_{0}=0, \tag{26}
\end{equation*}
$$

while in the space of variables $(u, v, \theta)$ is:

$$
\begin{equation*}
d \theta=-\frac{d u}{v} \tag{27}
\end{equation*}
$$

Since through (26) or (27) the invariant metric (25) is reduced to the Lobachevsky plane metric in Poincaré representation, it results:

$$
\begin{equation*}
\frac{d s^{2}}{f}=\frac{d k d \bar{z}}{(z-\bar{z})^{2}}=\frac{d u^{2}+d v^{2}}{v^{2}} \tag{28}
\end{equation*}
$$

In such a conjecture, $\theta$ from (27) defines the angle of the parallel transport of direction in the Levi-Civita sense ( $[7,8]$ ). Once the previous functionality is accepted, the infinitesimal generators of the group (21) become:

$$
\begin{equation*}
\bar{B}_{1}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}, \quad \bar{B}_{2}=z \frac{\partial}{\partial z}+\bar{z} \frac{\partial}{\partial z}, \quad \bar{B}_{3}=z^{2} \frac{\partial}{\partial z}+\bar{z}^{2} \frac{\partial}{\partial z} \tag{29}
\end{equation*}
$$

and satisfy the same commutation relations of (23) type.
Now, consider another group of $S L(2 R)$ - type given by means of infinitesimal generators

$$
\begin{equation*}
\bar{H}_{1}=\frac{\partial}{\partial h}+\frac{\partial}{\partial \bar{h}}, \quad \bar{H}_{2}=h \frac{\partial}{\partial h}+\bar{h} \frac{\partial}{\partial \bar{h}}, \quad \bar{H}_{3}=h^{2} \frac{\partial}{\partial h}+\bar{h}^{2} \frac{\partial}{\partial \bar{h}} \tag{30}
\end{equation*}
$$

which satisfy the commutation relations:

$$
\begin{equation*}
\left[\bar{H}_{1}, \bar{H}_{2}\right]=\bar{H}_{1}, \quad\left[\bar{H}_{2}, \bar{H}_{3}\right]=\bar{H}_{3}, \quad\left[\bar{H}_{3}, \bar{H}_{1}\right]=-2 \bar{H}_{2} \tag{31}
\end{equation*}
$$

Then, the Stoka system [9] for operators (29) and (30) takes the form:

$$
\begin{gather*}
\frac{\partial F}{\partial h}+\frac{\partial F}{\partial \bar{h}}+\frac{\partial F}{\partial z}+\frac{\partial F}{\partial \bar{z}}=0 \\
h \frac{\partial F}{\partial h}+\bar{h} \frac{\partial F}{\partial \bar{h}}+z \frac{\partial F}{\partial z}+\bar{z} \frac{\partial F}{\partial \bar{z}}=0  \tag{32}\\
h^{2} \frac{\partial F}{\partial h}+\bar{h}^{2} \frac{\partial F}{\partial \bar{h}}+z^{2} \frac{\partial F}{\partial z}+\bar{z}^{2} \frac{\partial F}{\partial \bar{z}}=0
\end{gather*}
$$

It is important to notice that this system has the rank 3; as such, only one independent integral exists. This is the cross-ration generated by means of the relation:

$$
\begin{equation*}
\frac{h-z}{h-\bar{z}}: \frac{\bar{h}-z}{\bar{h}-\bar{z}} \equiv \zeta^{2} \tag{33}
\end{equation*}
$$

where $\xi$ is real, and the square is taken in order to account for the fact that the cross-ratio (33) is always positive. Any joint invariant function, $F$, is here a
regular function of this ratio. In such a context, if $\zeta \equiv \tanh \phi$, where $\phi$ is arbitrary, then $z$ is related to $h$ through the linear relation:

| $z=\bar{u}+\bar{v} h_{0}$ | (34) |
| :--- | :--- |
| where | (35) |
| $h=\bar{u}+i \bar{v}, \quad i=\sqrt{-1}$ |  |
| $h_{0}=-i \frac{\cosh \phi-e^{-i \alpha} \sinh \phi}{\cosh \phi+e^{-i \alpha} \sinh \phi}, \Delta \phi=0$ |  |

$\Delta$ is the Laplace operator and $\alpha$ is real. Therefore, synchronization of phaseamplitude type of each complex system entity (mathematically described through parallel transport of direction in Levi-Civita sense) implies joint invariant function of two simultaneous isomorphic groups of $S L(2 R)$-type as solution of Stoka-type equation. Then, period doubling, damping oscillations, self-modulation and chaotic regimes emerge as natural behaviors in the complex system dynamics. (see Figures $4 \mathrm{a}-1$ for $\alpha=\omega t$, $\tanh \phi=0.1$ and Real $[(z-\bar{u}) / \bar{v}] \equiv$ Amplitude at various scale resolutions, given by means of the maximum value of $\omega$ ).



Figures 1. Various types of synchronization of complex system entity (3D, contour plot and time series representation of the function signaling): period doubling ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ), damped oscillation regime ( $\mathrm{d}, \mathrm{e}, \mathrm{f}$ ), signal modulation ( $\mathrm{g}, \mathrm{h}, \mathrm{i}$ ) and chaotic behavior ( $\mathrm{j}, \mathrm{k}, \mathrm{l}$ ) ).

## 3 Possible implications

These considerations might be applied in understanding brain's functionality. Indeed, human brain is, morphologically, a multifractal. Moreover, its own space (the one generated by the brain) is structurally, a multifractal in the most general sense given by Mandelbrot [2].

In such space, the only possible functionalities (which are compatible with the brain structure) are achieved on continuous but non-differentiable curves [3, 4, 5]. Brain's structural-functional compatibility (structural-functional duality) as a source of the cerebral dynamics at any scale is thus imposed. Accepting the structural-functional duality of the brain, the trajectory of motion realized on the structural component must be identified with an element from the functional part. If we admit that the anharmonic oscillations of the neurofibrils would be the source of the functional part of the brain, then the curve describing the motion of a neurofibril is a continuous non-differentiable curve. So this motion takes place in a fractal space, the one generated by the fractal structure of the brain, and thus it can be identified with the geodesic of the associated fractal space. At yet another scale, the neuron can be identified with its corresponding geodesic. More generally, the wave is identified with the corpuscle, the motion of the corpuscle in the field of its associated wave being obviously a continuous non-differentiable curve (fractal curve), whence the idea of geodesic.

We have presented the mathematics of this model in [10, 11] and, therefore, we shall not insist here on this aspect. We insist only on the fact that brain's spectral functionality is described through Schrödinger fractal type geodesics, while brain's structural functionality is described by means of geodesics. In our opinion, both functionalities (either the one which is responsible of the brain activity unpredictable character, or the other one which is responsible of the brain activity predictable character) act simultaneously. By their interconditioning there result brain coherence (or brain compatibility) of the two neuronal networks (the spectral one and the structural one).

The statements from Section 2 (refering to implicit and explicit duality, breaking of symmetries, role of measurement, spectral projection, etc.) can be found not only from the characteristics of chance ( $\mathrm{Df}=2$ ), but from the characteristics of determinism $(\mathrm{Df}=1)$. To achieve this it is possible to use the fractal component (Le Méhaute et al. [12], Le Méhaute [13, 14]) of category theory (Leinster $[15,16]$ ), introducing then the role of the Riemann Zeta function (Le Méhaute [17], Riot and Le Méhaute [18]) in connection with the Riemann conjecture (Riot and Le Méhaute [19]). The Riemann function carries with him a role of mediation between both limits and the role of the set of filters required in the frame of Scott topology (Steenrood [20]). The categorical approach confirms the epistemological break introduced by fractal geometry in the contemporary physics of complex systems (Riot et al. [21]).

## 4 Conclusions

The main conclusions of the present paper are the following:

1. Assimilating a complex system with a multifractal, we have shown that in the fractal theory of motion, its dynamics can be explained in the form of Schrödinger-type regimes at different scale resolutions;
2. In the stationary case of the fractal Schrödinger-type equation, we explicited the invariance groups, both that of the variables and that of the initial conditions, groups which are isomorphic. In our opinion, these groups contain system's potentiality in the form of implicit informational energy. The connection of the entities of the substructures to the complex system (which in our opinion corresponds to a mechanism of spontaneous symmetry breaking, a mechanism by which implicit information becomes explicit information), implies obtaining joint invariant functions in relation to the above mentioned isomorphic groups. Joint invariant functions are obtained based on Stoka's theorem and in this situation one can evidentiate diverse implicit-explicit information transition scenarios, in the form of period doubling, damped oscillations, quasi-periodicity and even chaos.

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# Dynamic Localized Autonomous Chaotic Orbital Patterns from Rotation-Translation Sequences 

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#### Abstract

Consider an ordered sequence of repeated operations given by a distancedependent rotation followed by a translation. Operating this sequence with special parameters and initial conditions provides for characteristic spatial density patterns in the plane. In this work we introduce an additional orbital rotation and find local chaotic orbital patterns and attractors in the plane. There are two ways to form a local density from discrete long-range jumps: either a jump-back boundary condition or rotating the jump direction. We focus on real time simulations, where the chaotic evolution and vivid dynamics (the live cycles of orbitals) with characteristic numbers or stability conditions is manifest. Stable and unstable chaotic orbital patterns or solitons emerge dynamically and fluctuate without any "hard" additional boundary or radial back-jump-condition. We show some typical orbital patterns and suggest a method of categorization.


Keywords: Chaos, Rotation, Discrete, Translation, Reflection, Closed Loop, Orbit, Soliton, Wavelet, Quantum, Attractor, Pattern, Sequence, Simulation, Commutation, Operator, Geometric Phase.

## 1 Introduction

Repeating a discrete sequence of rotation - translation - reflection operations can provide for a wide range of interesting and very complex pattern emerging from chaotic jumps, see Skiadas [1,2,3]. Counterintuitively, discrete long-range jumps often follow a continuous type of flow pattern, e.g. in [3] very similar to v. Kármán Streets, see fig. 1.


Fig. 1: Skiadas type v. Kármán Streets with rotation parameter $c_{1}=\pi$, power exponent $p_{1}=-2$, reflection $m=2$, and long jump back (red arrow).

To get a special pattern requires adjusting the rotation strength parameter and eventually a boundary distance condition parallel to the jump direction. We found that the correspondent pattern building process can be assigned to small nonlinear geometric (phase) shifts arising in the rotation-translation sequence on every loop Binder [4,5]. Applying this nonlinear concept it is possible to
generate a broad range of patterns, including periodic structures like waves, circles, saw-tooth or point-like discrete geometries by desktop computer simulations. The model can be generalized to multiple rotation-translation sequences on orthogonal rotation axes in higher dimensions Binder [5].
In this work we add to the basic rotation-translation-reflection model an extra orbital rotation orthogonal to the jump direction and wonder, if we can generate in these ways structures on closed orbits. This means, we add another rotation around the existing singularity (at the origin) and look for orbital structures. As a result, we find chaotic periodic structures emerging on the orbit without any additional constraint like a jump-back condition limit. These dynamical structures are more or less stable and often fluctuating like vivid entities. In this paper we point to some interesting patterns/simulations and try to make a categorization according to boundary conditions and characteristic parameter.

## 2 The Basic Operation Sequence

In the plane the chaotic model is based on a discrete iteration sequence of the polar vector

$$
\begin{equation*}
\vec{r}=\binom{x}{y} \tag{1}
\end{equation*}
$$

Its polar coordinates are given by the radius $r=|\vec{r}|$ and polar angle/direction $\varphi$, where

$$
\begin{equation*}
x=r \sin \varphi ; r^{2}=x^{2}+y^{2} ; \varphi=\operatorname{atan} 2(x, y) \tag{2}
\end{equation*}
$$

The iteration will be given by an ordered sequence composed by the two or three operations given by a polar rotation $\mathbf{R}$ and non-radial translation $\mathbf{T}$ and eventually a radial inversion I. The vector coordinate evolves in successive operations within one iteration sequence with numbering $t \rightarrow t+1$ according to

$$
\vec{r}_{t} \rightarrow \vec{r}_{R} \rightarrow \vec{r}_{T} \rightarrow \vec{r}_{I}=\vec{r}_{t+1}
$$

by the following relations:
I. A radius dependent polar/central rotation including reflection

$$
\begin{equation*}
\vec{r}_{R}=\mathbf{R}(\vec{r}, \Delta \phi) \tag{3}
\end{equation*}
$$

with rotation angle $\Delta \phi$ in eq. (3) composed by the following rotation and reflection components

$$
\begin{equation*}
\Delta \phi=\Delta \phi_{1}+\Delta \phi_{2} \tag{4}
\end{equation*}
$$

given by:

1. $\Delta \phi_{1}$ in eq. (4) is the Skiadas type rotation angle that has a powerlaw radial distance dependence with exponent $p_{1}<0$

$$
\begin{equation*}
\Delta \phi_{1}=c_{1} r^{p_{1}} \tag{5}
\end{equation*}
$$

2. $\Delta \phi_{2}$ in eq. (4) generalizes the reflection given by the difference $\sigma-\varphi$ multiplied by a reflection mode $m$

$$
\begin{equation*}
\Delta \phi_{2}=m(\sigma-\varphi), \tag{6}
\end{equation*}
$$

where $m=2$ is a reflection to the opposite side with respect to the initial polar angle $\varphi$ in the co-rotating frame. In our special case, the global direction $\sigma$ in eq. (6) sums up with a constant orbital rotation $c_{0}$ eventually driven by a radial power-law $c_{2} r^{p_{2}}$

$$
\begin{equation*}
\sigma=\sigma_{t}=\sigma_{t-1}+c_{0}+c_{2} r^{p_{2}} \tag{7}
\end{equation*}
$$

The total rotation is with in eqs. (4) - (7) given by

$$
\begin{equation*}
\Delta \phi=c_{1} r^{p_{1}}+m(\sigma-\varphi) \tag{8}
\end{equation*}
$$

and the rotation in eq. (3) provides for the new orientation angle

$$
\begin{equation*}
\varphi_{R}=\varphi+\Delta \phi \tag{9}
\end{equation*}
$$

II. The non-radial translation $\vec{r}_{T}=\mathbf{T}\left(\vec{r}_{R}, \varphi, \Delta r\right)=\mathbf{T}\left(\vec{r}_{R}, \overrightarrow{\Delta r}\right)$ of the rotated $\vec{r}_{R}$ by $\overrightarrow{\Delta r}$ in the initial $\varphi$-direction (and not in the actual $\varphi_{R}$ - direction of eq. (9))

$$
\begin{equation*}
\vec{r}_{T}=\mathbf{T}\left(\vec{r}_{R}, \overrightarrow{\Delta r}\right)=\vec{r}_{R}+\overrightarrow{\Delta r} \tag{10}
\end{equation*}
$$

where the translation in eq.(10) is given by

$$
\begin{equation*}
\overrightarrow{\Delta r}= \pm\binom{\Delta x}{\Delta y}= \pm|\overrightarrow{\Delta r}|\binom{\cos \varphi}{\sin \varphi}=c_{3} r^{p_{3}}\binom{\cos \varphi}{\sin \varphi} \tag{11}
\end{equation*}
$$

Eq.(11) generalizes the usual "jump" in $x$-direction, where the unit jump has $|\overrightarrow{\Delta r}|=1$ or always $c_{3}= \pm 1$ with $p_{3}=0$. In this paper we will consider for simplicity only unit jumps and in chapter 7 negative jumps with $c_{3}=-1$.
III. Finally there could be an additional inversion operation with respect to the origin

$$
\begin{equation*}
\vec{r}_{I}=\mathbf{I}\left(\vec{r}_{T}\right)=\vec{r}_{T} /\left|\vec{r}_{T}\right|^{2} \tag{12}
\end{equation*}
$$

with invariant direction angle but inverse length to get a rotation-translation-inversion sequence.

## 3 Categories of Jumping Patterns

Without orbital rotation the jumps would only go in one direction (usually the $x$ direction) and disappear to infinity. There are two ways to form a local density: either a jump-back boundary condition or rotating the jump direction. The v. Kármán Street pattern in fig. 1 and 2 and the typical $2 d$ wave pattern on the plane in fig. 3 have a jump-back distance condition in the $x$-direction, which means, if the distance to the origin in jump direction exceeds a limit, a jump back near to the origin follows.


Fig. 2: Long jump back with 6 arm-symmetric v. Kármán Street pattern, $c_{2}=$ $2 \pi / 6$, with $\mathrm{m}=2, \mathrm{c}_{1}=\pi, \mathrm{p}_{1}=-2$, and $\mathrm{p}_{2}=0$.

With a new extra orbital rotation orthogonal to the jump direction we get local chaotic patterns without any orbital conditions (no jump condition) similar to the orbital solutions we know from quantum mechanics. We will call them "Localized Autonomous Chaotic Orbital Patterns" or LACOP, where we get
many interesting orbital structures for integral $p_{1}, p_{2}, m$. It makes sense to group the jumping patterns according to the boundary conditions, where we have identified five categories given by a

1. random starts condition: long jump back (red arrow) randomly near to center, if distance or number of jumps exceeds a limit (see examples in figs. 1,2,6,7).
2. periodic boundary condition: a defined long jump back to the back side, if distance exceeds a limit, see fig.3.


Fig. 3: periodic boundary jump back conditions producing waves, where $p_{1}<0, p_{2}=0, m=2$.
3. inversion condition: inversion operation $\vec{r}^{\prime}=\vec{r} / r^{2}$ if distance $r$ exceeds a limit $r_{\text {max }}$.
4. no boundary condition: free orbital jumps around the center according to orbital quantum numbers and symmetries producing LACOP.
5. parameter conditions: relating the two parameter $c_{1}, c_{2}$ can define a family of patterns, e.g., if we define a small isotropic geometric phase shift gap $g \ll 1$ and relating the coefficients via the gap $g$ geometrically to the rotation-translation parameter according to

$$
\begin{equation*}
c_{1}=\arccos (1-g), c_{2}=\pi(1-g) / M . \tag{13}
\end{equation*}
$$

In this case we get with $=1, c_{3}=-1, p_{1}=0, p_{2}=1$ special radial conditions like spirals intersected by radial rays, see figs. 6 and 7 with random starts near to the center (condition 1).

## 4 Physics Relevance

It is interesting comparing these structures to Quantum Electrodynamic and spin symmetries. The parameter conditions in condition categories 5 and 6 . In
chapter 3 can be combined with condition categories $1,2,3,4$, where $p_{2}=-1$ with $m$ - pole provides for multipole type orbital ring clouds and field structures:


Fig. 4: $p_{1}=p_{2}-1$ with $p_{2}=2$, left: LACOP with $m=1$, right: random starts with $m=2$.

## 5 Simple $m=0$ LACOP

Both regular and highly non-linear or chaotic are the $m=0$ patterns in a wide parameter range, see fig.5:


Fig. 5: Vivid LACOP with regular orbital structure, examples of a regular pattern with $m=0, p_{1}=0, p_{2}=-1, p_{3}=0$.

## 6 Monopole $m=1$ Negative Jumps $c_{3}=-1$ with Random Start near to Center

Both, highly chaotic and regular are the $m=1, c_{3}=-1$ patterns in a wide parameter range, see figs. 6 and 7, where the geometric gap condition in eq. 13 produces radial rays intersecting spirals with random starts. It is not surprising to get Fresnel Charge structures in fig. 6 for $p_{2}=1$, since a Fresnel spiral has an inherent spiraling angular structure with $\varphi \propto \sum \sigma \propto r^{p_{2}+1}=r^{2}$.


Fig. 6: $m=1, c_{3}=-1, p_{1}=0, p_{2}=1$, geometric gap condition eq. 13 producing radial rays.


Fig. 7: $m=1, c_{3}=-1, p_{1}=0, p_{2}=1, g=0.001$, left and $\operatorname{mid} M=3$, right $M=13$.

With a second relation or constraint between the two parameter $c_{1}, c_{2}$ we can a fixed points for the parameter, e.g., combining eq.(13) with the simple isotropic gap condition $c_{1}=c_{2}$, we get the magic angle condition $M c_{1}=\cos \left(\pi c_{1}\right)$ Binder [5]. With no boundary condition 4. we get $M$-gonal fixed points on the monopole ring.

## 7 More Complex Vivid $m=2$ LACOP

Very interesting and exciting is the chaotic dynamics of the $m=2$ LACOP orbitals, see fig. 8 and some mixed examples in fig. 9 . There is in most cases no static or stationary solution, since the orbitals often show a chaotic variation in the orientation or orbital shape. Therefore, a stable LACOP should be properly initialized; usually by an high enough orbital rotation parameter $c_{2}$ (spin, energy) while increasing the $m$-parameter from 1 to 2 . In simulations the solutions (recorded as videos) appear to be like vivid orbitals with inherent chaotic dynamics especially in the substructure of orbital rings:


Fig. 8: Vivid LACOP steadily changing orbital structure (the two examples are snapshots from the same LACOP) with $m=2, p_{1}=-1, p_{2}=1, p_{3}=0$,

$$
c_{1} \approx 3 \pi, c_{2} \ll 1 .
$$



Fig. 9: typical LACOPs with a core and a hole in the center

$$
p_{1}<p_{2}, m>0 .
$$

## Conclusion

The resulting patterns often show a localized ring shape with several mixed orbits in a kind of hydrodynamic-type orbital shelf flow and an empty region or hole at the center. $m$ - poles or reflection modes with higher $m$ show more complex and instable pattern. A stable form must be initialized; otherwise the pattern collapses or expands to infinity. The emerging LACOP solitons or wavelets show always characteristic

- radial and orbital wave numbers,
- radial and orbital symmetries,
- parameter $m, c_{i}, p_{i}$,
- dynamics and fluctuations,
- geometric phase conditions.

Finally, we propose that this computer experiments show some relevance to quantum physical systems since we have

- a wave attractor showing quantization effects in terms of rotational units,
- a quantization of monopole and multipole charges,
- a basic non-zero quantum spin in the two main operators (rotationtranslation, non-commutative) with characteristic geometric phase shifts,
- point-like local events with emerging global wave-like probabilistic patterns.

There will be videos of simulations available on the internet with title "Dynamic Autonomous Chaotic Orbital Patterns" or tag "\#DACOPSimulation".

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# New discrete chaotic cipher key generation for digital embedded crypto-systems 

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#### Abstract

To benefit of embedded systems which are highly customized providing architectures to suit real-time computing, optimized unit size and low power consumption require highest levels of data communication security which are very useful advantage for telecommunications and networks and Internet of Things (IoT) communication applications that handle sensitive information. This paper presents an extracted new 5 -dimensional (5-D) discrete time chaos system to generate a robust chaotic cipher data stream to ensure encryption application for secure communication. Dynamical behaviors and security analysis are investigated and compared to current discrete chaotic maps proving its suitability for embedded data encryption systems. Field-Programmable Gate Array (FPGA) implementation design shows better performance and good security robustness compared with previous works while proving the performance improvement of the proposed cipher block in terms of throughput, used hardware logic resources, and resistance against most cryptanalysis attacks.


Keywords: Chaotic generator, Secure communication, Key space, Encryption, FPGA implementation.

## 1 Introduction

Due to advancement in technology, thousands of devices in home, industry and health care systems are connected to each other. With advent of IoT, this number is increasing exponentially [20]. With the provision of too many features, these devices still need more protection from cyber-attacks [9]. Due to the existing low security protocols, these devices are always on high threat, which makes the communication vulnerable and prone to foreign intrusion. In the literature, many schemes and protocols have been proposed to address the network security issue. In [21] authors have proposed an unclonable-based functions for authentication the protocol, for IoT devices. However, the proposed protocol lack some related computations [14]. In [16] authors have presented a simplified protocol to reduce the computations through the authentication phase But the complexity of the proposed scheme is not appropriate for the IoT devices [13]. Moreover, Random numbers are used in many cryptographic protocols, key management, identity authentication, image encryption, and so on [23]. As software generated random sequences are not truly random, fast entropy sources such as quantum systems or classically chaotic systems can be
viable alternatives provided they generate high-quality random sequences sufficiently fast [3]. In [2], authors have introduced a new hardware chaos-based pseudo-random number generator, which is mainly based on the deletion of an Hamilton cycle within the N -cube plus one single permutation. [7]initiates a systematic methodology for real-time chaos-based video encryption and decryption communications. Based on the fundamental anti-control principles of dynamical systems, a novel 6-dimensional real domain chaotic system is designed, and then the corresponding algorithm is developed. The proposed algorithm is utilized to design a real-time chaos-based secure video communication system, with a generalized design principle derived, which is implemented on an FPGA hardware. Additionally, some other research works have been proposed which include, applications of chaotic systems for speech signal encryption [18], e-mail and database encryption [19] and image encryption [8,10,12,15,6].Unfortunately due to different technical reasons, all these schemes are not useful for different IoT applications. Because the discovery of simple chaotic systems with complex dynamics has always been an interesting research work [11], we propose through this paper an extracted low resource consumption 5-D Chaotic System has been proposed for secure IoT communication. The proposed chaos-based cryptosystem is implemented by using Xilinx ZYNQ-XC7Z020 FPGA board. The rest of this paper is organised as follows. Section 2 describes the proposed 5-D map. Hardware implementation and performance analysis are presented in section 3. Finally, section 4 concludes this paper.

## 2 The proposed 5-D map

In [5] authors proposed a multidimensional chaotic map within good chaotic properties. From the proposed system, the extracted 5-D discrete time chaos system with nine nonlinear terms and five control parameters is described as follows:

$$
\left\{\begin{array}{l}
X(n+1)=1-a * X(n)^{2}+(Y(n) * Z(n) * W(n) * P(n))  \tag{1}\\
Y(n+1)=1-b * Y(n)^{2}+(X(n) * Z(n) * W(n) * P(n)) \\
Z(n+1)=1-c * Z(n)^{2}+(X(n) * Y(n) * W(n) * P(n)) \\
W(n+1)=1-d * W(n)^{2}+(X(n) * Y(n) * Z(n) * P(n)) \\
P(n+1)=e * X(n) * Y(n) * Z(n) * W(n)
\end{array}\right.
$$

Whereas $a, b, c, d$ and $e \in \mathrm{R}$ are the controllers and $X, Y, Z, W, P$ are the state variables respectively.

### 2.1 Bifurcation analysis

To investigate behaviours of the proposed system defined by the proposed 5-D map, we analyze the bifurcation diagrams related to parameters $a, b, c, d$ and $e$. According to the bifurcation study, chaotic behaviour of the proposed system appears for $a \in[0.8,1.8], b \in[0.1,1.4], c \in[0.7,1.9], d \in[0.3,1.6]$ and $e \in$ $[0.05,1.1]$ as shown in Figure 1 (the bifurcation study of the parameters b, d).


Fig. 1: The bifurcation graphs of the proposed 5 -D map with parameters b, d.

### 2.2 Signals analysis

To study the dynamical behaviours of the proposed model, the signal graph and the phase space trajectories defined by the state variables $(X, Y, Z, W, P)$ can be used as an indicator to determine that the motion of that system is chaotic. In the proposed work, the technique is based on the signal output and the projection of the trajectories onto the plane, which reflects the chaotic behaviour result of the proposed system as shown in Figures 2 and 3, respectively.


Fig. 2: Signals graphs of the proposed 5-D map

### 2.3 Sensitivity analysis

To evaluate the sensitivity to initial conditions of the proposed map, we consider a changing by $10^{-10}$ of the initial values $X(0), Y(0), Z(0), W(0)$ and $P(0)$, then for the parameters $a, b, c, d$ and $e$. The results shown in Figures 4, 5, 6,


Fig. 3: Trajectory graph of the proposed 5-D map

7 and 8 prove that after a few number of iteration all the signals are different from the initial ones.


Fig. 4: The proposed bench test platform


Fig. 5: The proposed bench test platform


Fig. 6: The proposed bench test platform


Fig. 7: The proposed bench test platform


Fig. 8: The proposed bench test platform

## 3 FPGA implementation of secure Peer-To-Peer communication

### 3.1 The proposed platform test bench

Because of flexibility, reliability, low cost, fast time-to-market, and long term maintenance, FPGA environment is considered more useful for the validation
of the proposed scheme. Initially the hardware description language (VHDL) is used to implement the proposed 5-D system as a chaos-based cryptosystem (called Chaos 5-D Generator). After that, the designed is integrated as a new core or module with the other components of the Xilinx ZYNQ-XC7Z020 FPGA board as given in Figure 9 [5].


Fig. 9: FPGA block design

To establish the final platform of Peer-To-Peer secure communication, we connect all the programmed FPGA boards through an Ethernet network (Fig.10).


Fig. 10: The proposed bench test platform

The proposed bench test runs as follows:

- The peers establish a new connection;
- If it is the first connection $(i=1)$, then key-Generator modules are reset;
- The key-Generator of both peers load the key corresponding to the sequence $i$ of the implemented 5 -D map;
- The generated keys are used to encrypt and send the data;
- The generated keys are used to decrypt and read the received data;

Lastly, the peers terminate the communication by closing the channel, and the key-Generator module saves the samples $(i+1)$ for the next communication.

### 3.2 Performance analysis

In such kind of cryptosystems, without any robust experimental solution to the chaotic synchronization issue [17], we introduce the control option for the chaos 5 -D generator. Hence for each established connection $i$, the proposed cryptosystem generates the same encryption key which corresponds to the sample $i$ of the proposed 5-D chaotic map.

Ensuring that both generators implemented on two different boards generate the same keys, thereby, the connected peers can encrypt/decrypt exchanged messages easily as shown in Figure 11.


Fig. 11: Exchanged messages between the peers

To show the performance of our system, the proposed model is compared with some of the state of the art works. Table 1 gives the details of the comparison. The comparison is made on the basis of LUTs, FF, DSP, maximum frequency and Slices. From table 1, it is evident that the proposed scheme has the best results with 2570 ( $4.83 \%$ from available) lookup tables (LUTs), 872 Slices ( $6,55 \%$ ), 2570 flip-flops (FF), 111 DSP multipliers ( $50,45 \%$ ) and no block RAMs, all at maximum frequency of $553,09 \mathrm{MHz}$. These results are never achieved before, which confirms the novelty and suitability of the proposed scheme.

## 4 Conclusion

With the development of chaotic theory and its applications in different domains, proposals of constructing new and higher dimensional chaotic systems
$\left.\begin{array}{|l|l|l|l|l|l|l|}\hline \text { Proposals } & \begin{array}{l}\text { Resources } \\ \text { Available }\end{array} & \begin{array}{ll}\text { LUTs } \\ 53200\end{array} & \text { FF } \\ 106400\end{array}\right)$

Table 1: FPGA resource utilization comparison
become one rising trend.
In this paper, we proposed an extracted 5-D discrete chaotic map for key stream cipher generation. The chaotic behaviour of the proposed map is investigated using the bifurcation and the trajectory analysis.
Compared to some well know chaotic systems, the proposed 5-D map presents better properties in terms of resource consumption and achieved frequency. Moreover, the proposed 5-D chaotic map provides an attractive trade-off between key space, resource utilization and memory consumption proving its suitability for securing communications of resource-constrained devices such as IoTs.

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# A Survey on Chaos-Based Cryptosystems: Implementations and Applications 

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#### Abstract

Chaos theory is considered as a tool for studying the systems that show divergence and disorder. After having used discrete mathematics to deduce nonconvergence situations, these theories are modeled in the form of a dynamic system and are applied in several domains such as electronic, mechanic, network security, etc. In network security domain, the development of new cryptosystems based on chaos is a relatively new area of research and is increasingly relevant. The essence of the theoretical and practical efforts in this field derive from the fact that these cryptosystems are faster than conventional methods, while ensuring performance of security, at least similar. In this paper, we discuss several proposals about chaos-based cryptosystem and pseudo-random number generator (PRNG). Moreover, topology and architecture of the proposed chaos systems are detailed. Finally, in order to show the more suitable system for encryption and secure communication, a synthesis comparison is presented and considered.


Keywords: Chaos, Network security, crypto-systems, Communication, PRNG.

## 1 Introduction

Nowadays, network communication is vulnerable to many threats and cyberattacks and it becomes more important for network experts to safeguard the network access [1]. Among the available security mechanisms, chaos-based cryptosystems are considered one of the most effective solution that provides the integrity, the authentication and the confidentiality. Recently, the development of new cryptosystems based on chaos is a relatively new area of research and is increasingly relevant.
In [2], an Field-Programmable-Gate-Array (FPGA) implementation of image encryption purpose using two chaotic discrete time systems. The proposed two phases algorithm is executed by using the well known Arnold Cat map and the generalized logistic map, respectively. Authors in [3] initiate a systematic methodology for securing real-time video communication. The proposed chaos-based cryptosystem have been implemented on an FPGA hardware platform via Verilog Hardware Description Language (Verilog HDL). [4] presents a Hardware implementation of a Pseudo chaos signal generator using three reconfigurable discrete time systems with a linear feedback shift registers (RLFSR).

The proposed technique was implemented using Verilog HDL codes, then analyzed using Xilinx Plan Ahead compiler and Model-sim software. In terms of network security protocols, [5] proposed a novel chaos-based mechanism that includes Pseudo-Random Key-Generator which can be used to secure a socketbased communication. The proposed key-generator, created by solving the Lorenz chaos-system, has the main task of delivering at each opened channel a new 32-bit key that is used for encrypting/decryption the exchanged data.

In this paper, we discuss several proposals about chaos-based cryptosystem and pseudo-random number generator (PRNG). Moreover, topology and architecture of the proposed chaos systems are detailed. Finally, in order to show the more suitable system for encryption and secure communication, a synthesis comparison is presented and considered.
The remainder of this paper is structured as follows. Section 2 describes the classification of the most used chaotic systems. Section 3 shows the hardware implementations of these chaotic systems as well as their purposes. Section 4 concludes this paper.

## 2 Background and description of chaotic systems

Due to the sensitivity and periodicity properties, chaotic systems have been involved mainly in key generation of the recently proposed cryptography schemes. Regarding their topology and mathematical model, we can classify all existing and newly proposed chaos systems in two main categories: continuous-time systems and discrete-time systems.

### 2.1 Continuous time systems

The continuous-time systems are described by a set of linear differential equation. Moreover, in order to ensure that the dynamical systems to be chaotic, the dimensions of the system's phase space must be at least equal to three (3). In the literature, there are several well known continuous-time systems such as Lorenz [6], Chen [7], Lu [8], etc.

Lorenz system The basic form of the Lorenz 3-D system is described by the following set of equation:

$$
\begin{aligned}
\dot{x} & =a(y-x) \\
\dot{y} & =y+b x-x z \\
\dot{z} & =x y-c z
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}$ and z are the state variables. $\mathrm{a}, \mathrm{b}$ and c are the system parameters. The chaotic behaviour (see Fig.1) appears for $a=10, b=28$ and $c=8 / 3$ with the initial conditions $x_{0}=0, y_{0}=5$ and $z_{0}=25$ [8].


Fig. 1. Trajectory graph of the Lorenz system

Van-der-Pol system The Van-der-Pol oscillator as given in [10], is described in two dimensions as follows:

$$
\begin{aligned}
& \dot{x}=a\left(x-(1 / 3) x^{3}-y\right) \\
& \dot{y}=(1 / a) x
\end{aligned}
$$

Where $x, y$ are the state variables, and a is the system controller. The phase portrait of the 2-D system is illustrated in figure 2 .


Fig. 2. Phase plan projection of the Van-der-Pol system

Chen system Based on the 3-D Lorenz system, Chen 3-D system is proposed and described by the following set of equations:

$$
\begin{aligned}
\dot{x} & =a(y-x) \\
\dot{y} & =(b-a) x+b y-x z \\
\dot{z} & =x y-c z
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}$ and z are the state variables. $\mathrm{a}, \mathrm{b}$ and c are the system parameters. The chaotic behaviour appears for $\mathrm{a}=35, \mathrm{~b}=28$ and $\mathrm{c}=8 / 3$ [19], while the phase plan projection is shown in figure 3 .


Fig. 3. Phase plan X-Y projection of the 3-D Chen system

Lu system The Lu system is known as the bridge between Lorenz system and Chen system [8]. Thereby, the mathematical model is given as follows:

$$
\begin{aligned}
& \dot{x}=a(y-x) \\
& \dot{y}=b y-x z \\
& \dot{z}=x y-c z
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}$ and z are the state variables. $\mathrm{a}, \mathrm{b}$ and c are the system parameters. The trajectory graph of the proposed system is given in figure 4 .


Fig. 4. Trajectory X-Y-Z of the 3-D Lu system

Linz-Sprott system Trying to simplify the formula of a chaotic system, Linz and Sprott [19] have proposed a new system which is defined as follows:

$$
\begin{aligned}
& \dot{x}=y \\
& \dot{y}=z \\
& \dot{z}=-a z-y-1+|x|
\end{aligned}
$$

Where $x, y$ and $z$ are the state variables and $a$ is the system's parameter. As shown in figure 5, the chaotic behaviour of the proposed system is achieved for $\mathrm{a}=0.6$.


Fig. 5. Trajectory X-Y-Z of the Linz-Sprott system

Four-Wing memristive hyperchaotic system Looking for higher dimensional chaotic system, authors in [13] have proposed a novel 4-D system which is described as follows:

$$
\begin{aligned}
& \dot{x}=a x+b y z \\
& \dot{y}=c y+d x z-p y W(w)-Q \\
& \dot{z}=e z+f x y+g x w \\
& \dot{w}=-y \\
& W(w)=m+3 n w^{2}
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}$ are the state variables. $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{m}, \mathrm{n}, \mathrm{p}, \mathrm{Q}$ are the controllers of the proposed system. In order to ensure the chaotic behaviour, the controllers parameters are defined as follows: $a=0.35, b=-10$, $\mathrm{c}=-0.6, \mathrm{~d}=0.3, \mathrm{e}=-1.6, \mathrm{f}=2, \mathrm{~g}=0.1, \mathrm{~m}=0.1, \mathrm{n}=0.01, \mathrm{p}=0.2$ and $\mathrm{Q}=0.01$. The trajectory graphs corresponding to the proposed system with the associated parameters, are shown in figure 6.


Fig. 6. Trajectory graphs of the proposed 4 -Wing system

New 3-D continuous time system Getting inspired from the Lorenz system[9], with only two (02) controllers, a novel 3-D system is proposed and defined as follows:

$$
\begin{aligned}
& \dot{x}=y-x-a z \\
& \dot{y}=x z-x \\
& \dot{z}=-x y-y+b
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}$ and z are the state variables. a and b are the system parameters. The chaotic behaviour of the proposed system is observed for the values $a=0.5$ and $b=1$ while the initial conditions are $x_{0}=y_{0}=z_{0}=0$ (See Fig.7).


Fig. 7. Phase plan projections of the proposed 3-D system

New 4-D continuous time system In [20], another new 4-D chaotic system is proposed based on the Rossler system, and defined by the following set of equations:

$$
\begin{aligned}
& \dot{x}=-y-z+d w \\
& \dot{y}=x+a y \\
& \dot{z}=b+z(x-c)-a(y-w) \\
& \dot{w}=a z-w
\end{aligned}
$$

Where $\mathrm{x}, \mathrm{y}, \mathrm{z}$ and w are the state variables. $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are the system parameters. By choosing $\mathrm{a}=0.4, \mathrm{~b}=0.6, \mathrm{c}=3$ and $\mathrm{d}=0.8$, the chaotic behaviour of the proposed system is showed by phase plan projection (see Fig.8)


Fig. 8. Phase plan projection of the proposed system


Fig. 9. Trajectory graph of the logistic map


Fig. 10. Signal graph of the logistic map

### 2.2 Discrete time systems

The discrete time chaos system is a dynamic system which works in increments and takes the conditions at a given time $t$ to change these conditions at a later time $t+\Delta t$. Hence, unlike the mathematical model of the continuous time systems, discrete time maps are described mathematically by an iterated function. Moreover, the dimension of the system's phase space could be only equal to one (01) to show chaos behaviour.

Logistic map In the literature, many proposals have used the well known logistic map such as in [11] for PRNG, [2] for image encryption,[15] for chaotic signal generating, etc. The mathematical description of this map is given as follows:

$$
x_{i+1}=a x_{i}\left(1-x_{i}\right)
$$

Where $x_{i}$ is the state variable and $a$ is the system controller.
To ensure the chaotic behaviour (see Fig. 9 and Fig.10) of this system, $a$ should be in the interval [3.57-4].

Hénon map A simple 2-D with quadratic non-linearity, Hénon system was the first map to show strange attractor with a fractal structure [14]. The mathematical description of this map is given as follows:

$$
\begin{aligned}
& x_{i+1}=a+y_{i}-x_{i}^{2} \\
& y_{i+1}=b x_{i}
\end{aligned}
$$

Where $x_{i}$ and $y_{i}$ are the state variables and $a, b$ are the system controllers. The obtained strange attractor of this map, is shown in figure 11 while the controllers are $\mathrm{a}=1.4$ and $\mathrm{b}=0.3$.


Fig. 11. Trajectory graph of the Hénon map

Rene-Lozi map By introducing the absolute value in the Hénon map, the Rene-Lozi map used in [12] for stream cipher purpose, is described as follows:

$$
\begin{aligned}
& x_{i+1}=1+y_{i}-a\left|x_{i}\right| \\
& y_{i+1}=b x_{i}
\end{aligned}
$$

Where $x_{i}$ and $y_{i}$ are the state variables and $a, b$ are the system controllers. Similarly to the Hénon map, it has been shown that for $\mathrm{a}=1.4$ and $\mathrm{b}=0.3$, chaotic behaviour of this map can appear (see Fig.12).


Fig. 12. Trajectory graph of the Rene-Lozi map

Bernoulli map Unlike all the discrete time maps, Bernoulli map is composed of two piece-wise linear parts which are separated by a discontinue space of points [13] (see Fig.13).
Mathematically, the Bernoulli map is defined as follows:

$$
x_{i+1}= \begin{cases}a x_{i}+0.5 & \text { if } x<0 \\ a x_{i}-0.5 & \text { if } x \geq 0\end{cases}
$$

Where $x_{i}$ is the state variable and $a$ is the control parameter.
The chaotic status of this map is ensure for all the values of the parameter $a$ inside the interval ]1.4-2] (see Fig.14).


Fig. 13. Trajectory graph of the Bernoulli map


Fig. 14. Signal graph of the Bernoulli map

Sine map the sine map is qualitatively similar to the logistic map, and the superficial similarity has resulted in a much deeper connection.
As indicated by its name, the sine map is defined by a sine function as follows:

$$
x_{i+1}=\operatorname{asin}\left(\pi x_{i}\right), 0 \leq x_{i} \leq 1, a>0
$$

Where $x_{i}$ is the state variable and $a$ is the system parameter. The projection graph which proves the behaviour of this map is shown in figure 15.

Tent map Regarding the slope of its mathematical function, tent map with only one state variable, is considered as a slope of two (02) model. Without


Fig. 15. Trajectory graph of the sine map
any control parameter, the tent map is defined as follows:

$$
x_{i+1}= \begin{cases}2 x_{i} & \text { if } 0 \leq x_{i}<1 / 2 \\ 2\left(1-x_{i}\right) & \text { if } 1 / 2 \leq x_{i} \leq 1\end{cases}
$$

Where $x_{i}$ is the state variable. Moreover, the trajectory graph of the tent map is shown in figure 16 .


Fig. 16. Trajectory graph of the tent map

All these systems have been used mainly for either generating random numbers, cipher keys or chaotic signals. They differ from each other in terms of dimension, control parameters and the purpose of use. In table 1 we summarize all these differences obtained regarding our study.

## 3 Hardware implementations and applications

FPGA-based prototyping is specifically geared toward meeting the design and verification demands created by the complexities of low and constrained resources devices. Moreover, FPGA-based prototyping allows designers to develop and test their systems and provides software developers early access to a fully functioning hardware platform long before silicon is available. In order to be implemented on FPGA, the continuous time systems need to be discredited numerically using some popular methods such as Euler and Runge-Kutta (RK)

| System | Reference | Type | Dimension | Controllers | Purpose |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Lorenz | $[8]$ | Continuous | 3-D | 3 | Image encryption |
| Van-der-Pol | $[10]$ | Continuous | 2-D | 1 | Random number generator |
| Chen | $[19]$ | Continuous | 3-D | 3 | Chaos signal generator |
| Lu | $[8]$ | Continuous | 3-D | 3 | Image encryption |
| Linz-Sprott | $[19]$ | Continuous | 3-D | 1 | Chaos signal generator |
| 4-Wing | $[13]$ | Continuous | 4-D | 11 | Random number generator |
| New 3-D | $[9]$ | Continuous | 3-D | 2 | Random number generator |
| New 4-D | $[20]$ | Continuous | 4-D | 4 | Image processing |
| Logistic | $[11]$ | Discrete | 1-D | 1 | Random bit Generator |
| Hénon | $[14]$ | Discrete | 2-D | 2 | Encryption |
| Rene-Lozi | $[12]$ | Discrete | 2-D | 2 | Stream cipher |
| Bernoulli | $[13]$ | Discrete | 1-D | 1 | Random number generator |
| Sine | $[15]$ | Discrete | 1-D | 1 | Chaos signal generator |
| Tent | $[16]$ | Discrete | 1-D | 0 | Chaos signal generator |

Table 1. Summary of the chaotic systems: description and purpose of application
methods. Euler's method is a straight-forward method that estimates the next point based on the rate of change at the current point and it is easy to code [24]. It is called also a single step method. While RK methods are actually a family of schemes derived in a specific style. Higher order accurate RK methods are multi-stage because they involve slope calculations at multiple steps at or between the current and next discrete time values [25]. The next value of the dependent variable is calculated by taking a weighted average of these multiple stages based on a Taylor series approximation of the solution. The weights in this weighted average are derived by solving non-linear algebraic equations which are formed by requiring cancellation of error terms in the Taylor series. Developing higher order RK methods is tedious and difficult without using symbolic tools for computation. The most popular RK method is RK4 since it offers a good balance between order of accuracy and cost of computation. RK4 is the highest order explicit Runge-Kutta method that requires the same number of steps as the order of accuracy (i.e. RK1=1 stage, RK2=2 stages, RK3 $=3$ stages, RK4 $=4$ stages, RK5 $=6$ stages, ...). Beyond fourth order the RK methods become relatively more expensive to compute. Among all the studied proposals, we have synthesised a brief comparison that includes mainly the used FPGA technology and the consumed resources. Table 2 summarizes the difference between different proposals regarding the chosen system as well as the resource consumption. However, we found that in the single-precision and the double-precision operations, there are more than 10-6 differences in less than 100 iterations, and the difference reaches more than one digit after 1000 iterations[23]. This is because the binary has a round-off error, so the binary cannot strictly obey the commutative law or the distribution law in floating-point operations.

| Reference | FPGA technology | Resources | Chaos system | Discretization |
| :--- | :--- | :--- | :--- | :--- |
| $[9]$ | Virtex-6 | LUTs $=1070$ Regs $=1196$ | New 3-D | Euler |
| $[10]$ | Virtex-6 | LUTs $=22674$ Regs $=21797$ | Van-der-Pol | RK4 |
| $[11]$ | Virtex-7 | LUTs $=510$ Regs $=120$ | Logistic | No Need |
| $[12]$ | Spartan-6 | LUTs $=562$ Regs $=386$ | Rene-Lozi | No Need |
| $[13]$ | ZYNQ-XC7Z020 | LUTs $=22556$ Regs $=26426$ | Four-wing | RK4 |
| $[14]$ | Virtex-5 | LUTs $=1496$ Regs $=432$ | Hénon | No Need |
| $[8]$ | Virtex-II | LUTs $=2490$ Regs $=1316$ | Lorenz/Lu | RK-4 |
| $[17]$ | Virtex-5 | LUTs $=2799$ Regs $=1722$ | Logistic | No Need |
| $[18]$ | Zynq-7000 | LUTs=856 Regs $=521$ | Hénon | No Need |
| $[22]$ | Stratix-IV | LUTs $=49005$ Regs $=611$ | New 3-D | Euler |

Table 2. Summary of the FPGA implementations

## 4 Conclusion

In this paper, we discuss several proposals about chaos-based cryptosystem and pseudo-random number generator (PRNG). Moreover, topology and architecture of the proposed chaos systems are detailed. Finally, in order to show the more suitable system for encryption and secure communication, a synthesis comparison is presented and considered.

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# Approximate Methods for Solving Hypersingular Integral Equations on Fractals 

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## Headings content

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3.1. Riemann integrals
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4. Approximate solution of hypersingular integral equations on prefractals
5. Solution of singular integral equations
6. Solution of hypersingular integral equations
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#### Abstract

The paper consists of three parts. The first one is devoted to approximate methods for evaluating Riemann integrals, singular and hypersingular integrals on closed non-rectifiable curves and fractals in the complex plane. An integral on non-rectifiable curves or fractals is defined as a double integral over a region that bounded by a non-rectifiable curve or a fractal. To evaluate double integral cubature formulas have been constructed.

The second part contains methods for solving hypersingular integral equations on prefractals. Issues of solvability of singular and hypersingular integral equations with fractal in the right-hand side have been studied in the third part. Singular and hypersingular integral equations that model aerodynamics problems have been investigated. In such cases right-hand side of equations describes the gas flow which is a fractal.


## Abbreviations

SI - singular integral
HI - hypersingular integral
SIE - singular integral equation
HIE - hypersingular integral equation

## 2. Introduction

### 2.1. Review of approximate methods for calculating hypersingular integrals and solving hypersingular

integral equations. Starting the middle of the last century, the methods of singular and then hypersingular integral equations have been increasingly used in the study and modeling of various problems in physics, natural science and technology: in aerodynamics, electrodynamics, elasticity theory, nuclear and atomic physics, geophysics, and mathematical physics. Analytical methods for solving singular and hypersingular integral equations are known only for very special cases. Thus numerical methods are widely employed for solving singular and hypersingular integral equations [1] - [7].

The development of approximate methods for solving singular integral equations (SIE) started in the 50s of the last century. The number of publications devoted to approximate methods for solving SIE and their generalizations and related Riemann and Hilbert boundary problems has not decreased up-to-date. Main approximate methods for solving SIE are presented in [1], [2], which contain extensive bibliography.

It is interesting to note that hypersingular integrals (HI) were introduced to mathematical world around the same time as singular integrals (SI). However the development of approximate methods for solving hypersingular integral equations (HIE) started later than the development of similar methods for solving SIE. Today HIE is the fast growing field in mathematics.

An intense development of approximate method for solving SIE and HIE is caused by their numerous applications. In particular, SIE and HIE are main mathematical engine in antenna theory, composite materials theory, metamaterials.

Main approximate methods for solving HIE can be found in the publications [3]-[7].
Effective approximate methods for evaluating SI and HI are required to implement numerical methods for solving SIE and HIE. Analytically singular and hypersingular integrals can be evaluated pretty rare. Lack of analytical methods require the development of numerical methods for evaluating SI and HI .

There are numerous publications devoted to evaluate SI and HI over smooth curves issues. The bibliography is presented in [1], [5], [8], [9], [10]. The authors do not know about works devoted to numerical methods forcalculation SI and HI and solution SIE and HIE over fractals.

Recently the need for study of physical and technical processes on fractals has appeared. First, it should be noted synthesis and analysis of the fractal antenna problems [11], and the microwave theory and technique. It is important to know, different antenna types are modeled by SIE and HIE. Obviously, the development of approximate methods to solve SIE and HIE on non-rectifiable curves and fractals for modeling electrodynamic processes in fractal antennas will be required.

The chapter is devoted to approximate methods for calculating singular and hypersingular integrals and solving singular and hypersingular integral equations over non-rectifiable curves and fractals.

### 2.2. Definitions.

Let $L$ be a contour on the complex plane. Let $A=[a, b]$ or $A=L$.
Definition 2.1. Class of Holder functions $H_{\alpha}(M ; A)(0<\alpha \leq 1)$ consists of functions $f(x)$ given on $A$
and satisfying at all points $x^{\prime}$ and $x^{\prime \prime}$ of this set the inequality $\left|f\left(x^{\prime}\right)-f\left(x^{\prime \prime}\right)\right| \leq M\left|x^{\prime}-x^{\prime \prime}\right|^{\alpha}$.
Definition 2.2. The class $W^{r}(M ; A)$ consists of functions defined on $A$, continuous and having continuous derivatives up to $(r-1)$-th order inclusive and piecewise continuous derivative $r$-th order satisfying on this set the inequality $\left|f^{(r)}(x)\right| \leq M$.

Definition 2.3. The class $W^{r} H_{\alpha}(M ; A)$ consists of functions $f(x)$ belonging to the class $W^{r}(M ; A)$ and satisfying the additional condition $f^{(r)}(x) \in H_{\alpha}(M)$.

Definition 2.4 [12]. Let $\varphi(t) \in W^{p-1} H_{\alpha}(M, A)$. The Integral $\int_{a}^{b} \frac{\varphi(\tau) d \tau}{(\tau-c)^{p}}, \quad a<c<b, p=2,3, \ldots$, in the sense of Cauchy - Hadamard principal value is called the limit: $\int_{a}^{b} \frac{\varphi(\tau) d \tau}{(\tau-c)^{p}}=\lim _{v \rightarrow 0}\left[\int_{a}^{c-v} \frac{\varphi(\tau) d \tau}{(\tau-c)^{p}}+\int_{c+v}^{b} \frac{\varphi(\tau) d \tau}{(\tau-c)^{p}}+\frac{\xi(v)}{v^{p-1}}\right]$, here $\xi(v)$ is a function satisfied the conditions: 1) the limit exists; 2) $\xi(v)$ has a continuous $p-1$ degree derivative at a neighborhood of zero.

Let us give the definitions of SI and HI on a closed non-rectifiable curves and fractals.
Let $\gamma$ be a simple closed curve in the complex plane forming the boundary of
$D$, and $D^{+}$and $D^{-}$be interior and exterior domains respectively. If $u(z)$ is continuous in $\bar{D}^{+}$and has intagrable partial derivatives in в $\quad D^{+}, \quad$ Stokes' formular occurs $\int_{\gamma} u(z) d z=-\int_{D^{+}} \frac{\partial u}{\partial z} \partial z d \bar{z}$, where $\bar{z}=x-i y, \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i \frac{\partial}{\partial y}\right)$.

This formula allows you to enter the definition of the integral over non-rectifiable curves and fractals:

$$
\int_{\gamma} u(z) d z=-\iint_{D^{+}} \frac{\delta u(z)}{\delta \bar{z}} d z d \bar{z},(2.1)
$$

here $u(z)$ is a continuation of $u(z)$ to the region $D^{+}$.
The [13] notes that this definition belongs to Whitney.
There are many methods of continuation. Here Whitney's continuation has been used [14], p.205.
The Whitney operator has the following properties:

1) if the function $u(z) \in H_{\lambda}(\gamma), z \in \gamma$, then its extension $\tilde{u}(z)$ satisfies the Holder condition in $D^{+}$;
2) in $C \backslash \gamma$, the continuation of $\tilde{u}(z)$ satisfies the estimate $|\operatorname{grad} \tilde{u}(z)| \leq C(\operatorname{dist}(z, \gamma))^{\lambda-1}$.

Stein [14] shows that Definition (2.1) does not depend on Whitney's operator selection. Thus for any $u_{1}(z)$ and $u_{2}(z)$ Whitney's continuation appears

$$
\iint_{D^{+}} \frac{\partial u_{1}(z)}{\partial \bar{z}} d z d \bar{z}=\iint_{D^{+}} \frac{\partial u_{2}(z)}{\partial \bar{z}} d z d \bar{z}
$$

It is known [15] the integral $\iint_{D^{+}} \frac{\partial u(z)}{\partial \bar{z}} d z d \bar{z}$ exists for $\lambda>\alpha(\gamma)-1$, where $\alpha(\gamma)$ is the cell dimension of the curve $\gamma$.

Definition 2.5. If a closed curve $\gamma$ has a cell dimension $\alpha(\gamma), f \in H_{\lambda}(\gamma)$ and $\lambda>\alpha(\gamma)-1$ occurs, then

$$
\int_{\gamma} f(z) d z=-\iint_{D^{+}} \frac{\partial f(z)}{\partial \bar{z}} d z d \bar{z}, \text { where } f(z) \text { is a Whitney continuation for } f
$$

In case of singularity of $f \quad, \quad f=f_{0} v$, here $f_{0} \in H_{\lambda}(\gamma), f_{0}(t)=0$, and $|v(z)| \leq c|z-t|^{-1},|\partial v / \partial \bar{z}| \leq c|z-t|^{-1}, z \in \bar{D}^{+} \backslash t$, then it occurs

Definition 2.6 [16]. If a closed curve $\gamma$ has a cell dimension $\alpha(\gamma)$, the inequalities $|v(z)| \leq c|z-t|^{-1},|\partial v / \partial \bar{z}| \leq c|z-t|^{-1}, z \in \bar{D}^{+} \backslash t \quad$ and $\quad \lambda>(\alpha(\gamma)) / 2 \quad$ are $\quad$ satisfied then $\int_{\gamma} f(z) d z=-\iint_{D^{+}} \frac{\partial\left(v f_{0}(z)\right)}{\partial \bar{z}} d z d \bar{z}$, where $f_{0}(z)$ is any Whitney continuation for $f$.

Consider the singular integral $S_{\gamma} f=\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau) d \tau}{\tau-t}, t \in \gamma$.
If $\gamma$ is a smooth curve, $S_{\gamma} f$ is regularized by

$$
\left(S_{\gamma} f\right)(t)=\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)-f(t)}{\tau-t} d \tau+\frac{f(t)}{\pi i} \int_{\gamma} \frac{d \tau}{\tau-t}=\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)-f(t)}{\tau-t} d \tau+f(t) .
$$

It yields us to the following statement.
Definition 2.7 [16]. A singular integral $S_{\gamma} f$ over a closed non-rectifiable curve is defined by $\left(S_{\gamma} f\right)(t)=f(t)-\frac{1}{\pi i} \iint_{D^{+}} \frac{\partial(f(z))}{\partial \tilde{z}} \frac{1}{z-t} d z d \bar{z}$, where $f(z)$ is a Whitney continuation for $f$.

Consider the hypersingular integral $\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau, t \in \gamma, p=2,3, \ldots$.
In case of a smooth closed curve $\gamma$ a hypersingular integral on the complex plane $C$ is defined by

$$
\begin{aligned}
& \frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau=\frac{1}{\pi i} \int_{\gamma} \frac{g(\tau, t)}{\tau-t} d \tau+f(t) \frac{1}{\pi i} \int_{\gamma} \frac{d \tau}{(\tau-t)^{p}}+ \\
& +\frac{f^{\prime}(t)}{1!} \frac{1}{\pi i} \int_{\gamma} \frac{d \tau}{(\tau-t)^{p-2}}+\ldots+\frac{f^{(p-1)}(t)}{(p-1)!} \frac{1}{\pi i} \int_{\gamma} \frac{d \tau}{\tau-t}=\frac{1}{\pi i} \int_{\gamma} \frac{g(\tau, t)}{\tau-t} d \tau+\frac{1}{(p-1)!} f^{(p-1)}(t),
\end{aligned}
$$

where $g(\tau, t)=\left(f(\tau)-f(t)-\frac{f^{\prime}(t)}{1!}(\tau-t)-\ldots-\frac{f^{(p-1)}(t)}{(p-1)!}(\tau-t)^{p-1}\right) /(\tau-t)^{p-1}$.
Using the last formula the definition follows.
Definition 2.8. A hypersingular integral over a non-rectifiable curve $\gamma$ is defined by $\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau=\frac{f^{(p-1)}(t)}{(p-1)!}-\frac{1}{\pi i} \iint_{D^{+}} \frac{\partial g(z)}{\partial \tilde{z}} \frac{1}{z-t} d z d \tilde{z}$, where $g(z)-$ is a Whitney continuation for $g(\tau, t)$.

On a smooth closed curve $\gamma$, the hypersingular integral is also defined by the expression $\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau=\frac{1}{(p-1)!} \frac{d^{p-1}}{d t^{p-1}} \frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{\tau-t} d \tau$.

Using this formula, we arrive at the following definition.
Definition 2.9. The hypersingular integral over the non-rectifiable contour or fractal $\gamma$ is defined by the formula $\frac{1}{\pi} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau=\frac{f^{(p-1)}(t)}{(p-1)!}-\frac{1}{(p-1)!} \frac{d^{p-1}}{d t^{p-1}} \frac{1}{\pi i} \iint_{D^{+}} \frac{\partial f(z)}{\partial \tilde{z}} \frac{1}{z-t} d z d \tilde{z}$, where $f(z)-$ any continuation of a Whitney type function $f(z)$.

It is easy to see that Definitions 2.8 and 2.9 are equivalent.

## 3. Approximate calculation of integrals on fractals

3.1. Riemann integrals. Let $f(z) \in H_{\lambda}(\gamma), \gamma$ be closed non-rectifiable curve.

There are two possibilities:

1) $f(z)$ is defined in domain $\overline{D^{+}}$with an intagrable partial derivative with respect to $\bar{z}$;
2) $f(z)$ is defined only on $\gamma$.

In the first case, the Stokes formula is used directly to calculate the integral $\int_{\gamma} f(z) d z$. One can put $\int_{\gamma} f(z) d z=-\iint_{D^{+}} \frac{\partial f}{\partial z} d z d \bar{z}$ and the problem is reduced to constructing a cubature formula for calculating the integral on the right.

In the second case, it is necessary to continue the function $f(z), z \in \gamma$, to the domain $D^{+}$. If $f(z) \in H_{\lambda}(\gamma), 0<\lambda<1$, then as the continuation operator we can take the zero Whitney operator [14], p. 204, associating the function $f(z)$ with the function $\tilde{f}(z)=\xi_{0}(f), z \in D^{+}$, by formula $\int_{\gamma} f(z) d z=-\iint_{D^{+}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} d z d \bar{z}$.

The specific function depends on the choice of the basic infinitely differentiable function $\varphi^{*}$, defined on a unit square. It is known [14] that the formula is valid for any basis functions.

Constructing the function $u(z), z \in D^{+}$given in [14] is rather complex. Below it is presented a numerical method for evaluating the integral $\int_{\gamma} f(z) d z$.

Let $G=[a, b ; c, d], \overline{D^{+}} \in G$. Let $h$ be a grid of a cubature formula. For simplicity assume $(b-a) / h=m_{1},(d-c) / h=m_{2}-$ are integer. Let $x_{k}=a+h k, k=0,1, \ldots, m_{1} ; y_{l}=c+h l, l=0,1, \ldots, m_{2}$ be nodes. By $z_{k l}$ denote the node $z_{k l}=x_{k}+i y_{l}, k=0,1, \ldots, m_{1}, l=0,1, \ldots, m_{2}$.

Let $\Delta_{k l}=\left[x_{k}, x_{k+1} ; y_{l}, y_{l+1}\right], k=0,1, \ldots, m_{1}-1, l=0,1, \ldots, m_{2}-1$.
Fix $\mathcal{E}(0<\varepsilon \ll h)$. Assign each point $z_{k l}$ a point $p_{k l} \in \gamma$ attains the distance from $z_{k l}$ to $\gamma$. Since it is rather difficult to find an accurate location of $p_{k l}$, it is sufficient to select any point $p_{k l}^{\prime} \in \gamma$ in $B\left(p_{k l}, \varepsilon\right)$. Assume $u\left(z_{k l}\right)=f\left(p_{k l}^{\prime}\right), k=0,1, \ldots, m_{1}-1, l=0,1, \ldots, m_{2}-1$.

Fix an arbitrary $l=0,1, \ldots, m_{2}$ and assume a sequence $u\left(z_{k l}\right), k=0,1, \ldots, m_{1}$. Using it we will calculate the derivative $\left.\frac{\partial u(z)}{\partial x}\right|_{k, l}$. There are various methods for derivatives calculating. The method based on hypersingular integrals is used below [17].

Consider the quadrature formula [17]

$$
\frac{\delta^{r} u\left(\mathrm{t}, \mathrm{y}_{l}\right)}{\delta t^{r}}=\frac{r!}{2 \pi i} \sum_{k=0}^{m_{1}-1} u\left(x_{k}, y_{l}\right)\left[\int_{x_{k}}^{x_{k+1}} \frac{d \tau}{(\tau-(t+i \delta))^{r+1}}-\int_{x_{k}}^{x_{k+1}} \frac{d \tau}{(\tau-(t-i \delta))^{r+1}}\right]+R_{m_{1}}(u)
$$

This formula allows you to calculate the derivatives of any finite order and has a sufficiently high accuracy and stability. The regularization parameter is $\delta$.

Similarly using the sequence $u\left(z_{k l}\right), k=0,1, \ldots, m_{2}$ derivatives $\left.\frac{\partial u\left(x_{k}, y\right)}{\partial y}\right|_{k, l}$ are calculated.
Each node $z_{k l}$ is assigned in the complex number $\left.\frac{\partial u(x, y)}{\partial \bar{z}}\right|_{x=x_{k}}, y=y_{l}=\left(\left.\frac{\partial u\left(x, y_{l}\right)}{\partial x}\right|_{x=x_{k}}+\left.i \frac{\partial u\left(x_{k}, y\right)}{\partial y}\right|_{y=y_{l}}\right) / 2, k=0,1, \ldots, m_{1}-1, l=0,1, \ldots, m_{2}-1$.

By $\Delta_{k l}^{*}$ denote rectangles $\Delta_{k l}$ having no intersection with $\gamma$. Let $\Omega_{*}=\bigcup_{k, l} \Delta_{k l}^{*}$.
Define the function

$$
w_{k, l}(z)=\left\{\begin{array}{l}
\left(\left.\frac{\partial u(x, y)}{\partial x}\right|_{x_{k}}, y_{l}+\left.i \frac{\partial u(x, y)}{\partial y}\right|_{x_{k}}, y_{l}\right) / 2, z \in \Delta_{k l}^{*} \\
0, z \in G \backslash \Delta_{k, l}^{*}, k=0,1, \ldots, m_{1}-1, l=0,1, \ldots, m_{2}-1
\end{array}\right.
$$

Let $w(z)=\sum_{k, l} w_{k, l}(z)$.
To evaluate the integral $\int_{\gamma} f(z) d z$ the following formula is used

$$
\int_{\gamma} f(z) d z=-\sum_{k=0}^{m_{1}-1} \sum_{l=0}^{m_{2}-1^{*}} w_{k, l} \iint_{\Delta_{k, l}} d z d \bar{z}+R_{m_{1}, m_{2}}(f)
$$

where $\sum \sum^{*}$ means summation over the rectangles $\Delta_{k, l}$, included in $\Omega_{*}$.
3.2. Singulal and hypersingular integrals. Consider the integral

$$
\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{\tau-t} d \tau=f(t)-\frac{1}{\pi i} \iint_{\Omega^{+}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} \frac{d \tau}{z-t}
$$

where $\tilde{f}(z)$ is a Whitney's continuation for $f(\tau)-f(t)$.
Similarly above, construct a function $w(z)$ approximating $\frac{\partial \tilde{f}(z)}{\partial \bar{z}}$. Then a cubature formular for calculating singular integral should be constructed using results in [8]. The calculation of the HI is carried out according to the formula

$$
\frac{1}{\pi i} \int_{\gamma} \frac{f(\tau)}{(\tau-t)^{p}} d \tau=\frac{f^{(p-1)}(t)}{(p-1)!}-\frac{1}{\pi i} \iint_{\Omega^{+}} \frac{\partial \tilde{g}(z)}{\partial \bar{z}} \frac{d \tau}{z-t}
$$

where $\tilde{g}(z)$ is a Whitney's continuation for $g(\tau, t)$. The function is given in Definition 2.8.
Then we construct a function $w(z)$, approximating $\frac{\partial \tilde{g}(z)}{\partial \bar{z}}$, and a cubature formula for calculating the singular integral.

## 4. Approximate solution of hypersingular integrals over prefractals

Let $C_{n}$ be $n$th prefractal of the Cantor set (in other words $n$th Cantor set iteration). Consider hypersingular integral equation

$$
\begin{equation*}
a(t) x(t)+b(t) \int_{C_{n}} \frac{x(\tau)}{(\tau-t)^{p}}+\int_{C_{n}} h(t, \tau) x(\tau) d \tau=f(t), t \in C_{n} \tag{4.1}
\end{equation*}
$$

Remark. The problem might be discussed for $t \in[0,1]$.
Two approximation schemes for solution of the equation (4.1) have been introduced and justified in [18].
Let's imagine one of them. For simplicity of notation, put $h(t, \tau)=0$.
Let $p=2$.
An approximate solution of the equation (4.1) is sought in the form of a spline

$$
\begin{gather*}
x_{n}(t)=\sum_{l=1}^{2} \sum_{i_{1}, \ldots, i_{n}} \alpha_{l, i_{1}, \ldots, i_{n}} \varphi_{l, i_{1}, \ldots, i_{n}}(t), t \in C_{n},(4.2)  \tag{4.2}\\
\varphi_{l, i_{1}, \ldots, i_{n}}(t)=\left\{\begin{array}{c}
g_{l, i_{1}, \ldots, i_{n}}(t), t \in \Delta_{i_{1}, \ldots, i_{n}}, \\
0, t \in[0,1] \backslash \Delta_{i_{1}, \ldots, i_{n}}, l=1,2, i_{j}=0,2, j=1, \ldots, n,
\end{array}\right.
\end{gather*}
$$

where functions $g_{l, i_{1}, \ldots, i_{n}}(t)$ are similar to basic functions

$$
\begin{gathered}
g_{1,0, \ldots, 0}(t)=\left\{\begin{array}{c}
1,0 \leq t \leq \frac{1}{3^{2 n}}, \\
\frac{3^{2 n} t-3^{n}+1}{2-3^{n}}, \frac{1}{3^{2 n}} \leq t \leq \frac{1}{3^{n}}-\frac{1}{3^{2 n}}, \\
0, \frac{1}{3^{n}}-\frac{1}{3^{2 n}} \leq t \leq \frac{1}{3^{n}},
\end{array}\right. \\
g_{2,0, \ldots, 0}(t)=\left\{\begin{array}{c}
0,0 \leq t \leq \frac{1}{3^{2 n}}, \\
\frac{3^{2 n} t-1}{3^{n}-2}, \frac{1}{3^{2 n}} \leq t \leq \frac{1}{3^{n}}-\frac{1}{3^{2 n}}, \\
1, \frac{1}{3^{n}}-\frac{1}{3^{2 n}} \leq t \leq \frac{1}{3^{n}} .
\end{array}\right.
\end{gathered}
$$

The coefficients $\alpha_{l, i_{1}, \ldots, i_{l}}$ are found from the system of equations:

$$
\begin{gathered}
a\left(t_{i_{1}, \ldots, i_{n}}^{l}\right) \alpha_{l, j_{1}, \ldots, j_{n}}+b\left(t_{i_{1}, \ldots, i_{n}}^{l}\right) \int_{C_{n}} \frac{x_{n}(\tau) d \tau}{\left(\tau-t_{i_{1}, \ldots, i_{n}}^{l}\right)^{2}}=f\left(t_{i_{1}, \ldots, i_{n}}^{l}\right),(4.3) \\
l=1,2, t_{i_{1}, \ldots, i_{n}}^{1}=\frac{2}{3} i_{1}+\frac{2}{3^{2}} i_{2}+\ldots+\frac{2}{3^{n}} i_{n}+\frac{2}{3^{2 n}} i_{n}, \quad t_{i_{1}, \ldots, i_{n}}^{2}=\frac{2}{3} i_{1}+\frac{2}{3^{2}} i_{2}+\ldots+\frac{2}{3^{n}} i_{n}+\frac{1}{3^{n}}-\frac{2}{3^{2 n}} i_{n},
\end{gathered}
$$

$i_{j}=0,2, j=1,2, \ldots, n$.
Under hypothesis $|b(t)| \geq \gamma>0, t \in[0,1]$ a unique solvability and convergence of the solution of the system of equations (4.3) to the solution of equation (4.1) has been proved.

A numerical method for solution hypersingular integral equation on Hilbert's curve has been constructed and justified in [18].

In the work cited above, a spline-collocation method for solving the following equation is constructed and substantiated $a\left(t_{1}, t_{2}\right) x\left(t_{1}, t_{2}\right)+b\left(t_{1}, t_{2}\right) \iint_{\Omega_{n}} \frac{x\left(\tau_{1}, \tau_{2}\right) d \tau_{1}, d \tau_{2}}{\left(\left(\tau_{1}-t_{1}\right)^{2}+\left(\tau_{2}-t_{2}\right)^{2}\right)^{p}}=f\left(t_{1}, t_{2}\right),\left(t_{1}, t_{2}\right) \in \Omega_{n}$, where $\Omega_{n}$ is the $n$-th prefractal of the Sierpinski carpet.

## 5. Solution of singular integral equations

Consider an SIE with a Hilbert integral kernel and constant coefficients $c$ and $d$

$$
\begin{equation*}
c x(s)+\frac{d}{2 \pi} \int_{0}^{2 \pi} x(\sigma) \operatorname{ctg} \frac{\sigma-s}{2} d \sigma=f(s) \tag{5.1}
\end{equation*}
$$

We denote by $W_{\alpha, \beta}^{c}(s)$ and $W_{\alpha, \beta}^{s}(s) \quad$ cosinosoidal and sinusoidal Weierstrass functions $W_{\alpha, \beta}^{c}(s)=\sum_{k=1}^{\infty} \alpha^{k} \cos \left(\beta^{k} s\right), W_{\alpha, \beta}^{s}(s)=\sum_{k=1}^{\infty} \alpha^{k} \sin \left(\beta^{k} s\right)$.

Hardy showed that the functions $W_{\alpha, \beta}^{c}(s)$ and $W_{\alpha, \beta}^{s}(s)$ for $0<\alpha<1, \beta>1, \alpha \beta \geq 1$ are continuous, nowhere non-differentiable functions.

In the equation (5.1) we put $f(s)=W_{\alpha, \beta}^{c}(s)$. Since the right-hand side of the equation (5.1) is a periodic function, we put $\beta=2$. Under the assumption that the parameters $\alpha$ and $\beta$ satisfy the conditions $1 / 2<\alpha<1, \alpha \beta \geq 1$, the function $f(s)$ is continuous nowhere non-differentiable function.

Its fractal dimension is $D=2+\frac{\ln \alpha}{\ln \beta}[19]$ and varies depending on the values of the parameters $\alpha$ and $\beta$ from 1 to 2.

We will seek an approximate solution of the Eq. (5.1) in the form of a series

$$
x(t)=\sum_{k=0}^{\infty} a_{k} \cos k t+\sum_{k=1}^{\infty} b_{k} \sin k t .(5.2)
$$

Substituting series (5.2) into the equation (5.1) and using the formulas [20]

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \sin k \sigma c t g \frac{\sigma-s}{2} d \sigma=\cos k s, \quad \frac{1}{2 \pi} \int_{0}^{2 \pi} \cos k \sigma c t g \frac{\sigma-s}{2} d \sigma=-\sin k s,(5.3)
$$

we get

$$
\begin{equation*}
c \sum_{k=0}^{\infty} a_{k} \cos k s+c \sum_{k=1}^{\infty} b_{k} \sin k s-d \sum_{k=1}^{\infty} a_{k} \sin k s+d \sum_{k=1}^{\infty} b_{k} \cos k s=\sum_{k=1}^{\infty} \alpha^{k} \cos \left(\beta^{k} s\right) \tag{5.4}
\end{equation*}
$$

From the equation (5.4) we have

$$
\begin{align*}
& a=0, \\
& a_{2^{k}}=\alpha^{k} c /\left(d^{2}+c^{2}\right), \\
& b_{2^{k}}=d \alpha^{k} /\left(d^{2}+c^{2}\right), k=1,2,3, \ldots,  \tag{5.5}\\
& a_{k}=b_{k}=0, k=\{1,2, \ldots\} \backslash 2^{l}, l=1,2, \ldots
\end{align*}
$$

Thus, the solution of the SIE (5.1) has the form

$$
\begin{equation*}
x^{*}(s)=\sum_{k=1}^{\infty} \frac{\alpha^{k} c}{c^{2}+d^{2}} \cos \left(2^{k} s\right)+\sum_{k=1}^{\infty} \frac{\alpha^{k} d}{c^{2}+d^{2}} \sin \left(2^{k} s\right)=\frac{c}{c^{2}+d^{2}} W_{\alpha, 2}^{c}(s)+\frac{d}{c^{2}+d^{2}} W_{\alpha, 2}^{s}(s) . \tag{5.6}
\end{equation*}
$$

Let us prove the validity of formula (5.6) under the assumption that the coefficients $a_{k}$ and $b_{k}$ are defined by formulas (5.5). For this, we investigate the smoothness of the function $x^{*}(t)$. We denote by $S_{2^{n}}\left(x^{*}\right)$ the sum

$$
\begin{align*}
& S_{2^{n}}\left(x^{*}\right)=\sum_{k=1}^{n}\left(\frac{\alpha^{k} c}{c^{2}+d^{2}} \cos \left(2^{k} s\right)+\frac{\alpha^{k} d}{c^{2}+d^{2}} \sin \left(2^{k} s\right)\right) \\
& \text { Then } \quad E_{2^{n}}\left(x^{*}\right) \leq\left|\sum_{k=n+1}^{\infty}\left(\frac{\alpha^{k} c}{c^{2}+d^{2}} \cos \left(2^{k} s\right)+\frac{\alpha^{k} d}{c^{2}+d^{2}} \sin \left(2^{k} s\right)\right)\right| \quad \leq C \alpha^{n} \\
& E_{n}\left(x^{*}\right) \leq \frac{C}{n^{\gamma}}, \gamma=\left|\log _{2} \alpha\right|
\end{align*}
$$

Here $E_{n}\left(x^{*}\right)$ is the best uniform approximation of the function $x^{*}(s)$ by trigonometric polynomials of order $n$.

The right-hand side in the equation (5.1) is the Weierstrass function with exponent 2 . So, one can put $\frac{1}{2}<\alpha<1$. Hence, $0<\gamma<1$. From Bernstein's converse theorems of the constructive function theory [21] implies that $x \in H_{\gamma}$.

Let us put $R_{2^{n}}(s)=x^{*}(s)-S_{2^{n}}\left(x^{*}\right)$ and estimate the inequality
$\left|\frac{1}{2 \pi} \int_{0}^{2 \pi} R_{2^{n}}(\sigma) \operatorname{ctg} \frac{\sigma-s}{2} d \sigma\right| \leq\left|\frac{1}{2 \pi} \int_{0}^{2 \pi}\right| R_{2^{n}}(\sigma)-\left.R_{2^{n}}(s)\right|^{1-\gamma}\left|R_{2^{n}}(\sigma)-R_{2^{n}}(s)\right|^{\gamma}\left|\operatorname{ctg} \frac{\sigma-s}{2}\right| d \sigma \leq$
$\leq C\left(\left|R_{2^{n}}(\sigma)\right|^{1-\gamma}+\left|R_{2^{n}}(s)\right|^{1-\gamma}\right) \leq \frac{c}{2^{n \gamma(1-\gamma)}}$
and $\lim _{n \rightarrow \infty}\left|\frac{1}{2 \pi} \int_{0}^{2 \pi} R_{2^{n}}(\sigma) \operatorname{ctg} \frac{\sigma-s}{2} d \sigma\right|=0$. Consequently, the permutation of the operators of summation and integration is justified.

The following statement is true.
Theorem 5.1. Let $c^{2}+d^{2} \neq 0$. Equation (5.1) has a unique solution $x^{*}(s)$, which is nowhere non-differentiable function.

Let us consider the singular integral equation

$$
\begin{equation*}
\frac{d}{2 \pi} \int_{0}^{2 \pi} x(\sigma) c t g \frac{\sigma-s}{2} d \sigma+h(s) \int_{0}^{2 \pi} k(\sigma) x(\sigma) d \sigma=W_{\alpha, 2}^{c}(s) \tag{5.7}
\end{equation*}
$$

We will seek an approximate solution to equation (5.7) in the form of series (5.2).
Substituting (5.2) into (5.7), we have

$$
\begin{align*}
& d\left(\sum_{k=1}^{\infty} b_{k} \cos k s-\sum_{k=0}^{\infty} a_{k} \sin k s\right)+\left(\sum_{k=0}^{\infty} v_{k} \cos k s+\sum_{k=1}^{\infty} w_{k} \sin k s\right) \times  \tag{5.8}\\
& \left.\times\left(\sum_{k=0}^{\infty} a_{k} \gamma_{k}+\sum_{k=1}^{\infty} b_{k} \delta_{k}\right)=W_{\alpha, 2}^{c}(s)\right)
\end{align*}
$$

where $\gamma_{0}=\frac{1}{2 \pi} \int_{0}^{2 \pi} k(\sigma) d \sigma$,

$$
\gamma_{k}=\frac{1}{\pi} \int_{0}^{2 \pi} k(\sigma) \cos k \sigma d \sigma, \delta_{k}=\frac{1}{\pi} \int_{0}^{2 \pi} k(\sigma) \sin k \sigma d \sigma, k=0,1, \ldots
$$

From (5.8) we obtain the following groups of equations

$$
\begin{align*}
& v_{0}\left(\sum_{l=0}^{\infty} a_{l} \gamma_{l}+\sum_{k=1}^{\infty} b_{l} \delta_{l}\right)=0 ; \\
& d b_{k}+v_{k}\left(\sum_{l=0}^{\infty} a_{l} \gamma_{l}+\sum_{k=1}^{\infty} b_{l} \delta_{l}\right)=0, k=1,2, \ldots ; \\
& -d a_{k}+w_{k}\left(\sum_{l=0}^{\infty} a_{l} \gamma_{l}+\sum_{k=1}^{\infty} b_{l} \delta_{l}\right)=\alpha^{j}, k=2^{j}, j=1,2, \ldots ;  \tag{5.9}\\
& -d a_{k}+w_{k}\left(\sum_{l=0}^{\infty} a_{l} \gamma_{l}+\sum_{l=1}^{\infty} b_{l} \delta_{l}\right)=0, k=\{1,2, \ldots\} \backslash\left\{2^{l}, l=1,2, \ldots\right\} .
\end{align*}
$$

If $v_{0} \neq 0$, then

$$
\begin{align*}
& \sum_{l=0}^{\infty} a_{l} \gamma_{l}+\sum_{l=0}^{\infty} b_{l} \delta=0 \\
& -a_{k}=\alpha^{j} / d, k=2^{j}, j=1,2, \ldots  \tag{5.10}\\
& a_{k}=0,\{k=1,2, \ldots\} \backslash\left\{k=2^{j}, j=1,2, \ldots,\right\} \\
& b_{k}=0, k=1,2, \ldots
\end{align*}
$$

From equation (5.8) we find $a_{0}: a_{0}=-\frac{1}{\gamma_{0} d} \sum_{l=1}^{\infty} \alpha^{l} \delta_{2^{l}}$.
It follows from equalities (5.10) that equation (5.7) has a unique solution if $\gamma_{0}=\frac{1}{2 \pi} \int_{0}^{2 \pi} k(\sigma) d \sigma \neq 0$ and $v_{0}=\frac{1}{2 \pi} \int_{0}^{2 \pi} h(\sigma) d \sigma \neq 0$. In other cases has a parameter-dependent solution.

## 6. Solution of hypersingular integral equations

Consider the hypersingular integral equation

$$
\begin{equation*}
\frac{1}{4 \pi} \int_{0}^{2 \pi} \frac{x(\sigma)}{\sin ^{2} \frac{\sigma-s}{2}} d \sigma=f(s), 0 \leq s \leq 2 \pi \tag{6.1}
\end{equation*}
$$

which simulates a number of aerodynamic problems. In these cases, $f(s)$ simulates a gas flow and, therefore, is a fractal.

Let us investigate the solvability of condition (6.1) under the assumption that the right-hand side is the Weierstrass function $W_{\alpha, 2}^{c}(s)$.

Equation (6.1) can be represented as

$$
S x \equiv \frac{d}{d s} \frac{1}{2 \pi} \int_{0}^{2 \pi} x(\sigma) \operatorname{ctg} \frac{\sigma-s}{2} d \sigma=W_{\alpha, 2}^{c}(s) .(6.2)
$$

The solution of equation (6.2) (and, hence, (6.1)) will be sought in the form of the series

$$
x^{*}(t)=\sum_{k=0}^{\infty}\left(a_{k} \cos 2^{k} s+b_{k} \sin 2^{k} s\right) .
$$

Acting formally, we arrive at the following equation

$$
-\sum_{k=1}^{\infty}\left(a_{k} 2^{k} \cos 2^{k} s+b_{k} 2^{k} \sin 2^{k} s\right)=\sum_{k=1}^{\infty} \alpha^{k} \cos \left(2^{k} s\right) .(6.3)
$$

It follows from formulas (5.3) that $\frac{1}{2 \pi} \int_{0}^{2 \pi} a_{0} \operatorname{ctg} \frac{\sigma-s}{2} d \sigma=0$.
Thus, the coefficient $\alpha_{0}$ turns out to be undefined and additional condition is required to determine it. As such condition, we can take $\int_{0}^{2 \pi} x(s) d s=0$. Then $a_{0}=0$. From equations (6.3) we have $a_{k}=\left(\frac{\alpha}{2}\right)^{k}, b_{k}=0, k=1,2, \ldots$. So, $x^{*}(t) \in W^{1} H_{\alpha}$.

Consider the hypersingular integral equation

$$
\frac{1}{4 \pi} \int_{0}^{2 \pi} \frac{x(\sigma)}{\sin ^{2} \frac{\sigma-s}{2}} d \sigma+h(s) \int_{0}^{2 \pi} k(\sigma) x(\sigma) d \sigma=W_{\alpha, 2}^{c}(s)(6.4)
$$

We will seek an approximate solution in the form of a series $x^{*}(s)=\sum_{k=0}^{\infty} a_{k} \cos k s+\sum_{k=1}^{\infty} b_{k} \sin k s$.
Substituting the series $x^{*}(s)$ into the equation (6.4) we have

$$
\begin{align*}
& -\sum_{k=1}^{\infty}\left(a_{k} k \cos k s+b_{k} k \sin k s\right)+  \tag{6.5}\\
& +\left(\sum_{k=0}^{\infty}\left(v_{k} \cos k s+w_{k} \sin k s\right)\right)\left(\sum_{k=0}^{\infty} a_{k} \gamma_{k}+\sum_{k=1}^{\infty} b_{k} \delta_{k}\right)=W_{\alpha, 2}^{c}(s) .
\end{align*}
$$

From equality (6.5) we obtain the following groups of equations. At first we introduce the notation

$$
A=\sum_{k=0}^{\infty} a_{k} \gamma_{k}+\sum_{k=1}^{\infty} b_{k} \delta_{k} .
$$

We have

$$
\begin{align*}
& v_{0} A=0, \\
& -b_{k} k+w_{k} A=0, k=1,2, \ldots \\
& -a_{k} k+v_{k} A=\alpha^{l}, k=2^{l}, l=1,2, . .  \tag{6.6}\\
& -a_{k} k+v_{k} A=0, k=\{1,2, \ldots\} \backslash\left\{2^{l}, l=1,2\right\} .
\end{align*}
$$

If $v_{0} \neq 0$, then $A=0$ and

$$
\begin{aligned}
& b_{k}=0, k=1,2, \ldots \\
& a_{k}=-\alpha^{l} / k, k=2^{l}, l=1,2, \ldots \\
& a_{k}=0, k=\{1,2, \ldots\} \backslash\left\{2^{l}, l=1,2, \ldots ;\right\}
\end{aligned}
$$

From the condition $A=0$, we have $a_{0}=\left(\sum_{l=1}^{\infty} \frac{\alpha^{l}}{2^{l}} \gamma_{2^{l}}\right) \frac{1}{\gamma_{0}}$.
Thus, for $v \neq 0$ and $\gamma \neq 0$, equation (6.4) has a unique solution

$$
x^{*}(s)=\left(\sum_{l=1}^{\infty} \frac{\alpha^{l}}{2^{l}} \gamma_{2^{l}}\right) \frac{1}{\gamma_{0}}-\sum_{l=1}^{\infty} \frac{\alpha^{l}}{2^{l}} \cos 2^{l} s .(6.7)
$$

Repeating the reasoning given above in $n^{0} 5$, we see that the series and the differentiated series on the right-hand side of (6.7) converge uniformly and $x^{*}(s) \in W^{1} H_{q}, q=\left|\log _{2} \alpha\right|$.

If $v_{0}=0$ we have a family of solutions depending on the parameter.

## 7. Conclusion

The approximate method for evaluating Riemann integrals, singular and hypersingular integrals on closed non-rectifiable curves and fractals has been proposed. For integrand continuation from fractal to the interior region the approximate computational scheme based on Whitney's continuation has been constructed. The approximate calculation of derivatives is based on using hypersingular integrals. It leads to two-dimensional integrals (for Riemann integrals) and to singular integrals (for singular and hypersingular integrals). Cubature formulas have been employed for evaluation of constructed integrals. The proposed method can be used to evaluate integrals over open curves.

The approximate method for solving hypersingular integral equations on the $n$-th prefractal of Cantor perfect setis presented. The spline-collocation method with first-order splines has been used. Justification of this method is based on theory of stability of ordinary differential equations systems. One of the main advantages of this method turns out to be its resistance to coefficients and right-hand side of equation disturbance. This method can be used for construction approximate solutions of singular and hypersingular integral equations on fractals of various types.

The solvability of singular and hypersingular integral equations with Weierstrass function in the right-hand side has been investigated.

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## Approximate solution of inverse problems of gravity exploration on fractals

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## Headings content

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2. Introduction
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#### Abstract

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The work is devoted to the approximate methods for solution direct and inverse problems of gravity exploration on bodies with a fractal structure. It is known that in order to construct mathematical models adequate to the geological reality, it is necessary to take into account the orderliness inherent in geological environments. One of the manifestations of orderliness is self-similarity, which remains during the transition from the microlevel to the macrolevel. Scaling of geological media can be traced in petrophysical data and in anomalous fields.

It should be noted that in real structures there is no infinite self-similarity and scaling must be considered in a certain range.

The work investigates analytical and numerical methods for solving inverse contact problems of the logarithmic and Newtonian potential in the generalized setting. In the case of a Newtonian potential, the problem is formulated as follows. It is required, having three independent functionals of the gravity field above the Earth's surface and additional information on the self-similarity of the disturbing body, to determine the depth, the density and the surface of the perturbing body.


## Introduction

For the effective solution of direct and inverse problems of gravity prospecting, the methods of modeling bodies that perturb the potential and gravitational fields of the Earth (perturbing bodies) are of great importance. In most works, disturbing bodies are modeled by a set of the simplest geometric bodies (bar, parallelepiped, ball) [1]. In the works [2], [3], modeling is carried out with spheroids. In recent years, a large number of studies have been carried out on the fractality of individual minerals and the entire Earth as a whole [4], [5], [6], [7]. Scaling of geological media can be traced in petrophysical data and in anomalous fields [8], etc. On the basis of the apparatus of fractional measure and fractional dimension, the processing of disturbances of the Earth's gravitational field is investigated [9].

Most minerals are porous. There are two types of porosity: the porosity of minerals and the porosity of liquids. Numerous studies have shown that in both mention cases, the porosity has a fractal structure.

In particular, the group of authors argues that sandstones have a fractal structure [4], [5], [10]. Hansen and Skjeltorp [6] investigated the fractal dimension $D$ of a flat sandstone sample and obtained $D=1,73$. Brakenseik [11] determined the fractal dimension of a two-dimensional oil cut. It is equal to $D=1,8$. In [12], the fractal dimension of the surfaces of porous ceramic materials is investigated.

In the monograph [7] the Menger's sponge is proposed as a mathematical model of porosity, which is constructed somewhat differently from the standard construction.

In this work, when constructing fractal models of geological environments, the authors proceed not from fractals, but, following [13], from additions to fractals, since areas (volumes) of additions tend to areas (volumes) of the original body.

Taking into account the fractal components of gravitational fields makes it possible to clarify the structure of the disturbing bodies.

Methods for solving contact inverse problems of logarithmic and Newtonian potentials in a generalized setting are analyzed [14]. The problem is formulated as follows. It is required, having three independent functionals of the gravity field over the Earth's surface $z=0$ and additional information about the self-similarity of the disturbing body, determine the depth $H$, the density $\sigma(x, y)$ and the surface $H-\varphi(x, y)$ of the disturbing body occupying the region $H \leq z(x, y) \leq H-\varphi(x, y)$.

Taking into account the fractal components of the gravitational and magnetic fields makes it possible to clarify the structure of the disturbing bodies.

The work is devoted to the approximate solution of direct and inverse problems of gravity prospecting on bodies with a fractal structure.

When solving inverse problems, a continuous method for solving nonlinear operator equations is used, which is presented in the next section.

## 3. Continuous operator method

Let $B$ be a Banach space, $a, z \in B, K$ be a linear operator mapping from $B$ to $B, \Lambda(K)$ be the logarithmic norm [15] of the operator $K$, and $I$ be the identity operator. We shall use the following notation: $B(a, r)=\{z \in B P z-a \mathrm{P} \leq r\}, S(a, r)=\{z \in B \cdot \mathrm{P} z-a \mathrm{P}=r\}, \mathrm{R} e K=K_{R}=\left(K+K^{*}\right) / 2$,
$\Lambda(K)=\lim _{h \downarrow 0}(\mathrm{P} I+h K \mathrm{P}-1) / h$.
Let a complex matrix $A=\left\{a_{i j}\right\}, i, j=1,2, \ldots, n$, be given in $n-$ dimensional space $R^{n}$ of vectors $x$ with the norms $\mathrm{P} x \mathrm{P}_{1}=\sum_{k=1}^{n}\left|x_{k}\right|, \mathrm{P} x \mathrm{P}_{2}=\left[\sum_{k=1}^{n}\left|x_{k}\right|^{2}\right]^{1 / 2}$, and $\mathrm{P} x \mathrm{P}_{3}=\max _{1 \leq k \leq n}\left|x_{k}\right|$.

The corresponding logarithmic norms of the matrix $A$ then read [16]:

$$
\Lambda_{1}(A)=\max _{j}\left(\mathrm{R} e\left(a_{j j}\right)+\sum_{i=1, i \neq j}^{n}\left|a_{i j}\right|\right), \Lambda_{2}(A)=\lambda_{\max }\left(\left(A+A^{T}\right) / 2\right),
$$

$$
\Lambda_{3}(A)=\max _{i}\left(\mathrm{R} e\left(a_{i i}\right)+\sum_{j=1, j \neq i}^{n}\left|a_{i j}\right|\right)
$$

Here $\lambda_{\max }(A)$ means the largest real part of eigenvalues of the matrix $A$.
Consider an equation

$$
A(x)-f=0,(3.1)
$$

where $A(x)$ is a nonlinear operator mapping from Banach space $B$ to $B$.
Let $x^{*}$ be a solution of the equation (3.1). In [17] the connection between stability of solutions of operator
differential equations in Banach spaces and resolving operator equations of the form (3.1) has been established. Here we shall summarize the results on the method.

Let us associate the equation (3.1) with the following Cauchy problem

$$
\begin{gathered}
\frac{d x(t)}{d t}=A(x(t))-f,(3.2) \\
x(0)=x_{0} \cdot(3.3)
\end{gathered}
$$

Theorem 3.1[17]. Let the equation (3.1) has a solution $x^{*}$ and on any differentiable curve $g(t)$ in Banach space $B$ the inequality is valid

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Lambda\left(A^{\prime}(g(\tau)) d \tau \leq-\alpha_{g}, \alpha_{g}>0\right. \tag{3.4}
\end{equation*}
$$

Then the solution of the Cauchy problem (3.2), (3.3) converges to the solution $x^{*}$ of the equation (3.1) for any initial approximation.

Theorem 3.2[17]. Let the equation (3.1) has a solution $x^{*}$ and for any differentiable curve $g(t)$ in a ball $B\left(x^{*}, r\right)$ the following conditions are satisfied:

1) for any $t(t>0)$

$$
\int_{0}^{t} \Lambda\left(A^{\prime}(g(\tau)) d \tau \leq 0\right.
$$

2) the inequality (3.4) is valid.

Then the solution of the Cauchy problem (3.2), (3.3) converges to a solution of the equation (3.1).
Note 1 . In the inequality (3.4) it is assumed that the constants $\alpha_{g}>0$ can differ for different curves $g(t)$.
Note 2. From inequalities (3.4) - (3.5) it follows that the logarithmic norm $\Lambda\left(A^{\prime}(g(\tau))\right.$ can be positive for some values of $\tau$; i.e. the Frechet derivative $A^{\prime}(g(\tau))$ can degenerate into an identically zero operator along the curve.

Note 3. An example in [18] (an approximate solution of a hypersingular integral equation) has demonstrated convergence of an iterative process based on a continuous operator method when the Frechet derivative vanishes at the initial approximation.

Note 4. Logarithmic norm has the property which is very useful for numerical analysis. Let $A, B$ be square matrices of order $n$ with complex elements and $x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right), \quad \xi=\left(\xi_{1}, \ldots, \xi_{n}\right)$, $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right)$ are $n$-dimensional vectors with complex components. Let us consider the following systems of algebraic equations: $A x=\xi$ and $B y=\eta$. The norm of a vector and its subordinate operator norm of the matrix are fixed; the logarithmic norm $\Lambda(A)$ corresponds to the operator norm.

Theorem 3.3[19]. If $\Lambda(A)<0$, the matrix $A$ is non-singular and $\mathrm{P} A^{-1} \mathrm{P} \leq 1 /|\Lambda(A)|$.
Theorem 3.4[19]. Let $A x=\xi, B y=\eta$ and $\Lambda(A)<0, \Lambda(B)<0$. Then

$$
\mathrm{P} x-y \mathrm{P} \leq \frac{\mathrm{P} \xi-\eta \mathrm{P}}{|\Lambda(B)|}+\frac{\mathrm{P} A-B \mathrm{P}}{|\Lambda(A) \Lambda(B)|} .
$$

Main properties of the logarithmic norm are given in [15].
The logarithmic norm of the operator K can have different (positive or negative) values in different spaces.
The continuous method for solving nonlinear operator equations admits the following generalization. Let us return to equation (3.1). Denote by $A^{\prime}\left(x_{0}\right)$ the Gateaux (Frechet) derivative on the element $x_{0}$. We introduce the equation

$$
\left(A^{\prime}\left(x_{0}\right)\right)^{*} A(x)-\left(A^{\prime}\left(x_{0}\right)\right)^{*} f=0 .(3.6)
$$

Equation (3.6) is associated with the Cauchy problem

$$
\left.\frac{d x}{d t}=-\left(A^{\prime}\left(x_{0}\right)\right)^{*} A(x)-\left(A^{\prime}\left(x_{0}\right)\right)^{*} f\right),(3.7)
$$

$$
x(0)=x_{0} \cdot(3.8)
$$

If $\left.\Lambda_{2}\left(A^{\prime}\left(x_{0}\right)\right)^{*} A^{\prime}\left(x_{0}\right)\right)>0$, then in some neighborhood $B\left(x_{0}, r\right)$ of the element $x_{0}$ the Euclidean logarithmic norm of the operator - $\left.A^{\prime}\left(x_{0}\right)\right)^{*} A(x)$ will be negative and $\mathrm{P} x(t) \mathrm{P}<\mathrm{P} x(0) \mathrm{P}$ for some interval $\mathrm{t} \in\left(t_{0}, t_{1}\right], \mathrm{t}_{0}=0$.

Let the inequality $\left.\Lambda_{2}\left(A^{\prime}\left(x_{0}\right)\right)^{*} A^{\prime}(x)\right)>0$ be satisfied on the segment $t \in\left[t_{0}, t_{1}\right], t_{0}=0$. (Here $x(t)$ is the solution to the Cauchy problem (3.7), (3.8)).

For $t \geq t_{1}$, consider the Cauchy problem

$$
\begin{gathered}
\frac{d x_{1}(t)}{d t}=-\left(A^{\prime}\left(x_{1}\right)^{*} A(x)-\left(A^{\prime}\left(x_{1}\right)\right)^{*} f\right),(3.9) \\
x_{1}\left(t_{1}\right)=x\left(t_{1}\right)(3.10)
\end{gathered}
$$

and define the segment $\left[t_{1}, t_{2}\right]$, in which the inequality $\left.\Lambda_{2}\left(A^{\prime}\left(x_{1}\right)\right)^{*} A^{\prime}\left(x_{1}\right)\right)>0$ occur.
Taking $x_{2}\left(t_{2}\right)=x_{1}\left(t_{2}\right)$ as an initial value when solving the Cauchy problem

$$
\begin{gathered}
\left.\frac{d x_{2}(t)}{d t}=-\left(A^{\prime}\left(x_{2}\right)\right)^{*} A(x)-\left(A^{\prime}\left(x_{2}\right)\right)^{*} f\right), \\
x_{2}\left(t_{2}\right)=x_{1}\left(t_{2}\right),(3.12)
\end{gathered}
$$

we have $\lim _{t \rightarrow \infty} \mathrm{P} \frac{d x(t)}{d t} \mathrm{P}=0$ and therefore $\lim _{t \rightarrow \infty} x(t)=x^{*}$
Assertions follow from this remark.
Theorem 3.5. Suppose that equation (3.6) has a solution $x^{*}$ and for any differentiable curve in the Banach space $B$ the inequality

$$
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Lambda\left(\left(A^{\prime}(g(\tau))\right)^{*} A^{\prime}(g(\tau))\right) d \tau \leq-\alpha_{g}, \alpha_{g}>0(3.13)
$$

occur. Then the solution to the sequence of Cauchy problems ((3.7), (3.8)), ((3.9), (3.10)), ((3.11), (3.12)), etc. converges to the solution $x^{*}$ of equation (3.6).

Theorem 3.6. Suppose that equation (3.6) has a solution $x^{*}$ and for any differentiable curve in the sphere $B\left(x^{*}, r\right)$ the inequalities

$$
\int_{0}^{t} \Lambda\left(\left(A^{\prime}(g(\tau))\right)^{*} A^{\prime}(g(\tau))\right) d \tau<0(3.14)
$$

and (3.13) occur. Then the solution of the sequence of Cauchy problems ((3.7), (3.8)), ((3.9), (3.10)), ((3.11), (3.12)), etc. converges to the solution $x^{*}$ of equation (3.6).

If the conditions of Theorems 3.5 and 3.6 are not satisfied, the regularization

$$
\frac{d x}{d t}=-\alpha x(t)-\left(\left(A^{\prime}\left(x_{0}\right)\right)^{*} A(x)-\left(A^{\prime}\left(x_{0}\right)\right)^{*} f\right), \alpha>0
$$

is carried out.

## 4. Direct tasks

Let us consider a geological deposit represented by the uniform body $D$ of arbitrary form. Assuming that the body has fractal dimension $D_{H}<3$, we will approximate it with its complement of the Menger sponge [7]. Let the body $D$ be situated in the cube $\Omega=[-a, a]^{3}$. Let us construct the $n$-th order prefractal ( $n$-th iteration of the fractal) for the Menger
sponge in the cube $\Omega$. During the construction of the first iteration the cube $\Omega$ is divided into 27 equal cubes with sides $r_{1}=2 a / 3$, and 7 central cubes are dropped.

During the construction of the second iteration every cube from the remaining 20 cubes is divided into 27 equal cubes with the sides $2 a / 9$. As the result we have 729 cubes including 400 central cubes (for every initial cube with the side $2 a / 3$ ) that are dropped. Repeating the described operations $n$ times we get the $n$-th Menger prefractal. As noted in the work [13], not classical fractals but their complements with respect to the initial domain should be used as the model for geological bodies. Consequently geological deposits are modeled with the set of cubes with different lengths of edges (and with different sizes).

When modeling granular and liquid media it seems that it is more efficient to model them with reduced copies of the first iteration of the Menger sponge. In that case we can construct the model using not only classical fractal but also complement to it.

Let us introduce the Cartesian three-dimensional rectangular coordinate system with down-directed $z$-axis and with the origin of coordinates placed at the Earth surface. Assume that the body $D$ occurs at sufficiently great depth z=H under the Earth surface.

As the parameter $H$ we fix the distance from the Earth surface to the average point (in vertical direction) of gravitating body.

In the introduced coordinate system the domain $\Omega$, which the body $D$ belongs to, rewrites as:

Let

$$
\Omega=\{(x, y, z):-a, x, a,-a, y,, a, H-a, z,, H+a\} .
$$

$$
\Delta_{i, j, k}=\left[x_{i}, x_{i+1} ; y_{j}, y_{j+1} ; z_{k}, z_{k+1}\right], \quad x_{i}=-a+a i / n, \quad i=0,1, \ldots, 2 n,
$$

$$
y_{j}=-a+a j / n, \quad j=0,1, \ldots, 2 n, z_{k}=H-a+a k / \mathrm{n}, \quad k=0,1, \ldots, 2 n
$$

We refer to as marked the cubes $\Delta_{i j k}$ that have nonempty intersection with the domain $D$. In the marked cubes we locate the first iteration of the Menger sponge fractal with the edge length $a / n$. Suppose that the body is modeled by the first iteration of the fractal. Denote the constructed model of the body by $D_{n}$. For computation of the perturbed field it is sufficient to compute the vertical component of the gravity field generated by the cell $\Delta_{i j k}$ at the point $(x, y, 0)$.

The cell $\Delta_{i j k}$ consists of 20 cubes with edges having the length $a / 3 n$. Assuming $n$ and $H$ being sufficiently big we may treat $\cos \left(\Theta\left(x^{\prime}, y^{\prime}, z^{\prime}\right)\right)$, where $\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \in \Delta_{i j k} \quad$ as constant within the limits of the cell. Here $\Theta\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$ is the angle between the radius-vector $M^{\prime} P\left(M^{\prime}=\left(x^{\prime}, y^{\prime}, z^{\prime}\right), P(x, y, 0)\right)$ and the $z$-axis.

Let us denote by $\mathrm{o}_{i j k}$ the center of the cell $\Delta_{i j k}$. Obviously,

$$
\mathrm{o}_{i j k}=(-a+a(i+1 / 2) / n,-a+a(j+1 / 2) / n, H-a+a(k+1 / 2) / n) .
$$

Let us also denote by $\theta_{i j k}$ the angle between the vector $\mathrm{o}_{i j k} P$ and the $z$-axis.
Thus the vertical component of the gravity force generated by the cell $\Delta_{i j k}$ at the point $P(x, y, 0)$ equals to $d V_{z}(i, j, k)=20 \gamma \rho(a / 3 n)^{3} \cos \left(\theta_{i j k}\right) /\left(r\left(\mathrm{o}_{i j k}, P\right)\right)^{2}=\frac{20 \gamma \rho a^{3}}{27 n^{3}} \cdot \frac{z_{k}+z_{k+1}}{2} /\left(r\left(\mathrm{o}_{i j k}, P\right)\right)^{3}$.

Here $\quad \gamma$ is the gravitational constant, $\rho$ is a density of body. Therefore the vertical component of the gravity force generated by the disturbing body $D$ at the point $(x, y, 0)$ equals to

$$
V_{z}(x, y, 0)=\sum_{i, j, k=0}^{2 n-1} 20 \gamma \rho_{i j k} a^{3}\left(\frac{z_{k}+z_{k+1}}{2}\right) / 27 n^{3}\left(r\left(\mathrm{o}_{i j k}, P\right)\right)^{3}, \rho_{i j k}
$$

is a density of cell. Consider the example.
Let us se t the following parameter values: $H=5, \quad a=1 / 4, \quad n=10$.
We perform calculations using the formula $d V_{z}(i, j, k)=\frac{20 \gamma \rho a^{3}}{27 n^{3}} \cdot \frac{z_{k}+z_{k+1}}{2} /\left(r\left(\mathrm{o}_{i j k}, P\right)\right)^{3}$.
Let us fix $i=j=k=n$, that corresponds to the central cell $\Delta_{i j k}$ in the domain $\Omega$. For illustrative purposes the product of the constants $\gamma$ and $\rho$ we set to $10^{6}$.

The field $d V(i, j, k)$ of the vertical component of anomalous gravity force generated by the described cell at the Earth surface is shown in the figure below.


Picture 1. The vertical component of anomalous gravity force generated by the cell of theMenger sponge first order prefractal.

For comparison we also introduce the plot of the vertical component of the anomalous gravity field generated by the continuous body occupying the domain

The computed field is depicted in the following figure.


Picture 2. The vertical component of anomalous gravity force generated by the elementary cell.
From the comparison of the computed fields it is obvious that the solution of the direct problem is essentially dependent on the chosen model for representation of the elementary cell.

## 5. Inverce tasks

This section examines the influence of the chosen model on the accuracy of the interpretation of the results.
Let in the domain $D\left\{D:-l \leq x \leq l,-l \leq y \leq l_{1}, H \leq z \leq H-\varphi(x, y)\right\}$ are distributed with density $\sigma(x, y, z)$ sources disturbing gravitational field of the Earth. The gravity field above the Earth's surface is determined by the equation

$$
\begin{equation*}
G \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{H-\varphi(\zeta, \eta)}^{H} \frac{\sigma(\zeta, \eta, \xi)(\xi-z) d \zeta d \eta d \xi}{\left((x-\zeta)^{2}+(y-\eta)^{2}+(\xi-z)^{2}\right)^{3 / 2}}=f(x, y, z) \tag{5.1}
\end{equation*}
$$

where $f(x, y, z)$ is the experimentally determined value, $G-$ gravitational constant, which for the convenience of further calculations will be set equal to $G=1 / 2 \pi$.

To describe the force of gravity on the Earth's surface in equation (5.1), one should set $z=0$.
Having calculated the integral on the left-hand side of Eq. (5.1) by parts and assuming that the density does not depend on $\xi$, we have

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta)\left[\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z-\varphi(\zeta, \eta))^{2}\right)^{-1 / 2}-\right.  \tag{5.2}\\
& \left.-\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}\right)^{-1 / 2}\right] d \zeta d \eta=f(x, y, z)
\end{align*}
$$

We represent equation (5.2) in the form

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta)\left[\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}\right)^{-1 / 2}(1+u)^{-1 / 2}-\right.  \tag{5.3}\\
& \left.-\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}\right)^{-1 / 2}\right] d \zeta d \eta=f(x, y, z),
\end{align*}
$$

where $u=\frac{\varphi^{2}(\zeta, \eta)-2(H-z) \varphi(\zeta, \eta)}{(x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}}$. Under the assumption that $|u|<1$, the function $\frac{1}{(1+u)^{1 / 2}}$ is expanded in the series

$$
\frac{1}{(1+u)^{1 / 2}}=1+\sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!} u^{n} .(5.4)
$$

Substituting (5.4) into (5.3) and using the uniform convergence of series (5.4), we have

$$
\begin{align*}
& \frac{1}{2 \pi} \sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta) \frac{\left(\varphi^{2}(\zeta, \eta)-2(H-z) \varphi(\zeta, \eta)\right)^{n} d \zeta d \eta}{\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}\right)^{n+1 / 2}}=  \tag{5.5}\\
& =f(x, y, z)
\end{align*}
$$

Let us approximate equation (5.5), limiting ourselves to one term on the left-hand side. As a result, we obtain the equation [14]

$$
\begin{equation*}
-\frac{1}{4 \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta) \frac{\left(\varphi^{2}(\zeta, \eta)-2(H-z) \varphi(\zeta, \eta)\right) d \zeta d \eta}{\left((x-\zeta)^{2}+(y-\eta)^{2}+(H-z)^{2}\right)^{3 / 2}}=f(x, y, z) \tag{5.6}
\end{equation*}
$$

Equation (5.6) contains three unknowns: the depth of the gravitating body $H$, the density of the body $\sigma(x, y)$ and the shape of the surface $H-\varphi(x, y)$. To find these unknowns, it is necessary, in addition to values of the gravity field on some surface, to have two more linearly independent sources of information. As these functionals, one can use values of the gravity field at three different levels, a combination of the values of the gravity field and its derivatives in different directions, etc.

Note. Having values of the gravity field at the same level, it is possible to restore values of the gravity field at several levels using the Poisson formula.

In the work [14], analytical and numerical methods are proposed for the simultaneous determination of the depth of the disturbing body, its density and the surface equation in contact problems of the logarithmic and Newtonian potential. In [14], the disturbing body was assumed to be solid.

Compared with iterative methods for solving equation (5.6), studied in [14], the preferable is the continuous operator method described in section 3. In both cases, the density is interpreted as a constant function within the unit cell, which simulates the gravitating body. In the case of modeling a gravitating body with fractals, the density in elementary cells is not constant. It is of interest to study the influence of fractals chosen for modeling disturbing bodies on the accuracy of determining their densities.

In [14] the following example was analytically solved.
Let in the domain $\Omega=\{5 \leq z(x, y) \leq 5-\varphi(x, y),-\infty<x, y<\infty\}$, there is a perturbing body with density $\sigma(x, y)$. Let the gravity force and its first two derivatives be known on the surface $z=0$ :

$$
\begin{gathered}
f_{0}(x, y, 0)=\frac{24 \pi}{\left(x^{2}+y^{2}+36\right)^{3 / 2}}-\frac{7 \pi}{5\left(x^{2}+y^{2}+49\right)^{3 / 2}}, \\
f_{1}(x, y, 0)=\left.\frac{\partial f(x, y, z)}{\partial z}\right|_{z=0}=\frac{432 \pi}{\left(x^{2}+y^{2}+36\right)^{5 / 2}}-\frac{4 \pi}{\left(x^{2}+y^{2}+36\right)^{3 / 2}}-\frac{147 \pi / 5}{\left(x^{2}+y^{2}+49\right)^{5 / 2}}-\frac{2 \pi / 25}{\left(x^{2}+y^{2}+49\right)^{3 / 2}} \\
f_{2}(x, y, 0)=\left.\frac{\partial^{2} f(x, y, z)}{\partial z^{2}}\right|_{z=0}=\frac{12960 \pi}{\left(x^{2}+y^{2}+36\right)^{7 / 2}}-\frac{1029 \pi}{\left(x^{2}+y^{2}+49\right)^{7 / 2}}- \\
-\frac{216 \pi}{\left(x^{2}+y^{2}+36\right)^{5 / 2}}+\frac{21 \pi / 25}{\left(x^{2}+y^{2}+49\right)^{5 / 2}}-\frac{4 \pi / 125}{\left(x^{2}+y^{2}+49\right)^{3 / 2}}
\end{gathered}
$$

It is necessary to find a depth of the gravitating body $H$, a density of the body $\sigma(x, y)$ and a shape of the surface $H-\varphi(x, y)$. To solve this problem, in addition to equation (5.6), two more equations are added

$$
\begin{align*}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2 H w_{1}(\xi, \eta)-w_{2}(\xi, \eta)}{\left((x-\xi)^{2}+(y-\eta)^{2}+H^{2}\right)^{3 / 2}} d \xi d \eta=f_{0}(x, y), \\
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left\{\frac{6 H^{2} w_{1}(\xi, \eta)-3 H w_{2}(\zeta, \eta)}{\left((x-\xi)^{2}+(y-\eta)^{2}+H^{2}\right)^{5 / 2}}-\frac{2 w_{1}(\xi, \eta)}{\left((x-\xi)^{2}+(y-\eta)^{2}+H^{2}\right)^{3 / 2}}\right\} d \xi d \eta=f_{1}(x, y),  \tag{5.7}\\
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}\left\{\frac{3 w_{2}(\zeta, \eta)-18 H w_{1}(\xi, \eta)}{\left((x-\xi)^{2}+(y-\eta)^{2}+H^{2}\right)^{5 / 2}}+\frac{30 H^{3} w_{1}(\xi, \eta)-15 H^{2} w_{2}(\xi, \eta)}{\left((x-\xi)^{2}+(y-\eta)^{2}+H^{2}\right)^{7 / 2}}\right\} d \xi d \eta=f_{2}(x, y) .
\end{align*}
$$

When obtaining system (5.7), the following formulas were used $w_{1}(x, y)=\sigma(x, y) \varphi(x, y)$, $w_{2}(x, y)=\sigma(x, y) \varphi^{2}(x, y)$.

Its exact solution was obtained: $H=5, \varphi(x, y)=\left(\frac{x^{2}+y^{2}+1}{x^{2}+y^{2}+4}\right)^{3 / 2}, \sigma(x, y)=\frac{\left(x^{2}+y^{2}+4\right)^{3 / 2}}{\left(x^{2}+y^{2}+1\right)^{3}}$.
When solving the system of equations (5.7) by the spline-collocation method with zero-order splines, an error is equal to $O\left(N^{-1}\right)$, where $h=N^{-1}$ is a step of the computational scheme by coordinates $x, y$. Hence it follows that the results of the approximate solution can be interpreted as follows. In area $(x, y):\left\{\frac{\left(x^{2}+y^{2}+4\right)^{3 / 2}}{\left(x^{2}+y^{2}+1\right)^{3}} \leq 1 / N\right\}$ let us put $\sigma(x, y)=0$. Domain $G$ defined by the inequality $\left\{\frac{\left(x^{2}+y^{2}+4\right)^{3 / 2}}{\left(x^{2}+y^{2}+1\right)^{3}} \geq 1 / N\right\}$ we will cover with elementary cells (cubes) with edges of length $d / N$, where $d$ is area diameter $G$. Place the first-order prefractal of the Mergel sponge in the elementary cells. Then, depending on the mineral filling the addition of the Margel sponge to the unit cell, the density of the body varies from $\sigma(x, y)$ to $27 \sigma(x, y) / 20$. Thus, when solving inverse problems on fractals, an additional problem arises of choosing an appropriate model (fractal, multifractal) for a gravitating body.

## 6. Conclusions

In this work by the example of the Menger sponge approximate methods for solution of direct and inverse problems of gravity exploration using fractals are investigated. As far as inverse geophysical problems belong to the class of ill-posed problems for their solution in this work we propose the generalization of continuous operator method for solution of nonlinear equations. The proposed method allows to obtain stable solution for inverse problems which are modeled with nonlinear convolutional equations. At the core of the method there are criteria for asymptotic stability of solutions of systems of ordinary differential equations. The method can be used for solution of numerous equations of mathematical physics. In solving direct and inverse problems using fractals we show the problem of dependency of interpretation of computational results on the chosen model.

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# The electromagnetic interaction among watery precipitations in the atmosphere" 

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#### Abstract

\section*{Watery Precipitations}

When a raindrop, mist or dwe falls through a thundercloud, it is subject to strong electromagnetic fields that pull and tug on the droplet, as well as a soap bubble in the wind. If the electric field and consequently the electromagnetic effect is strong enough, it can cause the droplet to burst apart, creating a fine, electrified mist. Droplets tend to form as perfect little spheres due to surface tension, the cohesive force that binds water molecules at a droplet's surface and pulls the molecules inward. The droplet may distort from its spherical shape in the presence of other forces, such as the force from an electric field. Sometimes, we have anomalous diffraction because of the different shapes of the droplets. While surface tension acts to hold a droplet together, the electric field acts as an opposing force, pulling outward on the droplet as charge builds on its surface.


## Main theme

## Interaction of a magnetic field with a charge object

How does the magnetic field interact with a charged object? If the charge is at rest, there is no interaction. If the charge moves, however, it is subjected to a force, the size of which increases in direct proportion with the velocity of the charge. The force has a direction that is perpendicular both to the direction of motion of the charge and to the direction of the magnetic field. There are two possible precisely opposite directions for such a force for a given direction of motion, extremely in cases of storm, followed by lightning and thunder. Certainly, the cold thermal currents contribute to the condensation of water vapor and the creation of various water sediments. This apparent ambiguity is resolved by the fact that one of the two directions applies to the force on a moving positive charge while the other direction applies to the force on a moving negative charge. Figure 1 illustrates the directions of the magnetic force on positive charges and on negative charges as they move in a magnetic field that is perpendicular to the motion. Depending on the initial orientation of the particle velocity to the magnetic field, charges having a constant speed in a uniform magnetic field will follow a circular or helical path.


Figure 1.
Electromagnetic field - Maxwell's Equations

$$
\begin{gather*}
\nabla \cdot \mathbf{D}=\rho  \tag{1}\\
\nabla \cdot \mathbf{B}=0  \tag{2}\\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}  \tag{3}\\
\nabla \times \mathbf{H}=\frac{\partial \mathbf{D}}{\partial t}+\mathbf{J} \tag{4}
\end{gather*}
$$

The meaning of symbols is: $\mathbf{D}=$ dielectric displacement, $\boldsymbol{\rho}=$ the density of charge, $\mathbf{B}=$ the magnetic density, $\mathbf{E}=$ electric field strength, $\mathbf{H}=$ magnetic field strength, $\mathbf{J}=$ current density and $\mathbf{t}=$ time.

## Interactions with the Atmosphere

Before radiation used for remote sensing reaches the Earth's surface it has to travel through some distance of the Earth's atmosphere. Particles and gases in the atmosphere can affect the incoming light and radiation. These effects are caused by the mechanisms of scattering and absorption. Scattering occurs when particles or large gas molecules present in the atmosphere interact with and cause the electromagnetic radiation to be redirected from its original path. How much scattering takes place depends on several factors including the wavelength of the radiation, the abundance of particles or gases, and the distance the radiation travels through the atmosphere. There are three (3) types of scattering which take place.

We have: Rayleigh scattering, Mie scattering, Nonselective scattering
Rayleigh scattering, finger2, occurs when particles are very small compared to the wavelength of the radiation. These could be particles such as small specks of dust or nitrogen and oxygen molecules. Rayleigh scattering causes shorter wavelengths of energy to be scattered much more than longer wavelengths. Rayleigh scattering is the dominant scattering mechanism in the upper atmosphere. The fact that the sky appears "blue" during the day is because of this phenomenon. As sunlight passes through the atmosphere, the shorter wavelengths (i.e. blue) of the visible spectrum are scattered more than the other (longer) visible wavelengths. At sunrise and sunset the light has to
travel farther through the atmosphere than at midday and the scattering of the shorter wavelengths is more complete; this leaves a greater proportion of the longer wavelengths to penetrate the atmosphere.


Fig. 2


Fig. 3
Mie scattering occurs when the particles are just about the same size as the wavelength of the radiation. Dust, pollen, smoke and water vapour are common causes of Mie scattering which tends to affect longer wavelengths than those affected by Rayleigh scattering. Mie scattering occurs mostly in the lower portions of the atmosphere where larger particles are more abundant, and dominates when cloud conditions are overcast.

The final scattering mechanism of importance is called nonselective scattering. This occurs when the particles are much larger than the wavelength of the radiation. Water droplets and large dust particles can cause this type of scattering. Nonselective scattering gets its name from the fact that all wavelengths are scattered about equally. This type of scattering causes fog and clouds to appear white to our eyes because blue, green, and red light are all scattered in approximately equal quantities (blue+green+red light $=$ white light). Fing. 3

Absorption is the other main mechanism at work when electromagnetic radiation interacts with the atmosphere. In contrast to scattering, this phenomenon causes molecules in the atmosphere to absorb energy at various wavelengths. Ozone, carbon dioxide, and water vapour are the three main atmospheric constituents which absorb radiation.

Ozone serves to absorb the harmful (to most living things) ultraviolet radiation from the sun. Without this protective layer in the atmosphere our skin would burn when exposed to sunlight.

You may have heard carbon dioxide referred to as a greenhouse gas. This is because it tends to absorb radiation strongly in the far infrared portion of the spectrum - that area associated with thermal heating - which serves to trap this heat inside the atmosphere. Water vapour in the atmosphere absorbs much of the incoming longwave infrared and shortwave microwave radiation (between $22 \mu \mathrm{~m}$ and 1 m ). The presence of water vapour in the lower atmosphere varies greatly from location to location and at different times of the year. For example, the air mass above a desert would have very little water vapour to absorb energy, while the tropics would have high concentrations of water vapour (i.e. high humidity). Fig. 4


Fig. 4

## Energy Well of Watery Precipitators



Fig. 5
Energy levels in atoms are known to be discreet rather than forming a continuous set. Light emission from a hot hydrogen gas therefore yields a spectrum consisting of individual lines at specific wavelengths rather than a continuous distribution of wavelengths. Understanding this discreet nature of the energy levels and the calculation of the energies requires the use of quantum mechanics as classical mechanics can not describe atomic systems correctly.

## The infinite quantum well

The infinite well represents one of the simplest quantum mechanical problems: it consists of a particle in a well which is defined by a zero potential between $x=0$ and $x=L x$ and an infinite potential on either side of the well. The potential and the first five energy levels are shown in the figure below:


Fig. 6
The energy levels in such a infinite well are given by:
$E_{n}=\frac{h^{2}}{2 m^{*}}\left(\frac{n}{2 L_{x}}\right)^{2}$, with $n=1,2, \ldots$
where $h$ is Planck's constant and $m^{*}$ is the effective mass of the particle. $n$ is the quantum number associated with the nth energy level, with energy En. Note that the lowest possible energy is not zero even though the potential is zero within the well. Also that the distance between adjacent energy levels increases as the energy increases. Two electrons with opposite spin can occupy each level as n and s are the only two quantum numbers needed to describe this system.


Position (nm)

Fig. 7
The finite quantum well


Fig. 8

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# Detection of Early Warning Signals for SelfOrganized Criticality in Cellular Automata 

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#### Abstract

Detection the precursors of critical transitions in complex systems is one of the most difficult and still unsolved problems. This problem has not received a final solution, not only for real complex systems, but also for model systems capable to self-organize into the critical state. The presented paper is devoted to early detection of time moments of self-organized critical transitions in cellular automata as a result of the analysis of the time series they generate for a number of grains falling from the grid. It was found that cumulative moments of probability distribution and cumulative scaling exponents are quite informative indicators for early detection of critical transitions. General features of the behavior of indicators when approaching a critical point are established for the time series generated by cellular automata with different rules.


Keywords: Cellular Automata, Sandpile Model, Self-Organized Criticality, Time Series, Probability Moments, Multifractality.

## 1 Introduction

More than thirty years development of the theory of self-organized criticality (SOC), explaining the emergence of power law for probability density function, $1 / \mathrm{f}$-noise and long-range spatial and temporal correlation in nonlinear systems far from equilibrium, has led to the emergence of the number of basic models, which have nontrivial scale-invariant dynamics under very simple local rules [1,2]. The basic models of SOC theory are sandpile models [3]. These models have become the most important tool for studying the mechanisms of the appearance of scale-invariant properties and power statistics.

The sandpile model is a conical pile of sand, on the center of which grains of sand are placed one by one. We will assume that the cohesion between grains of sand is large enough and only superficial movement of sand is possible. Then the state of the system is determined by the local slope of the surface of the sand pile ( $S$ ). If $S$ is small, then the sand is motionless. If $S$ exceeds a certain value $S_{c}$, then there is a spontaneous flow of sand $\eta$ over the surface, which increases continuously with increasing of $S$. This process corresponds to a second-order phase transition, in which the control parameter is $S$, the order parameter is $\eta$. The value $S_{c}$ separates subcritical (SubC) phase and supercritical (SupC) phase. A pile of sand in these phases is resistant to small disturbances. On the contrary, the SOC state is highly volatile. Adding just one
grain of sand to a pile in this state can lead to avalanches of sand of any size theoretically.

A fundamentally important property of systems, which are characterized by avalanche-like behavior, is their ability to self-organize into the critical state. In this case, it is not required to fine-tune the parameter $S$ to the value $S_{c}$. Such systems are capable to transit to the SOC state spontaneously, which is typical for most real and model complex systems, the behavior of which is determined by the nonlinear local rules.

Scale-invariant properties and power statistics are characteristics not only for the level of the structure of the complex system and its local microscopic interactions, but also for the level of time series generated by such systems [4]. For the sandpile model, such time series are the time series $\left(\eta_{t}\right)$, which demonstrate the stochastic dynamics of sand grains falling on the surface of the pile. When describing a pile of sand using cellular automata models, $\eta_{t}$ is the number of grains falling from the grid. The approach to the study of selforganized critical states of complex systems based on the analysis of generated time series has, at least, one significant advantage. The approach does not require the study of detailed interactions between elements of the real systems. Information about detailed interactions is usually inaccessible for research, for example, for social networks, or inaccessible, for example, for financial networks.

Detection of early warning signals for critical transitions is a challenging task not only for the real complex systems, but also for the model systems. The overwhelming majority of the papers known to us are devoted either to detection of early warning signals associated with the critical slowing down phenomenon [5-8], or to the solution of particular problems of the early warning [ 9,10 ]. In these studies, precursors of the critical transitions in real systems were established, which are associated with the change in the autocorrelation function, variance, skewness, and power spectral density of the observed time series when the system parameters approach their critical value.

To our knowledge, however, there is no study that investigates the search for precursors of the SOC transitions not only in real complex systems, but also in model systems, for example, in the self-organized criticality cellular automata. The detection of such precursors as a result of the analysis of the time series for the number of grains falling from the grid is the purpose of our study.

## 2 Methods

### 2.1 Time series generated by the cellular automata

It is easy to study the stochastic dynamics of the order parameter $\left(\eta_{t}\right)$ of the sandpile models using models of cellular automata in grids of size $L \times L$. The parameter $\eta_{t}$ is the number of sand grains falling from the grid at time $t$.

Random integers $z_{i, j}$ are generated in the grid cells to represent the local slope of the sand pile. The cells for which $z_{i, j} \geq z_{c}$, where $z_{c}$ is the critical value, are unstable and fall off according to the rules defined for each cellular automata. For the pile of sand, several different variants of the rules for shedding an unstable cell have been proposed. This paper considers six sandpile models, each of which belongs to one of two classes of self-organized critical models: conservative and dissipative models. Models with conservative rules are characterized by the fact that when unstable cells fall, the grains of sand removed from them are redistributed without loss and leave the grid only after reaching its edges. The boundary conditions of such systems are open. In dissipative models, after the shedding of the unstable cell, the number of grains of sand in it is zero. In the case of the supercritical number of sand grains, they are able to leave the grid also within its boundaries.

### 2.1.1 Conservative systems

Let us consider in more detail the historically very first model, called the BTWmodel [11]. Consideration of other models, including dissipative models, is limited to consideration of only the rules for shedding cells.

BTW-model is a cellular automaton on a square grid of size $L \times L$. A grain of sand is randomly added to a randomly selected cell $(i, j)$, increasing the number of grains of sand $\left(z_{i, j}\right)$ in the cell by one: $z_{i, j}+=1$. As a result, $z_{i, j} \rightarrow z_{i, j}+1$. If $z_{i, j} \geq 4$, then one grain of sand moves to the four nearest cells: $z_{i \pm 1, j \pm 1}+=1$. In this case, the number of grains of sand in the cell $(i, j)$ decreases by the value $z_{c}=4: z_{i, j}-=4$. The considered movement of sand grains can lead to loss of stability of neighboring cells, and, consequently, lead to the appearance of the avalanche with loss of stability. The introduction of the condition $z_{i, j}-=4$ leads to the saving of the number of sand grains.

Thus, the rules of the model are as follows:

$$
\begin{equation*}
z_{c}=4, z_{i, j}-=4, z_{i \pm 1, j \pm 1}+=1 \tag{1}
\end{equation*}
$$

In the Fig. 1 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the BTW-model are presented. The rest of the time series looks the same except for the iteration number (or point in time $t_{c}$ ) corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=1656$ for $20 \times 20$ grid, $t_{c}=2171$ for $30 \times 30$ grid, $t_{c}=3736$ for $40 \times 40$ grid, $t_{c}=5491$ for $50 \times 50$ grid, and $t_{c}=8234$ for $60 \times 60$ grid.


Fig. 1. Time series of the number of grains falling from the grid for the BTW-model


Fig. 2. Time series of the number of grains falling from the grid for the Manna model

Manna model [12] is the stochastic analogue of the BTW-model. The cell $(i, j)$ crumbles as a result of the stability loss, transferring a random number of grains of sand ( $\delta_{k} \geq 0$ ) to four neighboring cells.

Formally, the rules of the model are as follows:

$$
\begin{equation*}
z_{c}=4, z_{i, j}-=4, z_{i \pm 1, j \pm 1}+=\delta_{k}, \delta_{k} \geq 0, \sum_{k} \delta_{k}=4 \tag{2}
\end{equation*}
$$

In the Fig. 2 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the Manna model are presented. The rest of the time series looks the same except for the iteration number corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=892$ for $20 \times 20$ grid, $t_{c}=2510$ for $30 \times 30$ grid, $t_{c}=3335$ for $40 \times 40$ grid, $t_{c}=5671$ for $50 \times 50$ grid, and $t_{c}=7625$ for $60 \times 60$ grid.

DR-model [13] is the cellular automaton, the rules of which are formulated on a two-dimensional hexagonal lattice. It is a cellular automaton with open boundary conditions on the lower side and periodic boundary conditions on the left and right sides. A grain of sand is randomly added to a randomly selected cell $(i, j)$ of the top layer, increasing the number of grains of sand $\left(z_{i, j}\right)$ in the cell by one: $z_{i, j}+=1$. When the value in any cell exceeds one, this cell loses stability and crumbles, transferring one grain of sand to the two cells lying below. It is important that the DR-model rules are anisotropic, i.e. the avalanche of sand grains, spreading from the top to the bottom, never affects the same area twice.

Formally, the rules of the model are as follows:

$$
\begin{equation*}
z_{c}=2, z_{i, j}-=2, z_{i \pm 1, j \pm \frac{1}{2}}+=1 \tag{3}
\end{equation*}
$$

In the Fig. 3 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the DR-model are presented. The rest of the time series looks the same except for the iteration number corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=1491$ for $20 \times 20$ grid, $t_{c}=1821$ for $30 \times 30$
grid, $t_{c}=3241$ for $40 \times 40$ grid, $t_{c}=5200$ for $50 \times 50$ grid, and $t_{c}=7476$ for $60 \times 60$ grid.


PSV-model [14] is the stochastic analogue of DR-model. The cell $(i, j)$ crumbles as a result of the stability loss, transferring the random number of sand grains ( $\delta_{ \pm} \geq 0$ ) to the two cells lying below.

Formally, the rules of the model are as follows:

$$
\begin{equation*}
z_{c}=2, z_{i, j}-=2, z_{i \pm 1, j \pm \frac{1}{2}}+=\delta_{ \pm}, \delta_{ \pm} \geq 0, \delta_{+}+\delta_{-}=2 \tag{4}
\end{equation*}
$$

In the Fig. 4 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the PSV-model are presented. The rest of the time series looks the same except for the iteration number corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=1$ for $20 \times 20$ grid, $t_{c}=1$ for $30 \times 30$ grid, $t_{c}=1$ for $40 \times 40$ grid, $t_{c}=1$ for $50 \times 50$ grid, and $t_{c}=1$ for $60 \times 60$ grid.

### 2.1.2 Dissipative systems

The rules of the models considered above are conservative, i.e. when cells are shattered, the grains of sand removed from them are redistributed to neighboring cells without loss. The grains of sand leave the grid only when they reach its edges.

DFF-model [15] is a deterministic cellular automaton with a twodimensional orthogonal grid of size $L \times L$. The integers in the cells $z_{i, j}$ can be interpreted as the number of grains of sand that can participate in the pouring processes. There is no designated slope direction. If $z_{i, j} \geq 4$, then the cell $(i, j)$ is unstable and overturns. Overturn is zeroing of the number of sand grains in the cell with a simultaneous increase by 1 in the values in four cells that have a common side with this cell.

Formally, the rules of the model are as follows:

$$
\begin{equation*}
z_{c}=4, z_{i, j}-=0, z_{i \pm 1, j \pm 1}+=1 \tag{5}
\end{equation*}
$$

In the Fig. 5 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the DFF-model are presented. The rest of the time series looks the same except for the iteration number corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=1$ for $20 \times 20$ grid, $t_{c}=1$ for $30 \times 30$ grid, $t_{c}=1$ for $40 \times 40$ grid, $t_{c}=1$ for $50 \times 50$ grid, and $t_{c}=1$ for $60 \times 60$ grid.


The stochastic DFF-model with a random number of sand grains $\left(\delta_{k}\right)$ in four neighboring cells that have a common side with the cell is characterized by the following rules:

$$
\begin{equation*}
z_{c}=4, z_{i, j}-=0, z_{i \pm 1, j \pm 1}+=\delta_{k}, \delta_{k} \geq 0, \sum_{k} \delta_{k}=4 \tag{6}
\end{equation*}
$$

In the Fig. 6 the time series for the number of grains falling from the grid for a $40 \times 40$ grid of the stochastic DFF-model are presented. The rest of the time series looks the same except for the iteration number corresponding to the SOC state. $t_{c}$ depends on the grid's size: $t_{c}=1$ for $20 \times 20$ grid, $t_{c}=1$ for $30 \times 30$ grid, $t_{c}=1$ for $40 \times 40$ grid, $t_{c}=1$ for $50 \times 50$ grid, and $t_{c}=1$ for $60 \times 60$ grid.

### 2.2 Moments of probability density function for the time series

Earlier, we proposed the algorithm for detecting the self-organized critical state of the system. The algorithm is based on the analysis of scaling exponents of power laws for probability density function $(\alpha)$, power spectral density ( $\beta$ ), and autocorrelation function $(\gamma)$ for time series generated by SOC systems. The algorithm make it possible to identify the SubC phase and the SupC phase by belonging of $\alpha, \beta$ and $\gamma$ to certain intervals. The disadvantage of the algorithm is its applicability only to the analysis of scale-invariant probability density function and the impossibility of its application to the analysis of other heavy-tailed distributions. The heavy-tailed distributions are characteristic for the time intervals of the system evolution, corresponding to its SupC phase. A
famous example of a scale-invariant heavy-tailed distribution is the Pareto distribution.

Therefore, in order to go beyond the limitations of the algorithm associated only with the use of $\alpha$ as the only identifier of the SOC state, the SubC phase and the SupC phase, we used the main moments of probability density function as cumulative indicators.

For detection of early warning signals for self-organized criticality we used the following moments:
first raw moment (or mean) $\mu$,
second central moment (or variance) $\sigma^{2}$,
standardized third moment (or skewness) $\gamma$,
standardized fourth moment (or kurtosis) $\kappa$.

### 2.3 Scaling exponents for the time series

Even the description of model time series using the moments of their probability density function is exhaustive only for a very limited number of random processes. For example, realizations of Gaussian processes are fully described by second-order moments. Therefore, apart from the moments, other quantities should be used to describe the time series. These quantities include the scaling exponents for the time series, which determine the fractal dimensions of time series as geometric objects.

The most general approach to the study of scaling exponents of heterogeneous time series is their multifractal analysis. It is sufficient to calculate a single scaling exponent to describe the scale invariance of homogeneous model time series, since such time series demonstrate only one type of singular behavior constant in time. On the contrary, the nature of the singularity of inhomogeneous time series at different points in time may differ; therefore, the description of such time series cannot be performed using only scaling constant. Therefore, multifractal analysis, which allows to provide local analysis of heterogeneous time series, is a more informative approach.

We used multifractal detrended fluctuation analysis (MF-DFA) [16] for making of multifractal analysis of time series generated by self-organized critical cellular automata. The application of this method makes it possible to obtain estimates of the spectrum of scaling constant time series: $\{h(q)\}$.

In short, the algorithm of MF-DFA method is reduced to revealing the power law

$$
\begin{equation*}
F(q, s) \propto s^{h(q)} \tag{7}
\end{equation*}
$$

for the fluctuation function

$$
\begin{equation*}
F(q, s)=\left\{\frac{1}{2 N_{s}} \sum_{v=1}^{2 N_{s}}[\mu(v, s)]^{\frac{q}{2}}\right\}^{\frac{1}{q}} \tag{8}
\end{equation*}
$$

To calculate function (8) from a discrete time series $\eta_{i}$, a fluctuation profile is formed $X_{i}=\sum_{k=1}^{i}\left(\eta_{k}-\bar{\eta}\right)$, which is divided into $N_{s}$ non-intersecting intervals $V$ containing the equal number of points $S$. Further, for each of the intervals, the local trend $x_{v, i}$ and the deviation of the fluctuation profile from the local trend $\Delta Y_{v, i}=X_{v, i}-x_{v, i}$ are determined. The value $\mu(v, s)=\max \Delta Y_{v, i}-\min \Delta Y_{v, i}$ for each split interval.

A detailed description of the algorithm of the MF-DFA method, as well as its capabilities and limitations, are presented in the paper [16]. Therefore, we will restrict ourselves by considering the main features of the time series for which the power law is satisfied (7). For multifractal time series at $q>0$, the main contribution to function (8) is given by the partition intervals $v$ characterized by large values $\mu(v, s)$; at $q<0$, the main contribution to function (8) comes from the partition intervals $v$ characterized by small values $\mu(\nu, s)$. For monofractal time series $h(q)$ does not depend on $q$. This is due to the fact that the behavior of function (8) when changing the scale $s$ is the same for all intervals $v$.

## 3 Results and their Discussion

Cumulative mean and variance, as well as their corresponding time series, are presented in the Fig. 7 and the Fig. 8. These figures show moments and time series for the Manna model. The dimensions of cellular automata are $40 \times 40$. For other cellular automata and their other grid sizes, the mean and variance behavior are similar.

|  |  | 100 90 80 70 60 50 40 40 30 20 10 |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Fig. 7. Cumulative mean for the Manna model |  | Fig. 8. Cumulative variance for the Manna model |  |  |

Cumulative mean and variance are not informative indicators characterizing the transition of cellular automata in the SOC state. Indeed, these moments are increasing functions of time and there are no significant changes in them when passing through the SOC state.

Cumulative kurtosis and skewness are quite informative precursors for the transition of cellular automata into the SOC state. The cumulative kurtosis and skewness, as well as their corresponding time series, are presented in the Figures 9 and 10. These figures show the moments and time series for the Manna model. The dimensions of cellular automata are $50 \times 50$. For other cellular automata and their other grid sizes, the behavior of kurtosis and skewness is similar.


As the cellular automata approach to the SOC state, a noticeable decrease in kurtosis and skewness is observed up to the critical point. At the same time, a sharp increase in these cumulative moments is observed at the critical point.

The change in the cumulative moments when approaching the critical point has a simple explanation. Mean and variance increase as a result of the increase in the number of grains falling from the grid in the certain time interval $\Delta t_{c} \in\left(t, t_{c}\right)$ from the SubC phase, preceding the transition of the cellular automaton to the critical state. The decrease in the skewness in the interval $\Delta t_{c}$ is also a consequence of the increase in the number of the grains. In this interval, a right-sided asymmetry of the distribution is still observed, characterized by an elongated right "tail," which decreases as the critical point is approached. In other words, the shortening of the right "tail" of the distribution occurs in the interval $\Delta t_{c}$. At the critical point, a sharp lengthening of the right "tail" of the distribution occurs as a result of the accumulation of the number of the grains from the SupC phase. An increase in the number of the grains in the interval $\Delta t_{c}$ also leads to a decrease in kurtosis in this interval. The peak of the distribution near the mathematical expectation is sharp for the entire SubC phase, but as the critical point is approached, the peak of the distribution is smoothed out. At the critical point, there is severe increase in the sharpness of the distribution.

In the Fig. 11 the cumulative scaling exponents $h(q)$ at $q=1,3,5$ are shown. The behavior of scaling exponents is similar for all cellular automata and their sizes; therefore, we restrict ourselves by considering only two
automata. The scaling exponents $h(1)$ are shown in blue, in orange $-h(3)$ and in gray $-h(5)$.


The time series generated by cellular automata are multifractal time series. Moreover, scale invariance in the form (7) is characteristic only for $q>0$, for $q<0$ scale invariance is not observed. Therefore, there are only the scaling exponents describing the intervals of time series partitioning $v$ with large fluctuations. Intervals $v$ with small fluctuations are not typical for the studied time series.

As approaching to the critical point, the distance between the points $h\left(q_{i}\right)=|h(1)-h(3)|+|h(3)-h(5)|$ decreases and is the smallest at the critical point. In the SupC phase, the distance between the points is almost independent of the iteration. All this is demonstrated in the Figure 11. Recall that for the Manna model $(50 \times 50) t_{c}=5671$.

Thus, the moments of probability density function, primarily $\gamma$ and $\kappa$, as well as the scaling exponents $h(q)$, can be used as indicators of early warning for the SOC state in cellular automata.

## Conclusions

Analysis of the behavior in time of moments and scaling exponents made it possible to provide early detection of the self-organized critical state in cellular automata. For such the early detection, it is sufficient to carry out the statistical and multifractal detrended fluctuation analysis of time series for the number of grains falling from the grid generated by cellular automata.

The results obtained allow us to make the following conclusions:
(1) Self-organized critical cellular automata generate multifractal time series, in which subcritical and supercritical phase, and self-organized criticality state can be distinguished.
(2) The most informative indicators of early detection of self-organized criticality state are cumulative skewness and kurtosis.
(3) Multifractality of time series for number of grains falling from the grid makes it possible to use cumulative scaling exponents as indicators of early detection of self-organized criticality state.

In conclusion, we briefly consider the possible practical applications of the use of the proposed indicators for early detection of critical states. If real systems are able to self-organize into the critical state, then the cumulative moments of probability distribution and the cumulative scaling exponents can be used as early warning indicators for critical states. Self-organized criticality is characteristic of phenomena and processes of a very different nature: solar flares, earthquakes, floods, forest fires, the emergence and extinction of species, demographic, ecological, economic, social, informational processes. Early detection of the critical state means predicting the critical moment in time after which the system behaves in an unpredictable manner. In this case, the system is in the supercritical phase, which is characterized by avalanche-like dynamics.

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# Double Symmetry and Generalized Intermittency in Transitions to Chaos in Electroelastic Systems 

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#### Abstract

Mathematical models of a deterministic system of the type "analog generatorpiezoelectric transducer" are considered. A double symmetry, atypical for dynamical systems, is found in the alternation of scenarios of transitions from regular attractors to chaotic ones. For the considered system, the symmetry inside symmetry: the described above chains of scenarios is located at the "median" point of other wider symmetric chains of transition to chaos was found.

Also, for the first time for the considered system, a transition "chaotic attractor of one type-chaotic attractor of another type" through generalized intermittency was discovered. One of the distinctive features of such a transition is the appearance of coarse-grained (rough) laminar phase instead of laminar phase of usual intermittency. Keywords: nonideal electro-elastic systems, scenarios of transition to chaos, generalized intermittency.


## 1 Introduction

Consider a cylindrical piezoceramic transducer placed in an acoustic medium. Let us assume that the oscillations of a piezoceramic transducer are excited by an analog generator. Let's also assume that the power of the generator is comparable to the power consumed by the transducer. Under these assumptions, the "generator - piezoceramic transducer" system is a typical nonideal dynamic system according to Sommerfeld-Kononenko (Sommerfeld[1,2], Kononenko[3]). The mathematical model of such a system was described using a normal system of ordinary differential equations in Krasnopolskaya and Shvets[4].

The mathematical model of the "generator-transducer" system was derived for a real physical system based on the strict principles of the general theory of electroelastic systems in acoustic media. Subsequently, it was revealed that the "generator-transducer" system has a very wide variety of dynamic behavior. So in such a system, all the main types of regular attractors were discovered, such as equilibrium positions, limit cycles and invariant tori (Krasnopolskaya and Shvets[4], Balthazar et al.[5], Shvets and Donetskyi[6]). Chaotic attractors, including hyperchaotic ones, were also found in the "generatortransducer" system (Shvets and Krasnopolskaya[7]). Transitions to chaos (hy-
perchaos) through a cascade of period doubling bifurcations (Feigenbaum[8,9] and through intermittency (Manneville and Pomeau[10]) were identified. And finally, in paper Shvets and Donetskyi[6], self-excited, hidden and rare attractors were discovered in the "generator-transducer" system.

The above allows us to assert that the "generator-converter" system has greater variety of dynamic behavior than the classical Lorentz (Lorenz[11]) and Rössler (Rössler[12,13]) systems. Such system is the "library" of regular and chaotic dynamics and can be used as a basic one in the study of the general theory of dynamical systems.

## 2 Mathematical model

Using papers Krasnopolskaya and Shvets[4], Shvets and Donetskyi[6], we write the mathematical model of the "generator-converter" system in the form a normal system of differential equations:

$$
\begin{align*}
& \frac{d \xi}{d \tau}=\zeta \\
& \frac{d \zeta}{d \tau}=-\xi+\alpha_{1} \zeta+\alpha_{2} \zeta^{2}-\alpha_{3} \zeta^{3}-\alpha_{4} \beta \\
& \frac{d \beta}{d \tau}=\gamma,  \tag{1}\\
& \frac{d \gamma}{d \tau}=-\alpha_{0} \beta+\alpha_{5} \xi+\alpha_{6} \zeta-\alpha_{7} \gamma .
\end{align*}
$$

Here phase variables $\xi, \zeta$ describe the dynamics of piezoceramic transducer. Accordingly, phase variables $\beta, \gamma$ describe the dynamics of analog generator. The physical meaning of these variables and parameters $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{7}$ of the system (1) are described in detail in paper Krasnopolskaya and Shvets[4].

Since the system of equations (1) is a nonlinear system of differential equations, the study of its dynamic behavior, in the general case, can be carried out only by numerical methods. The methodology for conducting such research is described in the papers Shvets[14], Shvets and Krasnopolskaya[7].

## 3 Symmetry and double symmetry

Typical behavior for dynamical system is when, with increase(decrease) in the value of a bifurcation parameter, the following chain of transitions to chaos is observed: a cascade of bifurcations of period doubling of limit cycles, then chaos, then so called periodicity window, after which this chain repeats: cascade of period doubling $\rightarrow$ chaos $\rightarrow$ periodicity window $\rightarrow$ cascade of period doubling $\rightarrow \ldots$ This behavior is also known as Feigenbaum scenario. Accordingly, with a decrease(increase) in the value of a bifurcation parameter, different chain of transactions to chaos is observed. Namely limit cycle, then intermittency in chaos, then periodicity window, after which this chain repeats: limits cycle $\rightarrow$ intermittency in chaos $\rightarrow$ periodicity window $\rightarrow$ limit cycle $\rightarrow \ldots$ This


Fig. 1. Phase-parametric characteristic


Fig. 2. Phase-parametric characteristic
behaviour is known as Pomeau-Manneville scenario. In this system, however, there are regions of parameters for which violation of strict chain of transactions for either scenarios is observed.


Fig. 3. Phase portrait projections: at $\alpha_{2}=9.1$ (a); at $\alpha_{2}=9.13$ (b); at $\alpha_{2}=9.14$ (c); at $\alpha_{2}=9.15$ (c).

Let values of parameters be $\alpha_{0}=0.995, \alpha_{1}=0.0535, \alpha_{3}=9.95, \alpha_{4}=$ $-0.103, \alpha_{5}=-0.0604, \alpha_{6}=-0.12, \alpha_{7}=0.01$. And leave parameter $\alpha_{2}$ as bifurcation one. In Fig. 1, for these values of parameters, the phase-parametric characteristic of the system (1), the so-called bifurcation tree, is constructed. Steady-state periodic regimes correspond to individual branches of this tree, and chaotic ones correspond to densely black areas. A careful study of the phase-parametric characteristics allows us to understand the bifurcations occurring in the system. As one may notice, there is some symmetry value of bifurcation parameter ( $\alpha_{2} \approx 9.6455$ ), relative to which any chain of transitions to chaos is reflected. This means that with increase in the value of the bifurcation parameter both Feigenbaum scenario and Pomeau-Manneville one occur, which is violation of strict chain of transitions to chaos. Same behavior is also
true for the case of decrease in the value of the bifurcation parameter. We notice that such situation appears to be natural for this specific system, since it is not the first time when such symmetry in transition to chaos was established (Shvets and Donetskyi[6]). And as we will see further, not the last.

Consider couple more intervals of bifurcation parameter for which symmetric transition to chaos is observed along with some other interesting features.

Let us start with interval $9.075<\alpha_{2}<9.3$. As we can see from Fig. 2, there is a double symmetry in the alternation of scenarios of transitions to chaos. One of the symmetries is clearly seen over the entire range of variation of the bifurcation parameter. Inside this symmetry, in a much smaller interval, one more symmetry is visible. Such double symmetries (symmetries within symmetries) are quite atypical for dynamical systems. Just like before, we can see violation of strict chain of transitions to chaos both with increase and with decrease in the value of the bifurcation parameter.

In the Fig. 3 you can see couple of bifurcations of Feigenbaum scenario for the Phase-parametric characteristic presented in the Fig. 2. Namely, there are three first period doubling plotted in the Fig. 3a - Fig. 3c. And the chaos presented in the Fig. 3d.


Fig. 4. Phase-parametric characteristic

Another type of symmetry is realized on the interval of variation of the bifurcation parameter $9.646<\alpha_{2}<9.64625$. The phase-parametric characteristic of the system for this interval is shown in Fig. 4. Here, the transition to chaos occurs, in one bifurcation, through the intermittency both on the left
and on the right of the considered interval. Moreover, there are no periodicity windows inside the chaos. Accordingly, no other transitions to chaos are observed according to the Feigenbaum scenario.

## 4 Generalized intermittency and symmetry

Finally, consider the bifurcations that occur in the system on the interval $9.64624<\alpha_{2}<9.64665$. As the parameter $\alpha_{2}$ increases, a cascade of bifurcations of doubling the period of limit cycles begins in the system, which leads to the appearance of a chaotic attractor. Further, as $\alpha_{2}$ increases, the chaotic attractor is replaced with periodicity window. Then this chain of transitions is observed again: a cascade of period doubling bifurcations $\rightarrow$ chaos $\rightarrow$ a periodicity window, and so on. However, the sequence of such transitions is interrupted at $\alpha_{2} \approx 9.64631$. Further, an extremely interesting transition occurs from a chaotic attractor of one type to a chaotic attractor of another type according to the scenario of generalized intermittency. This scenario is described in detail in the papers Krasnopolskaya and Shvets[15], Shvets and Sirenko[16]. One of distinctive features of such a transition is the appearance of coarse-grained (rough) laminar phase instead of laminar phase of usual intermittency.


Fig. 5. Phase-parametric characteristic

We notice that all described above behavior is symmetric, i.e. exists some "median" value of bifurcation parameter $\alpha_{2}$, such that any transition to chaos


Fig. 6. Phase portrait projections: at $\alpha_{2}=9.6463$ (a); at $\alpha_{2}=9.64631$ (b). Distribution of invariant measure: at $\alpha_{2}=9.6463$ (c); at $\alpha_{2}=9.64631$ (d).
is reflected. But there is more than that, since the very first chain of transitions to chaos that happend prior the generalized intermittency is reflected too. It is worth emphasizing due to fact that regularly generalized intermittency is not a part of any other chain of transitions to chaos.

Let us illustrate the scenario of generalized intermittency using phase portraits and distributions of the invariant measure of the corresponding attractors presented in Fig. 6. In the Fig. 6a and Fig. 6c, the phase portrait projection and distribution of invariant measure are presented respectively, prior the generalized intermittency. After the bifurcation, chaotic attractor of one type disappears, and chaotic attractor of other type borns. Phase portrait projection, as well as distribution of invarian measure for this new attractor are presented in the Fig. 6b and Fig. 6d respectively. Behavior of newborn chaotic attractor consists of two main phases: the rough-laminar phase and turbulent one. In the rough-laminar phase, trajectory of the caotic attractor makes chaotic movements near localization of disappeared attractor. During these movements, trajectory of newborn attractor almost coincide with trajectory
of disappeared one. These correspond to the much darkened areas in Fig. 6d. Then, at the unpredictable moment of time, turbulent phase begins. During this phase, trajectory leaves localization region and moves to distant regions of the phase space. After some time, trajectory returns to rough-laminar phase. This process of switching phases is repeated infinitely many times.

## 5 Conclusions

Thus, the paper explored a number of symmetries in the alternation of scenarios of transitions to chaos in a nonideal dynamic system "piezoelectric converteranalog generator". The existence of double symmetry is established for such alternations of scenarios.

The possibility of transitions "chaotic attractor of one type - chaotic attractor of another type" according to the scenario of generalized intermittency was revealed for the first time.

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# Optimality Principles in Solution of Nonlinear Control Problems under Uncertainty Conditions 

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#### Abstract

We study the problem of estimating reachable sets of nonlinear dynamical control systems considered under assumption of uncertainty in system parameters and in initial system states. It is assumed that only bounding sets are available for unknown terms, and no additional statistical information is provided on their behavior. Basing on main results of the theory of trajectory tubes of control systems we find solutions for control problems under uncertainty and study their properties. The algorithms of constructing ellipsoidal estimates for the solution tubes are discussed and tested. Applications to the problems of behavior of competing firms, population growth models, environmental change, the development of certain competing industries, etc. are dicussed.


Keywords: Nonlinear control systems, Estimation problem, Set-membership uncertainty, Ellipsoidal calculus, Maximum principle, HJB equation.

## 1 Introduction

The nonlinear dynamical control systems with unknown but bounded uncertainties related to the case of a set-membership description of uncertainty have attracted the attention of researchers for many years, due to interesting mathematical formulations of theoretical problems and in connection with the study of real models of various nature, with the presence of elements of nonlinearity and uncertainty in their description. In this regard, it is important to highlight fundamental researches by Kurzhanski and Valyi[13], Kurzhanski and Varaiya[14], Schweppe[16], Chernousko[4], Polyak et al.[15] and other researchers. The important issue in nonlinear set-membership estimation is to develop retated techniques, which produce necessary external or internal estimates for unknown system characteristics. In this context, it is possible to point out not only the theoretical academic interest in the study of problems of this circle, but also the possibility of practical application as the basis of algorithmic support for a number of applied problems for the models of which there are both uncertainty and nonlinearity. It would be worth mentioning in this context, for example, researches by August et al.[1], Boscain et al.[2], Ceccarelli et al.[3].

In this paper the modified state estimation approaches which use the special structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus are presented.

Here we develop the techniques related to constructing external set-valued estimates of reachable sets for nonlinear control systems. The results presented here are related to the theory of trajectory tubes of differential control systems with uncertain parameters and are based on the following main principles:

- the set-membership estimation approach to deal with system uncertainty,
- the optimality principle used for studied control problems to analyze the properties of reachable sets,
- the Hamilton - Jacobi - Bellman (HJB) approach which is used to find the external set-valued estimates of uncertain solutions of dynamical systems.

These basic ideas and the above approach open the possibilities to get estimates for the solutions of some new classes of nonlinear control system studied under uncertainty conditions.

## 2 Main Notations and Formulation of the Problem

Introduce first some basic notations. Here we denote as $R^{n}$ the $n$-dimensional Euclidean space, comp $R^{n}$ stands for the set of all compact subsets of $R^{n}, R^{n \times m}$ is the set of all real $n \times m$-matrices.

The inner product of $x, y \in R^{n}$ is denoted as $x^{\prime} y=(x, y)=\sum_{i=1}^{n} x_{i} y_{i}$ and the norm of $x \in R^{n}$ is

$$
\|x\|=\|x\|_{2}=\left(x^{\prime} x\right)^{1 / 2}, \quad\|x\|_{\infty}=\max _{1 \leq i \leq n}\left|x_{i}\right| .
$$

We use also symbols $I \in R^{n \times n}$ for the identity matrix, $\operatorname{tr}(A)$ for the trace of $n \times n$-matrix $A$ (the sum of its diagonal elements) and $|A|$ for the determinant.

Also let $B(a, r)=\left\{x \in R^{n}:\|x-a\| \leq r\right\}$ be a ball in $R^{n}$ with a center $a \in R^{n}$ and a radius $r>0$ and

$$
E(a, Q)=\left\{x \in R^{n}:\left(Q^{-1}(x-a),(x-a)\right) \leq 1\right\}
$$

be an ellipsoid in $R^{n}$ with a center $a \in R^{n}$ and with a symmetric positive definite $n \times n$-matrix $Q$.

Consider the following nonlinear control system with uncertain terms

$$
\begin{equation*}
\dot{x}=A(t) x+f(x) d+u(t), \quad x_{0} \in \mathcal{X}_{0}, \quad t \in\left[t_{0}, T\right] \tag{1}
\end{equation*}
$$

where $x, d \in R^{n}, f(x)$ is the nonlinear function, which is quadratic in $x, f(x)=$ $x^{\prime} B x$, with a given symmetric and positive definite $n \times n$-matrix $B$.

We assume that the matrix $A(\cdot)$ in (1) contains uncertain elements, namely we have

$$
\begin{equation*}
A(t) \in \mathbf{A}=A^{0}+\mathbf{A}^{1} \tag{2}
\end{equation*}
$$

where $A^{0}$ is given $n \times n$-matrix and $\mathbf{A}^{1}$ is the following set

$$
\begin{gather*}
\mathbf{A}^{1}=\left\{\left\{a_{i j}\right\} \in R^{n \times n}: a_{i j}=0 \text { for } i \neq j,\right. \text { and } \\
\left.a_{i i}=a_{i}, \quad i=1, \ldots, n, \quad a=\left(a_{1}, \ldots, a_{n}\right), \quad(a, D a) \leq 1\right\} . \tag{3}
\end{gather*}
$$

We will assume here that $\mathcal{X}_{0}$ in (1) is an ellipsoid, $\mathcal{X}_{0}=E\left(a_{0}, Q_{0}\right)$, with a symmetric and positive definite matrix $Q_{0} \in R^{n \times n}$ and with a center $a_{0}$.

Let the absolutely continuous function $x(t)=x\left(t ; u(\cdot), A(\cdot), x_{0}\right)$ be a solution to dynamical system (1)-(2) with initial state $x_{0} \in \mathcal{X}_{0}$, control $u(\cdot)$ and with a matrix $A(\cdot)$ satisfying (2)-(3). We assume here that the solutions $\{x(t)\}$ are extendable up to the instant $T$ and are bounded $\|x(t)\| \leq K$ (with some $K>0$ ) (see e.g. Filippova and Berezina[5]).

The reachable set $\mathcal{X}(t)$ at time $t\left(t_{0}<t \leq T\right)$ of system (1)-(2) is defined as the following set

$$
\begin{equation*}
\mathcal{X}(t)=\left\{x \in R^{n}: \exists x_{0} \in \mathcal{X}_{0}, \exists u(\cdot) \in \mathcal{U}, \exists A(\cdot) \in \mathcal{A}, x=x\left(t ; u(\cdot), A(\cdot), x_{0}\right)\right\} \tag{4}
\end{equation*}
$$

This kind of nonlinear control systems with uncertain data was studied earlier in Filippova[6], here we consider another kind of techniques which allows us to find the external ellipsoidal estimate $E\left(a^{+}(t), Q^{+}(t)\right)$ (with respect to the inclusion of sets) of the reachable set $\mathcal{X}(t)\left(t_{0}<t \leq T\right)$.

The main problem considered here is related to the search for possibilities of determining the reachability sets of dynamical systems of the specified class. In cases when it is difficult to find the exact reachability sets of an indefinite system (or it takes a very long time to construct it), the statements and results proposed here may well turn out to be useful, especially in cases when approximate solutions of reachability and optimization problems are sufficient.

## 3 Hamilton - Jacobi - Bellman Inequalities in State Estimation

We develop here in some features the techniques of generalized solutions of Hamilton - Jacobi - Bellman inequalities to find the external set-valued estimates of reachable sets as level sets of a related cost functional.

The solution of problems of state estimation and control synthesis for systems described by ODEs with unknown but bounded disturbances may be transformed to the investigation of first order PDEs of the Hamilton- Jacobi Bellman (HJB) type and their modifications.

To investigate this possibility, consider the control system

$$
\begin{gather*}
\dot{x}=f(t, x, u(t)), \quad t \in\left[t_{0}, T\right]  \tag{5}\\
u(\cdot) \in U=\left\{u(\cdot): u(t) \in U_{0} \in \operatorname{comp} R^{m}, \quad t \in\left[t_{0}, T\right]\right\}  \tag{6}\\
x\left(t_{0}\right)=x_{0} \in X_{0} \tag{7}
\end{gather*}
$$

with a solution $x(t)=x\left(t, u(\cdot), t_{0}, x_{0}\right)$ and with the reachable set $X(t)=$ $X\left(t ; t_{0}, X_{0}\right)$ generated by the trajectory tube

$$
\begin{gather*}
X(\cdot)=X\left(\cdot ; t_{0}, X_{0}\right)=\bigcup\{x(\cdot)=  \tag{8}\\
\left.x\left(\cdot, u(\cdot), t_{0}, x_{0}\right) \mid x_{0} \in X_{0}, u(\cdot) \in U\right\}
\end{gather*}
$$

We will use here the following result (Kurzhanski[12]).

Lemma 1. Assume that there exists a function $\mu(t)$ integrable on $\left[t_{0}, T\right]$ and such that

$$
\begin{equation*}
V_{t}(t, x)+\max _{u \in U}\left(V_{x}, f(t, x, u)\right) \leq \mu(t), t_{0} \leq t \leq T \tag{9}
\end{equation*}
$$

Then the following external estimate of the reachable set $X(t)=X\left(t ; t_{0}, X_{0}\right)$ is true

$$
\begin{equation*}
X(t) \subseteq\left\{x: V(t, x) \leq \int_{t_{0}}^{t} \mu(s) d s+\max _{x \in X_{0}} V\left(t_{0}, x\right)\right\}, t_{0} \leq t \leq T \tag{10}
\end{equation*}
$$

Instead of (9), we may consider the following inequality of a more general type (Gurman[10], Gusev[11]).

$$
\begin{equation*}
V_{t}(t, x)+\max _{u \in U}\left(V_{x}, f(t, x, u)\right) \leq g(t, V(t, x)) \tag{11}
\end{equation*}
$$

with $g(t, V)$ integrable in $t \in\left[t_{0}, T\right]$ and continuously differentiable in $V$.
Consider the following ordinary differential equation

$$
\begin{equation*}
\dot{U}(t)=g(t, U), \quad U\left(t_{0}\right)=U_{0} \tag{12}
\end{equation*}
$$

which is called a comparison equation for (5)-(7).
Theorem 1. Assume that (11) and (12) are fulfilled. Assume also that

$$
\begin{equation*}
\max _{x \in X_{0}} V\left(t_{0}, x\right) \leq U_{0} \tag{13}
\end{equation*}
$$

Then the following upper estimate is valid

$$
\begin{equation*}
X(t) \subseteq\{x: V(t, x) \leq U(t)\}, \quad t_{0} \leq t \leq T \tag{14}
\end{equation*}
$$

## 4 Main Results

### 4.1 Ellipsoidal Estimates through HJB inequalities

Consider the following control system

$$
\dot{x}(t)=A x(t)+f(x(t)) d+u(t), \quad x \in R^{n}, \quad t_{0} \leq t \leq T
$$

with

$$
\begin{gathered}
x\left(t_{0}\right)=x_{0} \in X_{0}=E\left(a_{0}, Q_{0}\right), \quad u(t) \in U=E(\hat{a}, \hat{Q}), \\
f(x)=x^{\prime} B x, \quad d \in R^{n}
\end{gathered}
$$

Here $Q_{0}, \hat{Q}, B$ are symmetric positive definite $n \times n$ - matrices and $k_{0}^{+}$is such that $E\left(a_{0}, Q_{0}\right) \subseteq E\left(a_{0},\left(k_{0}^{+}\right)^{2} B^{-1}\right)$.

Consider the following HJB inequality

$$
\begin{equation*}
V_{t}(t, x)+\max _{u \in E(\hat{a}, \hat{Q})}\left(V_{x}, A x+\tilde{f}(x) d+u\right) \leq 0 \tag{15}
\end{equation*}
$$

with boundary condition

$$
\begin{equation*}
V\left(t_{0}, x\right)=\phi(x) \leq 0 \tag{16}
\end{equation*}
$$

where $\phi(x)$ is a given continuously differentiable function.

Theorem 2. Let

$$
\begin{equation*}
V(t, x)=\left(x-a^{+}(t)\right)^{\prime}\left(r^{+}(t)\right)^{-1} B\left(x-a^{+}(t)\right)-1 \tag{17}
\end{equation*}
$$

with $a^{+}(t)$ and $r^{+}(t)$ defined in Theorem 4. Then $V(t, x)$ satisfies the HJB inequality (15) with the boundary condition

$$
\begin{equation*}
V\left(t_{0}, x\right)=\left(x-a_{0}\right)^{\prime}\left(k_{0}^{+}\right)^{-2} B\left(x-a_{0}\right)-1 \leq 0 \tag{18}
\end{equation*}
$$

Moreover, the related upper estimate

$$
\begin{equation*}
X(t) \subseteq\{x: V(t, x) \leq 0\}, \quad t_{0} \leq t \leq T \tag{19}
\end{equation*}
$$

is true.
We observe that Theorem 2 allows us to find the solution of HJB inequality explicitly. It follows from the special form of the chosen initial function $V\left(t_{0} ; x\right)$ and from a type of studied control system.

In more general cases the use of appropriate approximations gives us the way to establish a similar connection between the techniques of ellipsoidal calculus for dynamic control systems with uncertainties and results based on comparison theorems of theory of Hamilton-Jacobi-Bellman equations and inequalities.

### 4.2 Example

Consider an example which show that in nonlinear case the reachable sets of the dynamical system of the studied type (with simultaneously presenting nonlinearity and uncertainty) may lose the convexity property with increasing time $t>t_{0}$. Nevertheless the related external estimates calculated on the basis of above ideas and results are ellipsoids (and therefore convex) and these ellipsoids contain the true reachable sets of the studied nonlinear system. The ellipsoidal estimates in some directions are tight that is, they cannot be further reduced, otherwise they will stop evaluating the reachable set and to give the guaranteed upper estimate. In this sense, the proposed estimates are accurate.

Example. Consider the following control system in the 3-dimensional Euclidean space $R^{3}$

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-a_{1} x_{1}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+u_{1}  \tag{20}\\
\dot{x}_{2}=a_{2} x_{2}+u_{2} \\
\dot{x}_{3}=a_{3} x_{3}+u_{3}
\end{array}\right.
$$

Here we take $x_{0} \in X_{0}=B(0,1), 0 \leq t \leq 0.4$ and $U=B(0,0.1)$. System coefficients $\left\{a_{1}, a_{2}, a_{3}\right\}$ are unknown but bounded,

$$
a_{1}^{2}+a_{2}^{2}+a_{3}^{2} \leq 1
$$

Applying Theorem 1 and using the numerical algorithm similar to those described in Filippova and Matviychuk[8,9] we can construct the upper ellipsoidal tube $E^{+}(t)$ (it is shown in blue colour in Fig.1) which estimate the real setvalued solution of the system $X(t)$ (shown in black in Fig.1).


Fig. 1. The reachable set (black colour) and its upper estimate (blue colour).

This example confirms that the upper ellipsoidal estimates in some directions are tight that is, they cannot be further reduced, otherwise they will stop evaluating the reachable set and will not give the guaranteed upper estimate.

In this sense, the proposed estimates are accurate. Further discussions and other numerical examples may be found also in

## 5 Conclusions

The problems of state estimation for nonlinear dynamical control systems with unknown but bounded initial state were considered.

The solution was studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets at instant $t$ to control system.

We presented the modified state estimation approach which uses the special structure of the control system and the techniques of generalized solutions of Hamilton - Jacobi - Bellman equations and inequalities and is based on the comparison method for analogies to related Lyapunov functions.

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# Piezo spintronic effect in DNA molecular chains 

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#### Abstract

Recent efforts have been focused on producing nanoscale spintronic systems based on molecular materials. Molecular spintronics is an exciting concept for spin-based quantum computing. Spintronics combines the electronic with electron spin, which is an attractive field for processing and transferring the information. On the other hand, creating the pure spin currents in response to strain can be studied in the content of the piezo spintronic effect. In this regard, we have tried to design a DNA-based piezo spintronic device. We have proposed a theoretical model for controlling the spin current in DNA based on coupling between mechanical distortions and spin degrees of freedom. We have used the chaos theory tools to study the spin transport properties in the system. The obtained results determine that different DNA sequences show distinct behavior with respect to the mechanical tension. Also, the regions in the parameter values in which the maximum spin current flows through the system can be investigated. The mechanical tension can adjust the spin current flowing through the system. Therefore, one can design and control a novel piezo spintronic nanodevice based on DNA sequences.


Keywords: Piezo spintronic, Mechanical tension, DNA chain, Spin current, Chaos theory tools.

## 1 Introduction

The spintronic field can detect, inject and manipulate electron spins into solidstate systems [1]. Researchers have recently shown, using experiments and theoretical work, that they can perform similar and even better functions in making spinning devices than inorganic metals and semiconductors [2]. This phenomenon, known as molecular spintronics, has grown exponentially over the past few decades for practical applications [3]. The molecular spintronic field uses the spin state of organic molecules to produce electromagnetic devices widely used in sensors, memory, and quantum computing [4]. Molecular spintronic devices can produce future spin valves and quantum computing devices [5]. In these devices, polar spin currents are transmitted through molecules [1]. In 2011, scientists discovered that the transfer of electrons through chiral molecules depends on the direction of the electron spin. It has recently been shown that the transmission of charge in these molecules is spin polarized [6].
In this work, we use a new effect, called piezo spintronic (piezo in Greek means stress), to generate spin current in DNA nanowires, which is based on mechanical connection and the degree of spin release [7]. This effect, unlike the effects of piezo magnetism and piezoelectricity, is a phenomenon limited to the simultaneous presence of systems with time-reversal (T), inversion (I), and symmetry failure [8]. This mechanism opens the way to obtain and measure net
spin currents. In essence, a crystal may exhibit piezoelectric, piezomagnetic, and piezo spintronic effects simultaneously. The piezo voltage is one of the most effective methods for controlling magnetic switching, in which the deformation of the crystal structure of the magnetic material changes the crystal magnetic anisotropy, which is directly related to a spin-orbit interaction in the crystal [9]. On the other hand, the DNA molecule is widely used as a complex nanostructure with high flexibility in nanotechnology [10]. The double-stranded DNA molecule is a piezoelectric material. Piezoelectric materials are a class of dielectrics that can be polarized by an electric field and mechanical stress [11]. This particular property of piezoelectric materials is due to the crystal structure of the material [12]. Piezoelectric materials are used in converters and devices that convert electrical energy into mechanical energy or vice versa. Piezoelectric materials have many applications in diodes, switches, memories, transistors, sensors, energy storage devices, etc. [13, 14, 15]. DNA molecule is a chiral molecule due to its asymmetric crystal structure (mirror asymmetry) that can exhibit conductivity, insulation, semiconductor, and superconductivity. On the other hand, chiral organic molecules are a good candidate for transmitting information encoded in spin and spin-polarized current sources [16]. The piezo spintronic effect is very similar to the polarization of charge currents caused by pressureinduced spin-orbit interactions. In this work, we show that many polar spin currents can be produced by applying external mechanical stress to the molecular junctions of the DNA chain. We also show that spin-dependent charge transport can be observed in DNA nanowires by applying the mechanical stress and in the presence of an external magnetic field. For this purpose, we designed a piezo spintronic nanostructure based on the DNA sequence according the Figure 1.


Fig. 1. A schematic illustration that shows the DNA nanowires immersed in a thermal bath and connected at both ends to the metal leads in the presence of an external mechanical stress.

## 2 Model and Methods

In the current work, we have studied the spin currents along DNA nanowires through the piezo spintronic effect using the Peyrard-Bishop-Holstein (PBH) model modified for the spin degree of freedom. PBH model considers the pairing of bases in the direction of hydrogen bonding and plots the DNA molecule as a
one-dimensional network [17]. The Hamiltonian of the system can be presented as follows:

$$
\begin{align*}
& \mathrm{H}=\mathrm{H}_{\mathrm{DNA}}+\mathrm{H}_{\mathrm{so}}+\mathrm{H}_{\text {lead }}+\mathrm{H}_{\mathrm{DNA}-\text { lead }}+\mathrm{H}_{\text {Bath }}+\mathrm{H}_{\mathrm{DNA}-\mathrm{Bath}}+  \tag{1}\\
& \mathrm{H}_{\text {fields }}
\end{align*}
$$

where, the first term is Hamiltonian related to DNA molecule written as follows:

$$
\begin{align*}
H_{D N A}=\sum_{i, j=1,2} \sum_{\sigma=\uparrow \downarrow}[ & \left.\varepsilon_{i, j} c_{i, j}^{+\sigma} c_{i, j}^{\sigma}+V_{i, i+1, j} c_{i+1, j}^{+\sigma} c_{i, j}^{\sigma}\right]+\sum_{i, \sigma} \lambda_{i} c_{1, i}^{+\sigma} c_{2, i}^{\sigma}  \tag{2}\\
& +\sum_{i, j}\left[2 i t _ { s o } \operatorname { c o s } \theta \left(c_{i, j}^{+\uparrow} c_{i+1, j}^{\uparrow}-c_{i, j}^{+\uparrow} c_{i-1, j}^{\uparrow}-c_{i, j}^{+\downarrow} c_{i+1, j}^{\downarrow}\right.\right. \\
& +c_{i, j}^{+\downarrow} c_{i-1, j}^{\downarrow}+D_{i, i+1} c_{i, j}^{+\uparrow} c_{i+1, j}^{\downarrow}-D_{i, i+1}^{+} c_{i, j}^{* \downarrow} c_{i+1, j}^{\uparrow} \\
& \left.\left.+D_{i-1, i}^{*} c_{i, j}^{+\downarrow} c_{i-1, j}^{\uparrow}-D_{i-1, i}^{+\uparrow} c_{i, j}^{\downarrow} c_{i-1, j}^{\downarrow}\right)\right]+ \text { H.c. }
\end{align*}
$$

where, $t_{s o}$ is a spin-orbit coupling constant, $\theta$ is the helix angle, and $i, j$ indicate the number of sites and strings, respectively. Also, $\varepsilon$ is electron energy and $c_{i, j}{ }^{\dagger}$ , $c_{i, j}$ are the electron creation and annihilation operators at the site $(i, j)$, respectively. $\lambda_{i}$ is the interaction coupling between the DNA chains and

$$
\begin{align*}
D_{n, n+1}=i t_{s o} \sin & \theta\{\sin [n \Delta \varphi]+\sin [(n+1) \Delta \varphi]  \tag{3}\\
& +i \cos [n \Delta \varphi]+i \cos [(n+1) \Delta \varphi]\}
\end{align*}
$$

where $\varphi=n \Delta \varphi$ is the angle in the cylindrical coordinate and $\Delta \varphi$ is defines the twist angle. To maintain inverse symmetry, we have

$$
D_{\mathrm{n}, \mathrm{n}-1}=D_{n-1, n}^{*}
$$

Also, $V_{i, i+l, j}$ shows the mutation between the nearest neighbors, which is written as follows:
$V_{\mathrm{i}, \mathrm{i}+1}=\mathrm{V}_{\mathrm{o}} e^{-\beta_{\mathrm{i}}}\left(\mathrm{y}_{\mathrm{i}+1}-\mathrm{y}_{\mathrm{i}}\right)$
where $V_{o}$ is the constant of the hopping integral and the $\beta_{i}$ indicates the intensity of the coupling.
$\mathrm{H}_{\text {so }}$ is a spin-orbit coupling Hamiltonian written as follows:

$$
\begin{gather*}
H_{S O}=\sum_{n}\left[2 i t_{s o} \cos \theta\left(c_{i}^{\dagger \uparrow} c_{i+1}^{\uparrow}-c_{i}^{\dagger \uparrow} c_{i-1}^{\uparrow}-c_{i}^{\dagger \downarrow} c_{i+1}^{\downarrow}+c_{i}^{\dagger \downarrow} c_{i-1}^{\downarrow}\right)\right.  \tag{4}\\
\quad+D_{i, i+1} c_{n}^{\dagger \uparrow} c_{i+1}^{\downarrow}-D_{i, i+1}^{*} c_{i}^{\dagger \downarrow} c_{i+1}^{\uparrow}+D_{i-1, i}^{*} c_{i}^{\dagger \downarrow} c_{i-1}^{\uparrow} \\
\left.\quad-D_{i-1, i} c_{i}^{\dagger \uparrow} c_{i-1}^{\downarrow}\right]
\end{gather*}
$$

$\mathrm{H}_{\text {lead }}$ is the Hamiltonian related to the electrodes expressed as follows:

$$
\begin{align*}
& H_{l e a d}=\sum_{j=1,2} \sum_{k, \sigma}\left(\varepsilon_{L_{j, k}}+\frac{e V_{b}}{2}\right) a_{L_{j, k}}^{+\sigma} a_{L_{j, k}}+\sum_{j=1,2} \sum_{k, \sigma}\left(\varepsilon_{R_{j, k}}-\right.  \tag{5}\\
& \left.\frac{e V_{b}}{2}\right) a_{R_{j, k}}^{+\sigma} a_{R_{j, k}}
\end{align*}
$$

where $V_{b}$ is the bias voltage applied to the system and $a_{\beta_{j, k}}^{+}, a_{\beta_{j, k}}$ are the operators of electron creation and annihilation in the electrode $\beta=R, L$, respectively. $H_{D N A-l e a d}$ is the Hamiltonian relating to the interaction of the DNA molecule with the electrodes written as follows:

$$
\begin{equation*}
H_{D N A-\text { lead }}=\sum_{j=1,2} \sum_{k, \sigma=\uparrow \downarrow}\left(t_{L} a_{L_{j, k}}^{+\sigma} c_{j, 1}^{\sigma}+t_{R} a_{R_{j, k}}^{+\sigma} a_{j, N}^{\sigma}+\text { H.c. }\right) \tag{6}
\end{equation*}
$$

$H_{B a t h}$ is the Hamiltonian of thermal bath defined as follows [18]:

$$
\begin{equation*}
H_{\text {Bath }}=\sum_{i=1}^{N} \hbar \omega_{i} b_{i}^{+} b_{i}+2 \sum_{i=1}^{N-1} \hbar \Omega_{i}\left(b_{i}^{+} b_{i+1}+b_{i+1}^{+} b_{i}\right)+\text { H.c. } \tag{7}
\end{equation*}
$$

where $b_{i}^{+}$and $b_{i}$ are the oscillator creation and annihilation operators at the $i$ site, respectively. $\Omega$ is the reciprocal coupling constant and $\omega$ is the oscillator frequency at the site. The Hamiltonian of the interaction of the thermal bath with the DNA molecule is written as follows:

$$
\begin{equation*}
H_{D N A-l e a d}=\sum_{j=1,2} \sum_{k, \sigma=\uparrow \downarrow}\left(t_{L} a_{L j, k}^{+\sigma} c_{j, 1}^{\sigma}+t_{R} a_{R_{j, k}}^{+\sigma} a_{j, N}^{\sigma}+\text { H.c. }\right) \tag{8}
\end{equation*}
$$

here, $t_{i}$ is related to the elements of the tunneling matrix. In recent equations, the phrase $H . C$. is entered for the effect of a Hermitian conjugate.
Finally, $H_{\text {fields }}$ is related to the external electric and magnetic field Hamiltonian written as follows:

$$
\begin{equation*}
\mathrm{H}_{\text {fields }}=\mathrm{H}_{\mathrm{E}}+\mathrm{H}_{\mathrm{B}} \tag{9}
\end{equation*}
$$

The Hamiltonian of an electric and a magnetic field is written as follows:

$$
\begin{equation*}
H_{\mathrm{E}}=-e \sum_{i, \sigma=\uparrow \downarrow \downarrow} E d \cos [(i-1) \Delta \varphi] c_{i}^{\sigma \dagger} c_{i}^{\sigma} \tag{10}
\end{equation*}
$$

$$
\begin{equation*}
H_{B}=\sum_{i, j}\left(-\mu_{B} B c_{i, j}^{\dagger \dagger} c_{i, j}^{\uparrow}+\mu_{B} B c_{i, j}^{\downarrow \dagger} c_{i, j}^{\downarrow}\right) \tag{11}
\end{equation*}
$$

where $d$ is the radius of DNA and $\mu_{\mathrm{B}}=\frac{e \hbar}{2 m c}=5 / 78838 \uparrow \times 10^{-5}$ is the constant of magneton Bohr.
In the current study, to apply a mechanical stress to the system, we corrected the electron onsite and electron hopping constants at site $n$ through the stress parameter as follows [19]:

$$
\begin{align*}
& \varepsilon_{n^{\prime}}=\varepsilon_{n}^{0}+\frac{e r\left(1-\sigma \frac{s_{1}}{10}\right)}{L\left(1+\frac{s_{1}}{100}\right)} \tan \alpha V_{s d} \cos \left(\frac{2 \pi i}{10}+\varphi_{0}\right)  \tag{12}\\
& V_{n^{\prime}+1}=V_{n+1} e^{\left(1+\frac{s_{1}}{100}\right)} \tag{13}
\end{align*}
$$

where $r=10$ is the radius of DNA, $\alpha$ is the angle of DNA rotation, $V_{s \mathrm{~d}}$ is the source voltage, $\sigma=0.5$ is the Poisson rate, and $S_{l}$ is the longitudinal stress applied to DNA [20].
Here, the evolution equations of our dynamical system can be derived through the Heisenberg equation $\dot{c}_{n}^{\sigma}=-\frac{i}{\hbar}\left[c_{n}^{\sigma}, H\right]$ for up spin and down spin operators, respectively [21]. On the other hand, the spin current corresponding to up and down spins can be extracted via the continuity equation as follows:

$$
\begin{align*}
& I^{\uparrow}(t)=\frac{-i e}{\hbar} \sum_{n}\left\{\begin{array}{c}
W_{n, n+1} c_{n}^{\dagger \uparrow} c_{n+1}^{\uparrow}+W_{n-1, n}^{*} c_{n-1}^{\dagger} c_{n}^{\uparrow}+ \\
D_{n, n+1} c_{n}^{\downarrow \dagger} c_{n+1}^{\downarrow}-D_{n-1, n} c_{n-1}^{\dagger \downarrow} c_{n}^{\downarrow}
\end{array}\right\}  \tag{14}\\
& I^{\downarrow}(t)=\frac{-i e}{\hbar} \sum_{n}\left\{\begin{array}{c}
W_{n, n+1}^{*} c_{n}^{\dagger \downarrow} c_{n+1}^{\downarrow}+W_{n-1, n} c_{n-1}^{\dagger \downarrow} c_{n}^{\downarrow}- \\
D_{n, n+1}^{*} c_{n}^{\dagger \dagger} c_{n+1}^{\uparrow}+D_{n-1, n}^{*} c_{n-1}^{\dagger \uparrow} c_{n}^{\uparrow}
\end{array}\right\} \tag{15}
\end{align*}
$$

Therefore, the net charge current $I_{C}$ and the net spin current $I_{S}$ can be defined as follows:

$$
\begin{align*}
& I_{c}=I^{\uparrow}+I^{\downarrow}  \tag{16}\\
& I_{s}=I^{\uparrow}-I^{\downarrow}
\end{align*}
$$

## 3 Results and Discussion

In this study, we have tried to investigate the spin transfer and generation of pure spin currents in different sequences of DNA in the presence and absence of mechanical stress and magnetic field. The system shows high sensitivity to the initial conditions since the dynamics behavior is nonlinear.

### 3.1 Mechanical stress effect

To investigate the effect of mechanical stress on the system in the absence of a magnetic field, we have applied micro-positive and negative mechanical stress to the system (negative stress means DNA compression and positive stress means DNA elongation). According to Figure 2, in $S_{1}=3$, the maximum spin current and in $S_{1}=-3$, the minimum spin current flow through the system.


Fig. 2. The spin current with respect to the mechanical stress parameter $(B=0)$.

### 3.2 Effect of different sequences on the spin current

The conductivity dependence of the DNA molecule on the type and length of its sequence are studied, previously [22]. One of the effective parameters on the electrical properties of DNA molecule is the variation the sequence type since different arrangement of adjacent pairs in the molecule changes the coupling and the energy of the pair [23].
In this study, we have chosen three types of sequences: CH22, AT-rich, and CGrich, with a length of 60 bp . Therefore, we have studied the spin transport yn system by applying stress $S_{1}=3$ and in the presence of a magnetic field $B=4.5$ $(m T)$. According to Figure 3, at $t=500(p s)$, the maximum spin current flows in CH 22 sequence. A moderate spin current value flows in the CG-rich sequence, and a minimum spin current flows in the AT-rich sequence. The result indicates the effect of type the sequence in the spin current flows through the molecule chain.


Fig. 3. The time-series of spin currents for CH 22 , AT-rich and CG-rich sequences in the presence of tension $S_{1}=3$ and $B=4.5(\mathrm{mT})$.

### 3.3 Voltage effect

The external electric field, or in other words the gate voltage, is an influential factor on the spin current flowing through the DNA molecule. DNA molecule behaves distinctly against variable voltage [24]. According to Figure 4, the spin current in terms of voltage shows an increase in spin current with increasing voltage in some regions can be called quasi-ohmic regions. In some regions, a decreasing spin current is observed by increasing the voltage which can be expressed as spin-polarized negative differential resistance (SPNDR) regions. In the interval 7-8 (mV), system shows the quasi-ohmic behavior while in the interval 11-12 (mV), the SPNDR behavior is observed.


Fig. 4. The I-V characteristic diagram for the spin current in the presence of a stress

$$
S_{1}=3(B=0)
$$

### 3.4 Spin current in the presence of simultaneous variation of mechanical stress and DNA twist angle

We have tried to examine the simultaneous effect of mechanical stress and DNA twist angle as the most effective parameters in piezo spintronic effect on the spin current flowing through the CH22 sequence (Fig. 5). Figure 5 shows the creation
of island-like areas in different parameter values. The simultaneous effect of the mechanical stress parameter and twist angle leads to the maximum and the minimum spin current flowing regions. It is clear in Fig. 5 that no significant current flows through the system as long as the mechanical tension have the zero or negative value, but by increasing the twist angle and applying mechanical stress $S_{1}=3$ and higher, an increase in net spin current is observed, so that at $S_{1}=5$ and $\alpha=0.27(\mathrm{rad})$, the maximum spin current flows through the system.


Fig. 5. The simultaneous effect of mechanical stress and DNA twist angle on spin current.

### 3.5 Spin current in the presence of simultaneous variation of the external electrical field and DNA twist angle

Figure 6 shows the simultaneous effect of DNA twist angle and applied voltage on the spin current flowing through the CH 22 sequence. We have observed island-like regions in which the simultaneous effect of the molecule's twist angle and voltage creates areas with a maximum and minimum spin current. It is clear that by increasing the applied voltage to the system in the presence of mechanical tension, the islands with maximum spin current increases. The result indicates the positive effect of simultaneous application of voltage and stress on the spin current in the system.


Fig. 6. The simultaneous effect of voltage and DNA twist angle on spin current.

## Conclusions

In this study, we have studied the piezo spintronic effect led to creating the net spin current in response to stress. We discussed the piezo spintronic response of DNA nanowires to create net spin currents. By studying the spin current in terms of voltage, we have observed quasi-ohmic and SPNDR regions. Finally, by examining the simultaneous variation of the DNA twist angle and the applied voltage to the system, we have observed the islands that represented the maximum and minimum spin current.
The results expand the field for spin mechanical systems since it provides a direct coupling between the spin current and the tension. In this work, we used a simplified model that considers the structure of DNA as a ladder. To continue the work, it is suggested that the natural structure of DNA, which is considered as a double helix with degrees of freedom of angle be considered.

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# The atom, from a mathematical-physical perspective 

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#### Abstract

In this paper, an exhaustive study on the problem of atomicity with respect to set functions is provided. Different types of atoms are discussed, the relationships among them are studied and several examples and physical possible implications and applications are obtained.


Keywords: Atom, Pseudo-atom, Minimal atom, Set function, Self-similarity.

## 1 Introduction

Using different notions, concepts and results, in this paper we shall try to answer the question "What is the atom?" from a mathematical-physical perspective, offering at the same time a series of possible interpretations and meanings that exceed its strict limits. We shall see that the mathematical perspective preserves the intimate, defining property of the atom, in its various forms and mathematical meanings of being, in a sense, the essential indestructible, indivisible, irreducible, minimal and self-similar unity. We emphasize that an atom is a mathematical object (an entity) that, in essence, has no other subobjects (subentities) than the object itself or the null subobject. The idea is also found in computer science, for example. In partially ordered sets, atoms are generalizations of the singletons (that is, sets containing only one element) of the sets theory. Moreover, in this sense, atomicity (the property of a mathematical object of being atomic), provides a generalization in an algebraic context of the possibility of selecting an element from a nonempty set. In mathematical logic, an atomic formula is a formula without a deep propositional structure, that is, a formula that does not contain logical connections, or, equivalently, a formula that does not have strict subformulas. Atoms are thus the simplest well-formed formulas of logic, the compound formulas being formed by combining atomic formulas using logical connections. Also, also in logic, an atomic sentence is a type of declarative sentence that is either true or false and that cannot be broken down into other simpler sentences. In some models of set theory, an atom is an entity (a mathematical object) that can be an element of a set but does not itself contain elements with similar properties (hence the "ultimate" character of an atom). In mathematical analysis, a set's property of being an atom is defined in relation to another mathematical object, namely, with respect to a set (multi)function.

## 2. The mathematical-physical perspective

### 2.1. Set functions

Let $\mathcal{C}$ be a ring of subsets of a non-empty abstract set $T$ and $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$be a set function which satisfies the condition $m(\varnothing)=0$. The following notions generalize the notion of a measure in its classic sense (as a foundation of measure theory). In mathematical analysis, a measure (in classic sense) is a function which „measures", assigning to certain sets of a class (family) of sets, a positive real number or $+\infty$. In this sense, a measure is a generalization of the concepts of length, area or volume. One particularly important example is the Lebesgue measure on a Euclidean space, which assigns the conventional length, area and volume of Euclidean geometry to appropriate subsets of the Euclidean space $\mathbb{R}^{n}$. For instance, the Lebesgue measure of the interval $[0,1]$ is its length in the ordinary sense of the word, namely, 1 (Royden, 1988; Fremlin, 2000). A measure must be additive, which means that the measure of a set representing the union of a finite (or countable) number of smaller sets that are pairwise disjoint is equal to the sum of the measures of these smaller subsets.

The notions that we shall introduce next have contributed to the development in recent years of the theory of non-additive measures, sometimes known as the fuzzy measures theory (Pap, 1995). These notions prove their utility due to the necessity to model phenomena from the real world, in circumstances in which the condition of additivity (either finite or countable), as an immediate property of a measure, is much too restrictive.

The set function $m$ is called:
(i) null-additive if $m(A \cup B)=m(A)$, for every sets $A, B \in C$, satisfying the condition $m(B)=0$;
(ii) null-null-additive if $m(A \cup B)=0$, for every sets $A, B \in \mathcal{C}$, satisfying the condition $m(A)=m(B)=0$;
(iii) diffused if $m(\{t\})=0$, whenever $\{t\} \in \mathcal{C}_{\text {; }}$
(iv) monotone if $m(A) \leq m(B)$, for every sets $A, B \in \mathcal{C}$, so that $A \subseteq B ;$
(v) null-monotone if for every two sets $A, B \in \mathcal{C}$, having the property that $A \subseteq B$, if $m(B)=0$ holds, then one necessarily has also $m(A)=0$;
(vi) finitely additive if $m(A \cup B)=m(A)+v(B)$, for every disjoint sets $A, B \in \mathcal{C}$;
(vii) subbaditive if $m(A \cup B) \leq m(A)+v(B)$, for every (disjoint or not) $A, B \in C$.

Example. (i) Let us suppose that $T=\left\{t_{1}, t_{2}, \ldots, t_{n}\right\}$, where for every $i \in\{1,2, \ldots, n\}, t_{i}$ represents a particle, and $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+}$is a set function representing the mass of the particle. In the macrosopic world, $m$ is a finitely additive set function. At quantum scale, however, this statement no longer remains valid due to the phenomena of annihilation. For instance, if $t_{1}$ and $t_{2}$ represents an electron and a positron, respectively, then $m\left(\left\{t_{1}\right\}\right)=m\left(\left\{t_{2}\right\}\right)=9,11 \times 10^{-31} \mathrm{~kg}$, $m\left(\left\{t_{1}, t_{2}\right\}\right)=m\left(\left\{t_{1}\right\} \cup\left\{t_{2}\right\}\right)=0$;
(ii) Entropy in Shannon's sense is a subadditive set function, taking real values (Gavriluț and Agop, 2016; Gavriluț, 2019).

### 2.2. Types of atoms

In the following, we shall present several types of atoms in their mathematical meaning, we shall establish some relationships among these types of atoms and we shall also highlight several possible interpretations. Unless stated otherwise, $\mathcal{C}$ will represent a ring of subsets of an arbitrary nonvoid set $T$ and $m: \mathcal{C} \rightarrow \mathbb{R}_{+y} \quad$ an arbitrary set function satisfying the condition $m(\emptyset)=0$. This abstract set function represents the generalization of the classic notion of a measure used in measure theory and it is the mathematical object through which the process of so-called "measurement" is performed.

## Atoms and pseudo-atoms

These are the main types of atoms from the mathematical perspective:
I. A set $A \in \mathcal{C}$ is called an atom of $m$ if $m(A)>0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, it holds either $m(B)=0$ or $m(A \backslash B)=0$.

We observe that, in a certain sense, an atom is a special set, of strictly positive ,,measure", having additionally the property that any of its subsets either has zero ,,measure", or the difference set between the initial set and its subset we refer to has zero ,,measure". An atom can be interpreted, from a physics viewpoint, as the correspondent of a black hole.
II. The set function $m$ is said to be non-atomic if it has no atoms, that is, for every set $A \in C$ with $m(A)>0$, there exists a subset $B \in \mathcal{C}(B \subseteq A)$ so that $m(B)>0$ and $m(A \backslash B)>0$.
III. A set $A \in \mathcal{C}$ is called a pseudo-atom of $m$ if $m(A)>0$ and for every subset $B \in \mathcal{C} \quad(B \subseteq A)$ one has either $m(B)=0 \quad$ or $m(B)=m(A)$.

In other words, a pseudo-atom is a special set, of strictly positive ,,measure", for which any of its subsets either has null ,,measure", or has the same „,measure" as the set itself. Thus, it can be stated that a pseudo-atom possesses the property that any of its subsets either has null ,,measure" (that is, it is negligible during the „,measurement" process), or it entirely ,,covers" the set (during the same „,measurement" process). In other words, assuming that the set function $m$ is monotone, then a pseudo-atom is a set of strictly positive ,,measure" and which does not contain any proper subset of strictly smaller and strictly positive ,,measure".
IV. The set function $m$ is said to be non-pseudo-atomic if it does not have pseudo-atoms, that is, for any set $A \in \mathcal{C}$ with $m(A)>0$, there exists a subset $B \in \mathcal{C}(B \subseteq A)$ so that $m(B)>0$ and $m(B) \neq m(A)$.

For instance, the Lebesgue measure on the real line is a measure (in the classic sense) which is non-pseudo-atomic (Royden, 1988), and therefore it does not have any pseudo-atom. The non-pseudo-atomic measures satisfy the following remarkable property, which we owe to Sierpinski, a property which states that if $m$ is a non-pseudo-atomic measure (in classic sense), defined on a $\sigma_{\text {-algebra }} \mathcal{A}$ (of subsets of an abstract space $T$ ), and $A \in \mathcal{A}$ is an arbitrary set so that $m(A)>0$, then for every element $b \in[0, m(A)]$, there exists a set $B \in \mathcal{A}$, so that $B \subseteq A$ and $m(B)=b$ (in other words, the set function $m$ takes a continuum of values, and thus it does not omit any intermediate value).
V. A set function $m$ is called purely-atomic if the space $T$ can be represented as a finite or countable union of atoms of $m$.

Examples. (i) Let be the set $T=\{1,2, \ldots, 9\}$. We define the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+}$as follows: $\forall A \subseteq T, m(A)=\operatorname{card} A$. Then $\forall i \in\{1,2, \ldots, 9\}$, the singleton $\{i\}$ is an atom of $m$ : $\forall i \in\{1,2, \ldots, 9\}, m(\{i\})=1>0$ and $\forall B \subseteq\{i\}$, we have either $B=\emptyset$, in which case $m(B)=0$, or $B=\{i\}$, in which case $m(\{i\} \backslash B)=m(\varnothing)=0$. So, in this case, any singleton is an atom.
(ii) Generally, there is no relationship between the notion of an atom and that of a pseudo-atom: Let us consider an abstract set $T=\left\{t_{1}, t_{2}\right\}$ and let also be the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+}$defined for every $A \subset T$ by $m(A)=\left\{\begin{array}{l}2, \text { if } A=T \\ 1, \text { if } A=\left\{t_{1}\right\} \\ 0, \text { if } A=\left\{t_{2}\right\} \text { or } A=\emptyset .\end{array}\right.$

Then $T$ is an atom and it is not a pseudo-atom for $m$. Indeed, $m(T)=2>0$. Let be an arbitrary subset $B$ of $T$. If $B=\emptyset$, then $m(B)=0$;

If $B=\left\{t_{1}\right\}$, then, by the definition, $m(T \backslash B)=m\left(\left\{t_{2}\right\}\right)=0$;
If $B=\left\{t_{2}\right\}$, then, by the definition, $m(B)=0$;
If $B=\left\{t_{1}, t_{2}\right\}(=T)$, then $m(T \backslash B)=m(\emptyset)=0$.
Therefore, $T$ is indeed an atom of $m$. On the other hand, let us note that there exists the singleton $\left\{t_{1}\right\}$ for which $m\left(\left\{t_{1}\right\}\right)=1 \neq 0$ and $m\left(\left\{t_{1}\right\}\right)=1 \neq 2=m(T)$. Consequently, $T$ is not a pseudo-atom of $m$.

However, we note that, if the set function $m$ is null-addtive, then any atom of $m$ is a pseudo-atom $\left({ }^{*}\right)$. Indeed, let us assume that $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is a null-additive set function, and that the set $A \in \mathcal{C}$ is an atom of $m$. We shall prove that $A$ is also a pseudo-atom of $m$ : Obviously, since $A$ este atom, then $m(A)>0$. If we consider an arbitrary set $B \in \mathcal{C}$, with $B \subseteq A$, from the fact that $A$ is an atom it follows that either $m(B)=0$ or $m(A \backslash B)=0$. In the latter case, since $m$ is null-additive, it follows that $m(A)=m((A \backslash B) \cup B)=m(B)$. Consequently, $A$ is a pseudo-atom of $m$. Conversely, if the set function $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is, moreover, finitely additive, then any pseudo-atom $A \in \mathcal{C}$ of $m$ is an atom, too, and this immediately yields based on the equality $m(A)=m((A \backslash B) \cup B)=m(A \backslash B)+m(B)=m(B), \quad$ which implies $m(A \backslash B)=0$.

That is why, in the framework of the classic measure theory (a measure always possesses the null-additive property), the notions of an atom and that of a pseudo-atom coincide. The converse of the above statement (*) does not generally hold since there exist pseudo-atoms which are not atoms:
(ii) Let $T=\left\{t_{1}, t_{2}\right\}$ be an abstract set, containing two arbitrary elements, and let us consider the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+}$, defined for every set $A \subseteq T$, by $m(A)=\left\{\begin{array}{l}1, \text { if } A \neq \emptyset \\ 0, \text { if } A=\emptyset .\end{array}\right.$

Then $m$ is null-additive and $T=\left\{t_{1}, t_{2}\right\}$ is a pseudo-atom of $m$, but it is not an atom of $m$. Let $A, B \subseteq T$ be so that $m(B)=0$. By the definition of $m$ we note that we must necessarily have $B=\emptyset$, whence $m(A \cup B)=m(A)$, and this proves that the set function $m$ is null-additive.

We prove now that $T=\left\{t_{1}, t_{2}\right\}$ is a pseudo-atom of $m$. Indeed, we have $m(T)=1>0$ and let $B \subseteq T$ an arbitrary subset. If $B=\emptyset$, then $m(B)=0$. If $B \neq \emptyset$, then the set $B$ either is a singleton, or is the set T , itself
consisting of two elements. In both situations, one has $m(T)=1=m(B)$, which proves that $T=\left\{t_{1}, t_{2}\right\}$ is a pseudo-atom of $m$.

Let us prove now that $T=\left\{t_{1}, t_{2}\right\}$ is not an atom of $m$. Indeed, $m(T)=1>0$ and there exists the singleton $\left\{t_{1}\right\}$ for which we have $m\left(\left\{t_{1}\right\}\right)=1 \neq 0$ and $m\left(T \backslash\left\{t_{1}\right\}\right)=m\left(\left\{t_{2}\right\}\right)=1 \neq 0$. Therefore, $T=\left\{t_{1}, t_{2}\right\}$ is not an atom of $m$.
(iii) The Dirac measure (or, the unit mass measure) (or, the $\delta$-measure) $\delta_{t}$ concentrated in an arbitrary fixed point $t$ of an abstract set $T_{s}$ is an example of a measure (in the classical sense) which is purely-atomic (Kadets, 2018). The Dirac measure is defined as follows: If $\mathcal{A}$ is a $\sigma$-algebra of subsets of $T$, then $\delta_{t}(A)=\left\{\begin{array}{l}1, t \in A \\ 0, t \notin A\end{array}, \forall A \in \mathcal{A}\right.$.

Obvioulsy, $T$ is an atom of $\delta_{t}$ (because $\delta_{t}(T)=1>0$ and $\forall A \in \mathcal{A}$, it holds either $\delta_{t}(A)=0$ or $\delta_{t}(c A)=0$, as $t \notin A$ or $t \in A$, that is, $t \notin c A$ ).

Let us recall now the following:
If $\mathcal{C}$ is a ring of subsets of an abstract space $T$ and if $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is a set function satisfying the condition $m(\emptyset)=0$, two sets $A_{1}, A_{2}$ are said to be equivalent if $m\left(A_{1} \Delta A_{2}\right)=0$.

We note that if the set function $m$ is additionally null-monotone and null-additive, then $m\left(A_{1}\right)=m\left(A_{2}\right)$ (which justifies the terminology, since the equivalence of the sets takes place in the sense of the ,"measurement" process). Indeed, since $m\left(A_{1} \Delta A_{2}\right)=m\left(\left(A_{1} \backslash A_{2}\right) \cup\left(A_{2} \backslash A_{1}\right)\right)=0$ and $m$ is null-monotone, it follows that $m\left(A_{1} \backslash A_{2}\right)=0$ and $m\left(A_{2} \backslash A_{1}\right)=0$, whence, because $m$ is null-additive and $m\left(A_{1}\right)=m\left(\left(A_{1} \backslash A_{2}\right) \cup\left(A_{1} \cap A_{2}\right)\right)=m\left(A_{1} \cap A_{2}\right), m\left(A_{2}\right)=$ $m\left(\left(A_{2} \backslash A_{1}\right) \cup\left(A_{1} \cap A_{2}\right)\right)=m\left(A_{1} \cap A_{2}\right)$
it follows that $m\left(A_{1}\right)=m\left(A_{2}\right)$. We note that, with respect to the Dirac measure $\delta_{t^{\prime}}$ the atom $T$ (the space itself, unreduced to a single point) is equivalent to the singleton $\{t\}, t \in T$ (Kadets, 2018). Indeed, we have $m(T \Delta\{t\})=0$ (so, with respect to the Dirac measure, the space ,,collapses" into a single point).

We shall prove in the following that, with respect to a monotone and null-additive set function, any set which is equivalent to an atom is itself an atom: Let us assume that the set $A_{1}$ is an atom and we prove that the set $A_{2}$, which is equivalent to the set $A_{1}$, possesses the same property. Indeed,
according to the above statements, we have $m\left(A_{2}\right)=m\left(A_{1}\right)>0$ and let $B \in \mathcal{C}, B \subseteq A$, be arbitrary. If $m(B)=0$, then the proof ends. If $m\left(A_{1} \backslash B\right)=0$, then, since $m$ is monotone and $m\left(A_{1} \Delta A_{2}\right)=0$, it follows that $m\left(A_{2} \backslash A_{1}\right)=0$.

On the other hand, again from the monotonicity of $m$ we have $m\left(A_{2} \backslash B\right) \leq m\left(\left(A_{2} \backslash A_{1}\right) \cup\left(A_{1} \backslash B\right)\right)=m\left(A_{1} \backslash B\right)=0$, based also on the fact that $m$ is null-additive and $m\left(A_{2} \backslash A_{1}\right)=0$. Consequently, $m\left(A_{2} \backslash B\right)=0$, and this finally proves that $A_{2}$ is an atom of $m$, too.

Let us also note that, with respect to a monotone and null-additive set function, any set which is equivalent to a pseudo-atom is, itself, a pseudo-atom: We assume that $A_{1}$ is a pseudo-atom and we prove that the set $A_{2}$, which is equivalent to the set $A_{1}$, possesses the same property. Indeed, from the above statements, we have $m\left(A_{2}\right)=m\left(A_{1}\right)>0$ and let $B \in \mathcal{C}, B \subseteq A$, be arbitrary. If $m(B)=0$, then the proof ends. If $m\left(A_{1}\right)=m(B)$, then, since $m\left(A_{2}\right)=m\left(A_{1}\right)=m(B)$, it follows that $A_{2}$ is also a pseudo-atom of $m$.

## Atoms and fractality

Next, we shall underline the fact that both the notion of atom and that of pseudo-atom (in the mathematical sense) possess a remarkable property, namely that of self-similarity (every part reflects the whole), a property which is a characteristic to fractals, both from a mathematical point of view and from the perspective of modern physics. This finding, among others, justifies the extension we illustrate in the last section, in which we address the necessity to introduce the notion of a fractal atom (Gavriluţ et al., 2019).

The self-similarity property of the atoms (pseudo-atoms, respectively)
(i) If $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is a null-monotone set function, with $m(\emptyset)=0, A \in C$ is an atom of $m$ and $B \in \mathcal{C}$ is a subset of $A$ having the property $m(B)>0$, then $B$ is also an atom of $m$ and, moreover, $m(A \backslash B)=0$. (which means that the ,,measure" of what remains when the set $B$ is removed from the set $A$ is null). Indeed, one has $m(B)>0$ and if we consider an arbitrary set $C \in \mathcal{C}$, with $C \subseteq B$, then, since $B \subseteq A$, it follows that $C \subseteq A$. If $m(C)=0$, the proof ends. Let us assume now that $m(C) \neq 0$. Because $A \in C$ is an atom al lui $m$, it follows that $m(A \backslash C)=0$. Since $B \backslash C \subseteq A \backslash C$ and $m$ is null-monotone it gets that $m(B \backslash C)=0$ and, therefore, $B$ is an atom of $m$. Moreover, since $A \in C$ is
an atom of $m$ and $B \in \mathcal{C}$ is a subset satisfying the property $m(B)>0$, then we must necessarily have $m(A \backslash B)=0$.
(ii) If $A \in \mathcal{C}$ is a pseudo-atom of $m$ and the set $B \in \mathcal{C}$ satisfies $B \subseteq A$ and $m(B)>0$, then $B$ is also a pseudo-atom of $m$ and, moreover, $m(B)=m(A)$ (the sets $A$ are $B$ are ,,identical" with respect to the „,measure" $m$ ). Indeed, we have $m(B)>0$ and, if we consider an arbitrary set $C \in \mathcal{C}$, with $C \subseteq B$, then, since $B \subseteq A$, it follows that $C \subseteq A$. If $m(C)=0$, the proof ends. Let us assume now that $m(C) \neq 0$. Since $A \in C$ is a pseudo-atom of $m$, it follows that $m(A)=m(C)$. On the other hand, since $A \in \mathcal{C}$ is a pseudo-atom of $m$, the set $B \in \mathcal{C}$ satisfies $B \subseteq A$ and $m(B)>0$, then $m(B)=m(A)$. In consequence, $m(B)=m(C)$, and this finally proves that $B$ is also a pseudo-atom of $m$.

Let us make, at the end of this section, the following observation: Assuming that a set function $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is monotone, null-additive and regular (meaning that, roughly speaking, we can, through it, approximate sets about which we have little information, with sets about which we have more information), one can prove that for each atom $A$ of $m$ (if it exists), there exists a unique element $a \in A$ so that $m(A)=m(\{a\})$ (Pap, 1995) (this means that the ,,measure" of the atom is equal to the measure of each ,,point" it contains, and this reflects the holographic perspective, according to which the information is concentrated in a single point.

## Minimal atoms

We shall now introduce a very special category of atoms, which we show to reflect the property of indivisibility (non-decomposability). Let $\mathcal{C}$ be an arbitrary ring of subsets of an abstract space $T$ and let $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$be a set function so that $m(\varnothing)=0$. A set $A \in \mathcal{C}$ is called a minimal atom of $m$ if $m(A)>0 \quad$ and for every subset $B \in \mathcal{C} \quad(B \subseteq A)$ it holds either $m(B)=0$, or $B=A$ (Ouyang et al., 2015).

In other words, a minimal atom is a special set, of stricly positive ,,measure", so that any of its subsets has either zero ,,measure", or identifies with the set itself. Thus, a minimal atom has the property that any of its subsets has either zero ,,measure" (that is, it is negligible during the „measurement" process), or identifies with the initial set (without the need of $a$,,measurement" process). The terminology is justified. Indeed, if $A \in \mathcal{C}$ is a minimal atom of $m$, then for $m$ there cannot exist other minimal atom $A_{1} \in \mathcal{C}$, which is
different from $A$ and satisfies $A_{1} \subset A$. Indeed, if we assume, on the contrary, that there exists another minimal atom $A_{1} \in \mathcal{C}$ which is different from $A$ and satisfies $A_{1} \subset A$, then, since $A_{1}$ is a minimal atom, we get that $m\left(A_{1}\right)>0$. Because $A_{1} \subsetneq A_{\text {, then }} A_{1}=A$, and this is false due to our assumption.

Example. Let $T=\{a, b, c, d\}$ be an abstract set, constituted of four distinct elements and let also be the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+p}$ defined for every $A \subseteq T$ by $m(A)=\left\{\begin{array}{l}5, \text { if } A=T \\ 2, \text { if } A \neq T \\ 0, \text { if } A=\emptyset\end{array}\right.$

We note that any singleton (i.e., a set containing only one element) is a minimal atom of $m$. Indeed, the ,,measure" $m$ of any singleton is, according to the definition, 2, so it is strictly positive and any subset is either void and hence has zero measure, or is the set itself.

In general, any minimal atom is, particularly, an atom and also a pseudo-atom. Indeed, if $A \in \mathcal{C}$ is a minimal atom of $m$, then $m(A)>0$ and for any of its subset $B \in \mathcal{C}(B \subseteq A)$ it holds either $m(B)=0$, or $B=A$. The latter posibility yields $m(A \backslash B)=0$ and $m(B)=m(A)$, so $A$ is both an atom and a pseudo-atom of $m$.

The following examples show that there is generally no relationship between the notions of atom/pseudo-atom and that of minimal atom:

Examples. (i) Let $T=\{a, b\}$ be an abstract set constituted of two distinct elements and let also be the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+}$defined as follows:

$$
\forall A \subseteq T, m(A)= \begin{cases}1, & \text { if } A=\{a\} \text { or } A=T \\ 0, & \text { otherwise }\end{cases}
$$

Then $T$ is an atom of $m$ : Obviously, $m(T)=1>0$. Let $B \subseteq T$ be an arbitrary set. If $B=\emptyset$, then $m(B)=0$. If $B=\{a\}$, then $m(T \backslash B)=m(\{b\})=0$. If $B=\{b\}, \quad$ then $\quad m(B)=0 . \quad$ If $B=T=\{a, b\}$, then $m(T \backslash B)=m(\emptyset)=0$.

But $T^{\text {is }}$ not a minimal atom of $m:$ Obviously, one has $m(T)=1>0$ and let $B \subseteq T$ be an arbitrary set. We observe that there exists the set $B=\{a\} \neq T$ for which $m(B)=1 \neq 0$. We also note that the set $\{a\}$ is an atom (we have $m(\{a\})=1>0$ and any subset $B \subseteq\{a\}$ either is void, so $m(B)=0$, or is the set $\{a\}$ itself, so $m(\{a\} \backslash\{a\})=0)$.

The set $\{a\}$ is also a minimal atom of $m$ since $m(\{a\})=1>0$ and any subset $B \subseteq\{a\}$ either is void, so $m(B)=0$, or is $\{a\}$ itself.
(ii) Let $T=\{a, b, c, d\}$ be an abstract set and let also be the set function $m: \mathcal{P}(T) \rightarrow \mathbb{R}_{+y}$ defined as follows: $\forall A \subseteq T_{v}$

$$
m(A)=\left\{\begin{array}{l}
5, \text { if } A=T \\
3, \text { if } A=\{a, b, c\} \text { or } A=\{a, b, d\} \text { or } A=\{a, c, d\} \\
2, \text { if } A=\{a, b\} \text { or } A=\{a, c\} \\
0, \text { otherwise. }
\end{array}\right.
$$

Then $\{a, b\}$ and $\{a, c\}$ are minimal atoms of $m$. We shall prove the statement, for instance, for the set $\{a, b\}$ : Indeed, we have $m(\{a, b\})=2>0$ and let $B$ be an arbitrary subset. If $B=\{a, b\}$, the statement is verified. If $B=\{a\}$ or $B=\{b\}$, then, according to the definition, we have $m(\{a\})=m(\{b\})=0$, so the statement is again verified. If $B=\emptyset$, then $m(B)=0$.

In the following, we note that if $m: \mathcal{C} \rightarrow \mathbb{R}_{+}$is a null-null-additive set function and $A, B \in \mathcal{C}$ are two different minimal atoms of $m$, then they must be necessarily disjoint, that is, $A \cap B=\emptyset$. Indeed, let us assume that, on the contrary, $A \cap B \neq \emptyset$. Since $A, B \in \mathcal{C}$ are two minimal atoms of $m$, $A \backslash(A \cap B)=A \backslash B \subseteq A$ and $A \cap B \subseteq B$, it follows that $[m(A \backslash B)=0$ or $A \backslash B=A]$ and $[m(A \cap B)=0$ or $A \cap B=B]$.
(i) If $A \backslash B=A$, then $A \cap B=\emptyset$, which is false since, according to our asumption, we have $A \cap B \neq \emptyset$.
(ii) If $m(A \backslash B)=0$ and $m(A \cap B)=0$, then, since $m$ is null-null-additive, one gets that $m(A)=m((A \backslash B) \cup(A \cap B))=0$, which is false, since $m(A)>0$, the set $A$ being a minimal atom of $m$.
(iii) If $m(A \backslash B)=0$ and $A \cap B=B$, then $B \subseteq A$, so, since $A$ is a minimal atom of $m$, one gets from the above observation that $B=A$, which is false.

Consequently, $A \cap B=\emptyset$.
The property we shall demonstrate next reflects the nondecomposability (non-partitionability) of the minimal atoms: A minimal atom $A \in \mathcal{C}$ of a null-null-additive set function $m$ cannot be partitioned in sets that are elements of $\mathcal{C}$. Indeed, if we suppose, on the contrary, that there exists a partition of a lui $A$, this means that there exists a family $\left\{A_{i}\right\}_{i \in[1,2, m p\}}$ of nonvoid sets of $\mathcal{C}$ so that $\bigcup_{i=1}^{p} A_{i}=A$ and the sets $A_{i}$ are pairwise disjoint.

Referring to the first set $A_{1}$, since $A \in \mathcal{C}$ is a minimal atom, it follows that we cannot have the situation $A_{1}=A$. Therefore, $m\left(A_{1}\right)=0$. Analogously, for the second set, $A_{2}$, we get that $m\left(A_{2}\right)=0$. Recurrently, it gets that $m\left(A_{3}\right)=\ldots=m\left(A_{p}\right)=0$. Since $m$ is null-null-additive, it follows that $m(A)=m\left(\bigcup_{i=1}^{p} A_{i}\right)=0$, which is obviously false.

Consequently, any minimal atom is non- decomposable.
In the following, we shall prove that the converse of this statement also holds, namely, we shall demonstrate that any non-decomposable atom $A \in \mathcal{C}$ is necessarily a minimal atom. Indeed, since the set $A$ is an atom, then $m(A)>0$. Since the set $A$ is not partitionable, there cannot exist two nonvoid disjoint subsets $A_{1}, A_{2} \in \mathcal{C}$ of $A$ so that $A=A_{1} \cup A_{2}$. Let be an arbitrary set $B \in \mathcal{C}_{y}$ with $B \subseteq A$. If $m(B)=0$, then the proof ends. If $m(B)>0$, since $B \subseteq A$, one gets that $B=A$ (otherwise, the family $\{A \backslash B, B\}$ is a partition of $A: A \backslash B, B \in \mathcal{C},(A \backslash B) \cap B=\emptyset,(A \backslash B) \cup B=A$, which is false). Consequently, $A$ is a minimal atom. From the two statements above, one arrives at the following conclusion: an atom is minimal if and only if it is not partitionable (it is non-decomposable).

In the following, we shall highlight that, in the case when the abstract set $T$ is finite, then any set $A \in \mathcal{C}$, satisfying the condition $m(A)>0$ possesses at least one set $B \in \mathcal{C}, B \subseteq A$, which is a minimal atom minimal of $m$. Moreover, in the particular case when $A$ is an atom of $m$ and the set function $m$ is null-additive, one gets that $m(A)=m(B)$ and the set $B$ is unique. Indeed, let us consider the family of sets $\mathscr{A}=\{M \in \mathcal{C}, M \subseteq A, m(M)>0\}$. Obviously, since $A \in \mathcal{C}$, then $\mathscr{A} \neq \emptyset$. We note that any minimal element $M \in \mathscr{A}$ of $\mathscr{A}$ is a minimal atom of $m$. Indeed, since $M$ is a minimal element, there cannot exist another set $D \in \mathscr{M}$ so that $D \subsetneq M$ (*).

Since $M \in \mathscr{A}$, this means that $M \in \mathcal{C}, M \subseteq A$ and $m(M)>0$.
We shall prove that $M$ is a minimal atom of $m$. Indeed, for any set $S \subseteq M, S \in \mathcal{C}$, we have either $m(S)=0$ or $m(S)>0$. In the latter case, we have either $S=M$ (which is suitable) or $S \neq M$, which contradicts the statement (**).

Let us assume, moreover, that the set $A$ is an atom of $m$ and $m$ is nulladditive. According to the considerations proved above, there exists at least one set $B \in \mathcal{C}, B \subseteq A$, which is a minimal atom of $m$. This means that $m(B)>0$ and, because $A$ is an atom, we must necessarily have
$m(A \backslash B)=0$. Since $m$ is null-additive, this yields $m(A)=m((A \backslash B) \cup B)=m(B)$.

It only remains to prove that the set $B$ is unique. Indeed, if we suppose, on the contrary, that there exist two different minimal atoms $B_{1}$ and $B_{2}$ of $m$, this would imply, as before, that $m\left(A \backslash B_{1}\right)=m\left(A \backslash B_{2}\right)=0$. If $m\left(B_{1} \cap B_{2}\right)=0, \quad$ then $m(A)=m\left(A \backslash\left(B_{1} \cap B_{2}\right) \cup\left(B_{1} \cap B_{2}\right)\right)=m\left(A \backslash\left(B_{1} \cap B_{2}\right)\right)=$ $m\left(\left(A \backslash B_{1}\right) \cup\left(A \backslash B_{2}\right)\right)$,
which is false. If $m\left(B_{1} \cap B_{2}\right)>0$, since $B_{1}$ and $B_{2}$ are minimal atoms of $m$, it results that $B_{1}=B_{1} \cap B_{2}=B_{2}$, which is again false. Finally, we shall prove that, if the set $T$ is finite, the set function $m$ is null-additive, and $\left\{A_{i}\right\}_{i \in\{1,2, \ldots, p\}}$ is the family of all different minimal atoms which are contained in a set $A \in \mathcal{C}$, satisfying $m(A)>0$ (we proved in the above considerations that such atoms exist), then $m(A)=m\left(\bigcup_{i=1}^{p} A_{i}\right)$.
(This means that the set $A$ identifies itself, from the ,,measure" $m$ viewpoint, with the union of all different minimal atoms which it contains, therefore the minimal atoms are the only ones that matter from the ,,measurement" point of view).

Let us note that $m\left(A \backslash \bigcup_{i=1}^{p} A_{i}\right)=0$ (if, on the contrary, one has $m\left(A \backslash \bigcup_{i=1}^{p} A_{i}\right)>0$, from the statement proved above it would follow that there exists at least one set $B \in \mathcal{C}, B \subseteq A \backslash \bigcup_{i=1}^{p} A_{i} \subseteq A$, which is a minimal atom of $m$, and this is false since $A_{1}, \ldots, A_{p}$ are the only different minimal atoms contained in $A$ ). Since $m\left(A \backslash \bigcup_{i=1}^{p} A_{i}\right)=0$ and $m$ is null-additive, it follows that $m(A)=m\left(\left(A \backslash \bigcup_{i=1}^{p} A_{i}\right) \cup\left(\cup_{i=1}^{p} A_{i}\right)\right)=m\left(\cup_{i=1}^{p} A_{i}\right)$.

We finally note the following:

1. Any minimal atom is also an atom and a pseudo-atom (which justifies the terminology);
2. If the set function is null-additive, then any of its atoms is a pseudoatom, too;
3. If, moreover, the set function is finitely additive, then the converse of the above statement is also valid, therefore any pseudo-atom is particularly an atom.

Consequently, for a finitely additive set function (which is automatically null-additive), the notion of atom and that of pseudo-atom coincide.

### 2.3. Extensions of the notions of atom

Generalizations of the mathematical notion of an atom have been made, so far, in two major directions. A first direction is given by the fact that, instead of set functions, which are indispensable to the process of the so-called "measurement", one could generally operate with set multifunctions (that is, functions that associate a set to another set). Thus, results with a higher degree of generalization and abstraction can be obtained. The second direction is given by the correlation that can be made by placing the notion of (minimal) atom within the fractal sets theory, thus resulting in the notion of fractal (minimal) atom (Gavriluţ et al., 2019; Gavriluţ and Agop, 2016).

## The first direction: Set-valued approach

Let be an abstract nonvoid set $T, \mathcal{C}$ a ring of subsets of $T, X$ a real linear normed space with the origin $\theta$ and $\mathcal{P}_{0}(X)$, the family of all nonvoid subsets of $X$. By a set multifunction we mean a function (or, application) which associates a set to another set, in contrast with the notion of a function, which associates a point to another point. So, in what follows, let $\mu: \mathcal{C} \rightarrow \mathcal{P}_{0}(X)$ be an arbitrary set multifunction, with $\mu(\emptyset)=\theta$.

The notions of atom, pseudo-atom, minimal atom introduced with respect to a set function $m$ can be generalized in this context, with respect to the set multifunction $\mu$, as follows. We say that a set $A \in \mathcal{C}$ is:
(i) an atom of $\mu$ if $\mu(A) \supseteq\{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A_{p}$ we have either $\mu(B)=\{\theta\}$ or $\mu(A \backslash B)=\{\theta\}$;
(ii) a pseudo-atom of $\mu$ if $(A) \supsetneq\{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A_{j}$ it holds either $\mu(B)=\{\theta\}$, or $\mu(A)=\mu(B)$;
(iii) a minimal atom of $\mu$ if $\mu(A) ?\{\theta\}$ and for every set $B \in \mathcal{C}$, with $B \subseteq A$, one has either $\mu(B)=\{\theta\}$ or $A=B$.

Detailed considerations on atomicity with respect to set multifunctions can be found in Gavriluţ and Agop, 2016 and also in Gavriluţ et al. 2019.

The second direction: Towards a fractal theory of atomicity

The main idea in the quantum theory of measure and in generalized quantum mechanics is to provide a description of the world in terms of histories. A history is a classical description of the system considered for a certain period of time, which may be finite or infinite. If one tries to describe a particle system, then a history will be given by classical trajectories. If one deals with a field theory, then a history corresponds to the spatial configuration of the field as a function of time. In both cases, the quantum theory of measure tries to provide a way to describe the world through classical histories, extending the notion of probability theory, which is obviously not enough to shape our universe.

On the other hand, ordinary structures, self-similar structures etc. of nature can be assimilated to complex systems, if one considers both their structure and functionality. The models used to study the complex systems dynamics are built on the assumption that the physical quantities that describe it (such as density, momentum, and energy) are differentiable. Unfortunately, differentiable methods fail when reporting to physical reality, due to instabilities in the case of complex systems dynamics, instabilities that can generate both chaos and patterns.

In order to desribe such dynamics of the complex systems, one should introduce the scale resolution in the expressions of the physical variables describing such dynamics, as well as in the fundamental equations of the evolution (density, kinetic moment and equations of the energy). This way, any dynamic variable which is dependent, in a classical sense, both on the space and time coordinates, becomes, in this new context, dependent on scale resolution as well. Therefore, instead of working with a dynamic variable, we can deal with different approximations of a mathematical function that is strictly nondifferentiable. Consequently, any dynamic variable acts as the limit of a family of functions. Any function is non-differentiable at a zero resolution scale and it is differentiable at a non-zero resolution scale. This approach, well adapted for applications in the field of complex systems dynamics, in which any real determination is made at a finite resolution scale, clearly involves the development of both a new geometric structure and a physical theory (applied to the complex systems dynamics) for which the motion laws, that are invariant to the transformations of spatial and temporal coordinates, are integrated with scale laws, which are invariant to transformations of scale. Such a theory that includes the geometric structure based on the assumptions presented above was developed in the scale relativity theory and, more recently, in the scale relativity theory with constant arbitrary fractal dimension. Both theories define the class of fractal physics models. In this model, it is assumed that, in the complex systems dynamics, the complexity of interactions is replaced by nondifferentiability. Also, the motions forced to take place on continuous, differentiable curves in a Euclidean space are replaced by free motions, without constraints, that take place on continuous, non-differentiable curves (fractal curves) in a fractal space. In other words, for a time resolution scale that seems large when compared to the inverse of the largest Lyapunov exponent, deterministic trajectories can be replaced by a set of potential trajectories, so
that the notion of "defined positions" is replaced by the concept of a set of positions that have a definite probability density. In such a conjecture, quantum mechanics becomes a particular case of fractal mechanics (for the structural units motions of a complex system on Peano curves at Compton scale resolution). Therefore, the quantum theory of the measure could become a particular case of a fractal measure theory. One of the concepts that needs to be defined is that of a fractal minimal atom, as a generalization of the concept of a minimal atom (Gavriluţ et al., 2019).

## Conclusions

An exhaustive study on the problem of atomicity with respect to set functions is provided. Different types of atoms are discussed, the relationships among them are studied and several examples and physical possible implications and applications are obtained.

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# Nonlinear Phenomena in the Dynamics of a Class of Rolling Pendulums: A Trigger of Coupled Singularities <br> Plenary Review Lecture 

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#### Abstract

In the introductory part of the plenary lecture, an overview of nonlinear differential equations of heavy ball and heavy thin dick rolling along curvilinear paths and surfaces of different shapes were presented. This is reason that this content is omitted from present review paper. In the introductory part of this article, we will present nonlinear phenomena of motion of a heavy material point moving along a rotating, smooth circle around a vertical, central or eccentric axis, as well as around an eccentric oblique axis relative to the vertical, at a constant angular velocity. Using chomear and nonlinear approximations of the nonlinear differential equation in the vicinity of singular points of the observed dynamics, the analysis of the local dynamics of the heavy material point system along the rotating circle around the oblique axis is given. A mathematical analogy between this model and the model of the dynamics of a thin heavy disk rolling in a rotating circle around an eccentric-centric oblique axis is pointed out. Using linear and nonlinear approximations of the nonlinear differential equation in the vicinity of the singular points of the observed dynamics, the analysis of the local dynamics of the heavy material point system along the rotating circle around the oblique axisat a constant angular velocity is given. A mathematical analogy between this model and the dynamics model of a thin heavy disk rolling in a rotating circle around an eccentric-centric verticaloblique axis is pointed out. The central and main subject of the paper is the identification and presentation of nonlinear phenomena in the nonlinear dynamics of a class of generalized rolling pendulums, whose heavy bodies roll along curvilinear paths, lying in a vertical plane, rotating around a vertical axis, at a constant angular velocity. The bifurcation parameter of coupled rotations is identified. The bifurcation of the position of stable equilibrium of the generalized rolling pendulum and the corresponding representative singular points of the type of the stable center is described, as well as the stratification and transformation of phase trajectories in the phase portrait of nonlinear dynamics of the generalized rolling pendulum in the Earth's gravitational field, and along curvilinear route in rotate vertical plane around vertical axis at a constant angular velocity. Additionally a theorem of trigger of coupled singularities and a homoclinic orbit in the form of the number "eight" is graphically proofed. A series of graphs of characteristic equation oh nonlinear dynamics as well as series of phase portraits for different coefficients of curvilinear paths described by parabola, br-quadratic parabola or polynomials of the eighth degree is presented and sets of transformed phase trajectories and homoclinic oebits in the form of the number "eight" are presented, which include one or more triggers of coupled singular points in nonlinear dynamics of relative rolling thin


heavy disk along these curvilinear trace in rotate vertical plane around vertical axis at a constant angular velocity.

Keywords: Generalized rolling pendulum, Bifurcation, Trigger of coupled singularities, Curvilinear rolling puth, Phase trajectory portrait, Homoclibic orbit, Theorem, mathematical analogy.

## 1 Introduction

This review paper presents the main content of the Plenary Lecture, which Author held at the international conference "CHAOS 2021 Conference", traditionally organized by Professor Christos H Skiadas, CHAOS Conference Chair. He is the "spiritus movens" of the high scientific level program of the series of these good conferences and the accompanying series of publications of papers presented at them.

In the introductory part of the Plenary lecture, an overview of nonlinear differential equations of heavy ball [17, 22, 23]] and heavy thin dick rolling along curvilinear paths [18, 20, 21, 24] and surfaces of different shapes [23 ] were presented.

Also, an overview of nonlinear differential equations and nonlinear equations of phase trajectories were given for a number of special rolling points on spherical surfaces, on a cone and on a torus (see References [24]).

For a number of nonlinear dynamics of ball and thin disk rolling along curvilinear paths, phase portraits were presented with the definition of the term generalized rolling pendulum (see References [20]).

This review paper of mine contains my author's original scientific results, one part of which has already been presented or published, and some of which have now been shown for the first time and have not been published before. The presentation begins with an introduction to the nonlinear dynamics of rheonomic discrete systems with coupled rotations, both with two degrees of mobility and one degree of freedom of movement. Such systems are described by one rheonomic coordinate and one independent generalized coordinate. The rheonomic coordinate introduces a kinematic constrain, ie kinematic excitation, into the system, and with the help of an independent generalized coordinate, which describes a degree of freedom of movement of the rheonomic system, a nonlinear differential equation of nonlinear dynamics of the rheonomic system is formed.

## I.1. Models of fascinating nonlinear dynamics of abstraction of real systems and phenomenological mapping of local dynamics around stationary states

We first present several models of abstraction of nonlinear dynamics of real rheonomic mechanical systems, which simultaneously represent models of motion of a heavy meterail point in a circle, which rotates with a constant angular velocity around the centric/eccentric vertical or oblique axis (see

Refences $[4,8,9,11-14,16,19])$, but in terms of mathematical analogy and models of abstraction of real systems of rigid bodies (rolling without sliding a thin heavy disk), which perform dynamics with coupled rotations around two or three passing (no intersecting) axes. In the conditions of rotation around the hair, in relation to the vertical, the axis, there is another member that explicitly depends on both the generalized coordinate and the time.

In this part, we focus on determining the linearization and the linear and nonlinear approximation of the nonlinear differential equation around stationary singular points. The stability as well as the instability of the system dynamics around stationary singular points-positions of relative rest (equilibrium) of the material point on the rotating circle were, also, analyzed.

The appearance of triggers of coupled singularities $[1,6,7,10,13,15$, 26,35 ] is especially pointed out, as well as the fact that the sources of chaos dynamics in such systems are. It is indicated how the character of local properties of linear and nonlinear dynamics around stationary singular points is examined.

Figure 1 shows three models of structurally the same system, with one axis around which the circle rotates at a constant angular velocity and along which a heavy metal point moves.

The difference between these three models is in the different position of the fixed axis, around which the circular line rotates, in relation to the horizon, as well as the position of the axis in relation to the center of the circular line. Depending on the position of the axis around which the circular line rotates at a constant angular velocity, the nonlinear dynamics of the relative motion of a material point along a circular, ideally smooth line is described by the following nonlinear differential equations $[4,19]$ :
$\ddot{\varphi}+\Omega^{2}(\lambda-\cos \varphi) \sin \varphi=\Omega^{2} \lambda \operatorname{ctg} \alpha \cos \varphi \cos \Omega t$
(Model in Figure 1.a.1*, за $\varepsilon=0$ )
$\ddot{\varphi}+\Omega^{2}(\lambda-\cos \varphi) \sin \varphi=0$
(Model in Figure 1.a.2*, за $\varepsilon=0$ )
$\ddot{\varphi}_{2}+\Omega^{2}\left\langle\lambda-\cos \varphi_{2}\right\rangle \sin \varphi_{2}-\Omega^{2} \varepsilon \cos \varphi_{2}=0$
(Model in Figure 1.a.3*, за $\varepsilon \neq 0$ )
These differential equations (1)-(3) in Petrović's terminology would be phenomenological differential equations (see References [4, 19]).

Figure 1 shows models of a heavy material point moving in a rotating circle, at a constant angular velocity, around an eccentric axis obliquely positioned relative to the horizon (a.1*), or a vertical centric (a.2*) or vertical eccentric axis (a. $3^{*}$ ); Figure 1.b.2* shows the phase portraits of the nonlinear dynamics of a heavy material point moving along a rotating circle, at a constant angular velocity, around a centric (b.2*) or eccentric vertical axis (b.3*). Both phase portraits are in the case of the appearance of triggers of coupled singularities. Each of the triggers of coupled singularities contains a homoclinic phase trajectory in the shape of the number "eight" which intersects at a singular point of the unstable saddle type, and surrounds a stable center-type singular


Figure 1. Models of a heavy material point moving in a rotating circle, at a constant angular velocity, around an eccentric axis obliquely positioned with respect to the horizon (a.1*), or a vertical centric (a. $2^{*}$ ) or vertical eccentric axis (a. 3*); b* Phase portrait of the nonlinear dynamics of a difficult material point, moving in a rotating circle, at a constant angular velocity, around a centric (b.2*) or eccentric vertical axis (b.3*)
point on the left and onthe right side of the unstable saddle type singular point. The trigger of coupled singularities in the phase portrait in the Figure (1.b.2*), ie (1.b.3*), was created by bifurcation of a singular point of the stable center type. At certain values of the bifurcation parameter of the system, this singular point of the stable center type was transformed into a singular point of the unstable saddle type, and two new singular points of the stable center type appeared around it and a new separatrix-homoclinic phase trajectory appeared in the form of number "eight" with self-intersection at a singular point of the unstable saddle type and surround singular points of stable centre types. At certain values of the bifurcation parameter of the system, in the phase portrait of the material point nonlinear dynamics along the rotating circle there is no trigger structure of coupled singularities and there is no transformation of the singular point of the stable center type into the singular point of the unstable saddle type. These cases of phase portraits, which do not contain in their structure a substructure of triggers of coupled singularities, were not considered to be presented here.

## I.2. Linearization of a nonlinear differential equation around singular points and local properties of system dynamics

We start with the linearization of the nonlinear differential equation (1) around the singular points, obtained for the differential equation (2) obtained from the previous one for $\alpha=\frac{\pi}{2}$. In particular, we will consider cases of linearizations $[4,19]$ around singularity: $1 *$ for, $\lambda>1$ around singularity $\varphi=0$ and $2 *$ for $\lambda<1$, around singularity $\varphi= \pm \arccos \lambda$.

1* In the case that $\lambda>1$, and we examine small forced oscillations around a singular point $\varphi=0$ of the center type, by linearization around it we obtain a linearized differential in the following form

$$
\begin{equation*}
\ddot{\varphi}+\Omega^{2}(\lambda-1) \varphi \approx \Omega^{2} \lambda \operatorname{ctg} \alpha \cos \Omega t \tag{4}
\end{equation*}
$$

in which we performed linearization by the following approximations $\sin \varphi \approx \varphi+\ldots$ and $\cos \varphi \approx 1+\ldots$, and measured the coordinate from the singularity, around when we performed linearization, and introduced the assumption that the coordinate is small.

From the linearized differential equation (4) we can conclude that for the case $\lambda>1$, around a singular point $\varphi=0$ for small force amplitudes $\Omega^{2} \lambda \operatorname{ctg} \alpha$ and small values of initial angle coordinate $\varphi_{0}=\varphi(0)$ and small initial angular velocity $\dot{\varphi}_{0}=\dot{\varphi}(0)$, that the circular frequency of natural oscillations is $\omega \approx \Omega \sqrt{\lambda-1}$, around that singular point, while the frequency $\Omega$ of forced oscillations corresponds to angular velocity of a rotating circle in a rheonomic system.

We can also conclude that when $\Omega=\Omega_{\text {rez }}$ and $\lambda=2$ when there is a resonant state for small oscillations around the singular point $\varphi=0$ for $\lambda>1$, so the linearization of the differential equation can be accepted only in a very short time interval, while the amplitude of forced oscillation, which increases with time in which the system is exposed coercion, does not go beyond the assumed limits, which allow linearization.

In the resonant case, the linearization around a singular point $\varphi=0$, when $\Omega_{r e z}^{2}$ and $\lambda=2$, for given initial conditions, the solution of the linearized differential equation is of the form:

$$
\begin{align*}
& \varphi(t) \approx \varphi_{0} \cos \frac{\omega_{0} t \sqrt{2}}{2}+\frac{\sqrt{2} \dot{\varphi}_{0}}{\omega_{0}} \sin \frac{\omega_{0} t \sqrt{2}}{2}+\left[\frac{\omega_{0} t \sqrt{2}}{2} \sin \frac{\omega_{0} t \sqrt{2}}{2}\right] \operatorname{ctg} \alpha  \tag{5}\\
& \text { where } \Omega_{r e z}^{2}=\frac{g \sin \alpha}{2 \ell}=\frac{\omega_{0}^{2}}{2}, \omega_{0}^{2}=\frac{g \sin \alpha}{\ell} \cdot \lambda=\frac{\omega_{0}^{2}}{\Omega^{2}}, \lambda_{r e z}=2 .
\end{align*}
$$

2 * For the case when $\lambda<1$, we will report the linearization around the singular points $\varphi_{s}= \pm \arccos \lambda$, so instead of the coordinate $\varphi$ in the nonlinear differential equation (1), we enter $\varphi_{s}+\varphi$ by writing:

$$
\begin{equation*}
\ddot{\varphi}+\Omega^{2}\left[\lambda-\cos \left(\varphi_{s}+\varphi\right)\right] \sin \left(\varphi_{s}+\varphi\right)=\Omega^{2} \lambda \operatorname{ctg} \alpha \cos \left(\varphi_{s}+\varphi\right) \cos \Omega t \tag{6}
\end{equation*}
$$

now the coordinate $\varphi$ is measured from these singular points $\varphi_{s}= \pm \arccos \lambda$, as the beginnings in which that coordinate is zero. After linearization of the nonlinear differential equation (6) around singular points $\varphi_{s}= \pm \arccos \lambda$, the linearized differential equation takes the form:

$$
\begin{equation*}
\ddot{\varphi}+\Omega^{2}\left(1-\lambda^{2}\right)\left[1+\frac{\lambda \operatorname{ctg} \alpha}{\sqrt{1-\lambda^{2}}} \cos \Omega t\right] \varphi \approx \Omega^{2} \lambda \operatorname{ctg} \alpha \cos \Omega t \tag{7}
\end{equation*}
$$

Obtained by linearization, nonlinear differential equations around singular points $\varphi_{s}= \pm \arccos \lambda$, the previous differential equation is reolinear and Mathieu-Hill type:

$$
\begin{equation*}
\frac{d^{2} \varphi}{d \tau^{2}}+(\tilde{\lambda}+\tilde{\gamma} \cos \tau) \varphi=h \cos \tau \tag{8}
\end{equation*}
$$

whose shape coefficients are:

$$
\omega=\Omega \sqrt{\left(1-\lambda^{2}\right)}, \tilde{\lambda}=1-\lambda^{2} \text { и } \tilde{\gamma}=\lambda \sqrt{1-\lambda^{2}} \operatorname{ctg} \alpha=h \sqrt{\tilde{\lambda}}, \tau=\Omega t,
$$

A general solution to the Mathieu-Hill differential equation can be found in References to Mathieu's differential equation [31, 32, 34] or Floquet, Annales de l'Ecole Normale, 1883..

$$
\begin{equation*}
\varphi(t)=A e^{\mu t} p_{1}(t)+B e^{-\mu t} p_{2}(t) \tag{9}
\end{equation*}
$$

in which $A$ and $B$ there are also integration constants; $\mu$ is a characteristic exponent, and $p_{i}(t), i=1,2$ are periodic functions of the period $2 \pi$, which depend on the parameters $\tilde{\lambda}$ and $\tilde{\gamma}$.

The main problem in studying the stability around these singular points, which in the basic autonomous nonlinear system described by the autonomous nonlinear differential equation (2) represent stable centers (stable relative positions of the relative equilibrium of the rheonomic system, when the axis of rotation is vertical), and in the nonautonomous system described by nonlinear differential equation (1) and approximation (7) and (8), respectively, open more complex questions of motion stability testing around these same singular points. References to the solution of the solution of the Mathieu-Hill differential equation of the form (8) can be found in references [30, 31]. Using the Ince-Strutt stability map, we can determine the parameter areas by $\tilde{\lambda}$ and $\tilde{\gamma}$, in which the solutions are stable or unstable, and then conclude about the character of the stability of the system motion around the relative equilibrium position or singular points of the nonlinear differential equation (1).

## I.3. Phenomenological approximate mapping of nonlinear dynamics

In this previously presented example, linearization of nonlinear differential equations, we have shown that the analysis of nonlinear dynamics of a nonlinear system can be performed by decomposing the analysis to the analysis of local dynamics in the vicinity of singular points by linear mappings of dynamic phenomena into approximate local dynamics or linear or rheolinear and in depending on the initial conditions to study the properties of these dynamics. Knowing the properties of local dynamics or stable singular or unstable singular points, we can assemble-compose a whole of global motion and rare nonlinear phenomena of nonlinear system dynamics.

This is a phenomenological mapping [28-30] of partial, linear and reolinear phenomena from local dynamics to global nonlinear system dynamics by analyzing the phenomenon of dynamics in local domains and synthesizing the results of these analyzes into global ones.

We see that by approximations around some singularities, harmonic natural or forced oscillations occur with the possibility of the occurrence of the basic resonant state of local dynamics, ie in rheolinary natural or forced dynamics, with the possibility of the appearance of the parametric resonant state. Both the basic resonant state and the parametric resonant state, depending on the initial conditions, as well as the length of the time interval in which the nonlinear dynamical system is subjected to that local state of dynamics, come out of that state of dynamics, when the descriptions do not applicable, and stady must ro continue in global.

The ideas of Mihailo Petrović and his mathematical or qualitative phenomenology [28-30] can also be recognized in these approaches. And if we take into account that these models of nonlinear dynamics of a heavy material point have the same mathematical description as the motion of a rolling rigid body (homogeneous heavy thin disk), which performs coupled rotations around the passing (no intersecting) axes, then Mathematical phenomenology and phenomenological mapping become the right tool for
searching for models of abstraction of nonlinear dynamics of real mechanical systems, which are described by one or more nonlinear differential equations.

## 2 Bifurcation and trigger of coupled singularities in the dynamics of generalized rolling pendulums along curvilinear route in a rotating vertical plane at a constant angular velocity about a vertical axis

In a series of References $[18,20,21,24]$ of the author of this paper, the results of research of nonlinear dynamics of special cases of generalized rolling pendulums on curvilinear line in a rotating vertical plane, at a constant angular velocity $\Omega$ around the vertical axis are presented, and a given series of phase trajectory portraits in phase planes. Each of these phase portraits contains at least one trigger of coupled singularities, consisting of a singular point of the unstable saddle type, and two singular points of the stable center type, surrounded by a single-separator phase trajectory in the form of number "eight", which intersects at a singular point of the unstable saddle type. The angular velocity $\Omega$ of rotation of the vertical plane around the vertical axis appears as a bifurcation parameter, whose change can achieve the disappearance of the trigger of coupled singularities, or the appearance of that trigger in the phase portrait, or the appearance of bifurcation of a stable type sigular position, and two new singular points of the stable center type appear around it, and in the phase portrait a separatrix phase trajectory in the form of a number of "eight" that surrounds them and self-intersect at singular point which has lost stability and bifurcated into unstable saddle-type singular point.

In such a system, there is now a phenomenon of a bifurcation $[1,2,3,4]$, because the trigger of coupled singularities is now in results caused by the property of nonlinearity in the form of bifurcation and nonlinear dynamics of such a system. And with the existence of a bifurcation parameter with the change of which the trigger of coupled singularities appears or disappears.

In such a system, there is at least one parameter with the change of which such a trigger of coupled singularities would disappear or appear, which is caused by the properties of both, the curvilinear path and the existence of extremums in a set of one maximum and two minimums of the curvilinear trajectory in the rotate vertical plane at constant angular velocity $\Omega$ around vertical axis. Bifurcation and trigger of coupled singularities are nonlinear properties of nonlinear dynamics of generalized rolling pendulum.

For example, the ordinary nonlinear differential equation of non-linear rolling dynamics, non-slip, heavy homogeneous thin disk, radius $r$, in a circle, radius $R$, in a rotating vertical plane at a constant angular velocity $\Omega$ about the vertical central axis, is [21]:

$$
\begin{equation*}
\ddot{\varphi}+\frac{\Omega^{2}}{\kappa}(\lambda-\cos \varphi) \sin \varphi=0 \tag{10}
\end{equation*}
$$

which $\varphi$ is feberalized independent coordinate, $\kappa=1+\frac{\mathbf{i}_{\mathbf{C}}^{2}}{\mathbf{r}^{2}}$ is the coefficient of disk rolling and $\lambda=\frac{g}{(R-r) \Omega^{2}}$ is the bifurcation parameter (see References [12, 15] for details).

In Figure 2, a*and $b^{*}$, shown graphic representation of the transformations, by changing the bifurcation parameter $\lambda=\frac{g}{(R-r) \Omega^{2}}$, of the separatrix phase trajectories of the phase portrait of the non-linear rolling dynamics of a heavy homogeneous thin disk, radius $r$, in a circle, radius $R$, in a rotating vertical plane at a constant angular velocity $\Omega$ around the vertical central axis. These graphs are also presentation continuous process of bifurcations followed by change of bifurcation parameter $\lambda=\frac{g}{(R-r) \Omega^{2}}$ depending of angular velocity $\Omega$ of vertical plane rotation around vertical axis, and also of difference between radiuses of rolling disk $r$ and circle trace $R$.


Figure 2. Transformation of a singular point of the stable center type into an unstable singular point of the unstable saddle type by bifurcation into a trigger of coupled singularities with a homoclinic phase orbit in the form of the number "eight"; a* bifurcation of a stable center-type sinar point into a trigger of coupled singularities and $b^{*}$ stratification of phase trajectories by changing the bifurcation pathmeter

The nonlinear differential equation of rolling dynamics of a heavy homogeneous disk in a circle in a rotating vertical plane of the central vertical axis at a constant angular velocity is thematically analogous to the nonlinear differential equation of motion of a heavy material point along a smooth circular
line in a rotating vertical plane around a vertical axis at a constant angular velocity. We have shown a phase portrait in Fig. 1. b.2*, for the case when a tiger of conjugate singularities is visible.

Figure 2 shows the transformation of a singular point of the stable center type into an unstable singular point of the unstable saddle type by bifurcation into a trigger of coupled singularities with a homoclinic phase orbit in the shape of the number "eight". Figure 2.a* shows the bifurcation of the singular point of the stable center type into the trigger of coupled singularities, and Figure 2.b* shows the stratification of phase trajectories by changing the bifurcation parameter. (see also Reference [1-3]).

## 3 Nonlinear differential equations and phase trajectory equation of nonlinear dynamics of a class of rolling pendulums

Reference [20] by Hedrih (Stevanović) K.R., titled by "Generalized rolling pendulum along curvilinear trace: Phase portrait, singular points and total mechanical energy surface", is the full paper of Plenary Lecture given in Minisymposium on Computational Aspects of Classical and Celestial Mechanics, Stability and Motion Control at CASTR (Computer Algebra Systems in Teaching and Research - CASTR'2017. Thus paper contains description of a generalized rolling pendulum along curvilinear trace consisting by three circle arches in vertical plane. Sets of three non-linear differential equations of dynamics of described generalized rolling pendulum along each of three circle arches, is presented. Three integrals of previous three nonlinear differential equations present a set of three equations of each of three phase trajectory branches which correspond to dynamics of described generalized rolling pendulum along each of three circle arches. Phase portrait, set of singular points and total mechanical energy surface are graphically presented for particular case of geometrical parameters of the system. Paper contains basic elements of the methodology for investigation of the vibro-impact dynamics of the system with two rolling bodies along defined curvilinear trace in successive collisions.

Reference [21] by Hedrih (Stevanović) K.P. titled by "Rolling heavy disk along rotating circle with constant angular velocity" is a paper of Plenary lecture in section of Mini-symposium on Computational Aspects of Classical and Celestial Mechanics, Stability and Motion Control included in the Conference Program Computer Algebra Systems in Teaching and Research CASTR'2015. In Abstract of this reference we read following: "Non-linear differential equation of non-linear dynamics of a rolling heavy disk along rotating circle trace, with constant angular velocity, about central axis in vertical direction is derived. For this case, corresponding equation of phase trajectory portraits depending on kinetic parameters of the system are obtained. Existence of trigger (see References [20, 21]) of coupled three singular points and homoclinic orbit in the form of number "eight" depending on system kinetic parameters and appearance of the bifurcation of relative equilibrium positions are investigated. Functional dependence between angle of disk relative arbitrary position on rotating circle trace and time of motion duration is derived. For
obtaining this solution, an elliptic integral is derived. For solving elliptic integral, series of transformations are introduced and functions under the elliptic integral are expanded in three series along angle of disk relative arbitrary position on rotate circle trace. By use obtained functional dependence between time of disk rolling and angle of disk relative position, discussion of different period duration of rolling disk oscillations along rotating circle trace about vertical central axis is done depending of initial conditions and constant angular velocity of the circle rotation".

Next Reference [24] titled by "Vibro-impact dynamics of two rolling heavy thin disks along rotate curvilinear line and energy analysis" written by Hedrih (Stevanović) K.P. is published as original article in Journal Nonlinear Dynamics, Springer Nature.

Reference [25] by Hedrih (Stevanović) K.P. titled by "Dynamics of a rolling heavy thin disk along rotate curvilinear trace on vertical plane about vertical axis" is an extended abstract of first presentation lecture in session of General Mechanics at Congress of Serbian Society of Mechanics helped in Sremski Karlovci 2019. In the lecture Nonlinear differential equation of dynamics of a rolling, without slipping, heavy thin disk along rotate general curvilinear trace, in vertical plane, around vertical axis with constant angular velocity, is presented. First integral of this nonlinear differential equation is presented. First integral present the nonlinear equation of the phase trajectory in phase plane of a rolling, without slipping, heavy thin disk along rotate general curvilinear trace, in vertical plane, around vertical axis with constant angular velocity.

Based on the new author's authentic research and new results, which represent new contributions by generalizing previous results, without the articlec being clear and we think that there is no need to historically present the results of other authors, which are mostly in the field of mathematics.

We observe an axially symmetric heavy rigid body with one central plane of symmetry, which is in the case of a thin disk of radius $r$, mass $M$, the axial mass inertia moment $\mathbf{J}_{C}$ for the central axis parallel to the rolling axis. See Figure 3.


Figure 3. Geometric parameters of a rolling, without slipping, of heavy thin rigid discs on a rotating curvilinear trace in a vertical plane around a vertical axis with constant angular velocity

Suppose there is a curvilinear trace, determined by $y=f(x)$, such that the radius of curvature of each of its concave arch is larger than the radius of the contour of the disc circle in the plane of symmetry, by which the disk rolls, without slipping, along the curvilinear trace, rotating, around the vertical axis with constant angular velocity $\Omega$, in the rotating vertical plane.

If we introduce the coefficient of disk rolling, without slipping, in the form $\kappa=\frac{\mathbf{J}_{P}}{M r^{2}}=\frac{\mathbf{i}_{P}^{2}}{r^{2}}=\frac{\mathbf{i}_{C}^{2}}{r^{2}}+1=\kappa, \quad[31-34]$, which for thin disk is $\kappa=\frac{3}{2}$, then the nonlinear ordinary differential equation of rolling, without sliding, a heavy rigid thin disk along a curvilinear line route in rotate vertical plane around vertical axis at a constant angular velocity $\dot{\vartheta}=\Omega$, is in the following form:

$$
\begin{equation*}
\ddot{x} F(x, r)+\frac{1}{2} \dot{x}^{2} F^{\prime}(x, r)-\frac{r^{2}}{2} \Omega^{2} \frac{\mathbf{J}_{z}^{\prime}(x, M, r)}{\mathbf{J}_{P}}+\frac{g}{\kappa} f_{C}^{\prime}(x)=0 \tag{11}
\end{equation*}
$$

where $\quad \vartheta=\Omega t$ is rheonomic coordinate, as a kinematica eccitation, $x$ independent generalized coordinate, correspond to one degree of freedom of this rheonomic system with two degrees of mobility. Next, $f_{C}(x)$ and $F(x, r)$ are expressed by dollowijg expressions (for details see Reference [4] by Hedrih (Stevanović) KR.):

$$
\begin{align*}
& F(x, r)=\left\langle 1+\left[f^{\prime}(x)\right]^{2}\right\rangle\left\{1-\frac{r f^{\prime \prime}(x)}{\left[1+\left[f^{\prime}(x)\right]^{2}\right]^{\frac{3}{2}}}\right\}^{2}  \tag{12}\\
& f_{C}(x)=y+r \frac{1}{\sqrt{1+\left[f^{\prime}(x)\right]^{2}}}  \tag{13}\\
& \mathbf{J}_{z}(x, M, r)=\mathbf{J}_{C z}+M\left[x_{C}\right]^{2}=M \frac{r^{2}}{4}+M\left[x-r \frac{f^{\prime}(x)}{\sqrt{1+\left[f^{\prime}(x)\right]^{2}}}\right]^{2} \tag{14}
\end{align*}
$$

Expression (14) presents axial mass inertia moment of thin disk for vertical axis of vertical plane rotation in which curvilinear trace of disk rolling lies.

Then, previous integral of nonlinear differetial equation (11) finally take the following form:

$$
\begin{align*}
& {[\dot{x}(x)]=} \\
& \mp \sqrt{\left[\dot{x}\left(x_{0}\right)\right]^{2}+\Omega^{2} \frac{r^{2}\left[\mathbf{J}_{z}(x, M, r)-\mathbf{J}_{z}\left(x_{0}, M, r\right)\right]}{\mathbf{J}_{P} F(x, r)}-\frac{2 g}{\kappa F(x, r)}\left[f_{C}(x, r)-f_{C}\left(x_{0}, r\right)\right]} \tag{15}
\end{align*}
$$

where $f_{C}(x)$ is expressed by (13) and $F(x, r)$ is expressed by (12) and . $\mathbf{J}_{z}(x, M, r)$ is expressed by (14).

We can now write an expression for the relative angular velocity $\omega_{P}(x, \dot{x})$ of thin rigid disk relative rolling, without slipping, along a curved line route $y=f(x)$ in rotate vertical plane around vertical axis with constant angular velocity $\Omega$, based on a expression: $\left[\omega_{P}(x, \dot{x})\right]^{2}=\frac{1}{r^{2}} \dot{x}^{2} F(x, r)$ and previous obtained integral (15) in the following form: $\left[\omega_{P}(x, \dot{x})\right]=\frac{1}{r} \dot{x} \sqrt{F(x, r)}$, and finally in the form:

$$
\begin{align*}
& {\left[\omega_{P}(x)\right]=} \\
& \pm \frac{1}{r} \sqrt{\left[\dot{x}_{0}^{2} F\left(x_{0}, r\right)\right]+\Omega^{2} \frac{r^{2}\left[\mathbf{J}_{z}(x, M, r)-\mathbf{J}_{z}\left(x_{0}, M, r\right)\right]}{\mathbf{J}_{P}}-\frac{2 g}{\kappa}\left[f_{C}\left(x_{0} r\right)-f_{C}(x, r)\right]} \tag{16}
\end{align*}
$$

where $f_{C}(x)$ is expressed by (13) and $F(x, r)$ is expressed by (12) and . $\mathbf{J}_{z}(x, M, r)$ is expressed by (14).

## 4 Characteristic equations of nonlinear dynamics of a class of rolling pendulums and bifurcatuion of relative stable equilibrium positions

In this paper, main attention is paid to a more detailed analysis of the characteristic equation $K(x)$ of dynamics of the generalized rolling pendulum, along trajectory in rotate vertical plane at a constant angular velocity $\Omega$ about vertical axis, which was performed in [24] in the form:

$$
\begin{align*}
& K(x)=f^{\prime}(x)\left\{1-r \frac{f^{\prime \prime}(x)}{\left[1+\left[f^{\prime}(x)\right]^{2}\right]^{\frac{3}{2}}}\right\}-  \tag{17}\\
& -\frac{2 \kappa}{3 g} \Omega^{2}\left\langle x-\frac{r f^{\prime}(x)}{\sqrt{1+\left[f^{\prime}(x)\right]^{2}}}\right\rangle\left\langle 1-\frac{r f^{\prime \prime}(x)}{\left[1+\left[f^{\prime}(x)\right]^{2}\right] \sqrt{1+\left[f^{\prime}(x)\right]^{2}}}\right\rangle=0
\end{align*}
$$

and in which: $y=f(x)$ in general, or in particular cases $y=f(x)=k x^{2}$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2} \quad$ or $\quad y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right) \quad$ or $f(x)=-k x^{2}\left(x^{2}-a^{2}\right)\left[c^{4}-\left(x^{2}-b^{2}\right)^{2}\right]$ is equation of the curvilinear path, where $a, b, c$ and $k$ are known constants, and with the following relation $a<b$.

For various changes of values of the angular velocity $\Omega$ of rotation of the vertical plane, in which the curvilinear path along which the generalized
rolling pendulum rolls is located, the numerical analysis shows the obtaining of the zero (roots) of characteristic equation (17), or singular points and triggers, each of coupled three singular points.

## 5 Nonlinear phenomena in the dynamics of a class of rolling pendulums: Bifurcations and trigger of coupled singular points

In this part of our paper, using the equation of a curvilinear trajectory in a rotating vertical plane, around a vertical axis at a constant angular velocity $\Omega: y=f(x)$ in general, for in particular cases $y=f(x)=k x^{2}$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2} \quad$ or $\quad y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right) \quad$ or $f(x)=-k x^{2}\left(x^{2}-a^{2}\right)\left\langle c^{4}-\left(x^{2}-b^{2}\right)^{2}\right\rfloor$, where $a, b, c$ and $k$ are known constants, and with the following relation $a<b$, we will analyze the zeros of the graph of the characteristic equation $K(x)=0$, defined by expression (17).


Figure 4. Graphs of the curvilinear route, defined by parabola in the form $y=f(x)=k x^{2}$ as well as the frequency function $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17): case with a trigger of coupled three singular points

The following, Figures 4-10, show, in pairs, the characteristic curves of the shape of the curvilinear trajectory, $y=f(x)=k x^{2}$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$ or $f(x)=-k x^{2}\left(x^{2}-a^{2}\right)\left[c^{4}-\left(x^{2}-b^{2}\right)^{2}\right]$, with the corresponding extreme points, the minimum and maximum, and the graphs of the corresponding characteristic equations $K(x)=0$ with the corresponding zeros, for the cases of the indicated curvilinear paths the chosen angular velocity $\Omega$ of rotation of the system around the vertical axis.

Im Figures 4 and 5, graphs of the curvilinear route, defined by parabola in the form $y=f(x)=k x^{2}$ as well as the frequency function $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17) are presented for different geometrical and kinetic parameters. We can see different cases of characteristic function graphs depending of the value of angular velocity $\Omega$, which appears as a bifurcation parameter. In Figure 4 a graph of characteristic equation poses thee zero points around minima of curvilinear trace of disk rolling and dynamics system is with a trigger of coupled three singular points. Analogous graph of characteristic equation $K(x)$ is presented in Figure $5 . \mathrm{b}^{*}$ for different values of angular velocity $\Omega$, also a trigger of coupled three singular points exists. The graph of characteristic equation $K(x)$ presented in Figure 5.a* for zero values of angular velocity $\Omega=0$, is without any trigger of coupled three singular points and in minima of curvilinear parabolic trace of disk rolling no bifurcation.


Figure 5. Graphs of the curvilinear route, defined by parabola in the form $y=f(x)=k x^{2}$ as well as the frequency function $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17): a* case without a trigger of couple three singular points and $b^{*}$ case with a trigger of coupled three singular points

Im Figures 6 and 8, graphs of the curvilinear route, defined by bequadratic parabola in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ as well as the
characteristic frequency function $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17) are presented for different geometrical and kinetic parameters. We can see different cases of characteristic function graphs depending ofhe value of angular velocity $\Omega$, in which appears two bifurcation in each of two minima of curvilinear trace of disk rolling.

In Figure 6, graphs of the curvilinear route, defined by equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$, as well as the frequency function expressed by expression $h(x)=K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity around the vertical axis defined by the equation (17) are presented. In Figure 6 graph of characteristic equation poses thee zero points around each of two minima of curvilinear trace of disk rolling and dynamics system is with two triggers, each of coupled three singular points. Analogous graph of characteristic equation $K(x)$ is presented in Figure 7 for different values of angular velocity $\Omega$, also two triggers, each of coupled three singular points. The graph of characteristic equation $K(x)$, for zero values of angular velocity $\Omega=0$, is without any trigger of coupled three singular points and in each of two minima of curvilinear be-parabolic trace of disk rolling no bifurcation.


Figure 6. Graphs of the curvilinear route, defined by be-quadratic parabola in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ as well as the frequency functions $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17): case with a trigger of coupled three singular points


Figure 7. Detail of the graph of the frequency functtion $K(x)$ of nonlinear rolling dynamics of a rigid heavy thin disk, along a curvilinear path in a rotating vertical plane with constant angular velocity $\Omega$ around the vertical axis, defined by the equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ : detail shows the phenomenon of bifurcation of a stable singular point centre type into unstable saddle-type brush and basket of two new stable singular points center type around - appearance of a trigger of coupled three singular points


Figure 8. Graphs of the curvilinear route, defined by be-quadratic parabola in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ as well as the frequency functions $K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17): case with a trigger of coupled three singular points

In Figure 7, detail of the graph of the frequency functtion $h(x)=K(x)$, from Figure 6 , of nonlinear rolling dynamics of a rigid heavy thin disk, along a curvilinear path in a rotating vertical plane with constant angular velocity $\Omega$ around the vertical axis, defined by the equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ : detail shows the phenomenon of bifurcation of a stable singular point centre type into unstable saddle-type brush and basket of two new stable singular points center type around - appearance of a trigger of coupled singularities, are visible around a of two minima in trace of rolling in the form of be-quadratic parabila..

$T_{,}, s=1,2,3,4$, trigger of coupled three singular points, bifurcation of a stable centre type singular
paint into three singular points, one instoble soddie singular point and two new stable centre rype points
Figure 9. Two graphs of the curvilinear route, defined by polynomial equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, as well as the frequency function $h(x)=K(x)$ of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis defined by the equation (17): case with a series of the four triggers, each of coupled three singular points

In Figure 9, two graphs of the curvilinear route, defined by polynomial expression $\quad y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, as well as the frequency function $h(x)=K(x)$, defined by the equation (17), of the nonlinear rolling dynamics of a rigid heavy thin disk, in a rotating vertical plane with a constant angular velocity $\Omega$ around the vertical axis are presented. Series of triggers each of three coupled singular poits are visible around each of four minimum $C_{s}, s=1,2,3,4$ of curvilinear route of disk rolling. There are four minima
$C_{s}, s=1,2,3,4$ in which and around which appear in total four bifurcations and four triggers of coupled singularities, and three maxima $S_{s}, s=1,2,3$ around which no appearing bifurcations. Four bifurcations and four triggers of coupled singularities occur $C_{s} \rightarrow T_{s}, s=1,2,3,4$ in each point of the four minimums $C_{s}, s=1,2,3,4$ are visible in this Figure 9.

$C_{2}, s=1.2,3,4$, stoble centre type singuipr points $\quad S_{n}, s=1,2,3$, unstabie sodale type singular points
$T_{n}, s=1,2,3,4$, trigger of coupled three singular points, bifurcation of a stable centre type singular paint into three singuiar points, one unstable saddle singular paint and two new stabie centre type points
Figure 10. Detail, of Figure 9, of the graph of the frequency function $h(x)=K(x)$ of nonlinear rolling dynamics of a rigid heavy thin disk, along a curvilinear path in a rotating vertical plane with constant angular velocity $\Omega$ around the vertical axis, defined by the equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$ : detail shows the phenomenon of bifurcation of two stable singular points each of a stable centre type into unstable saddle-type brush and basket of two new stable singular points center type around each - appearance of two triggers each of coupled three singular points

In Figure 10, detail, of Figure 9, of the graph of the frequency function $h(x)=K(x)$ of nonlinear rolling dynamics of a rigid heavy thin disk, along a curvilinear path in a rotating vertical plane with constant angular velocity around the vertical axis, defined by the equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$ is presented. Detail shows the phenomenon of bifurcation of two stable singular points each of a stable centre type into unstable saddle-type brush and basket of two new stable singular points center type around each - appearance of two triggers each of coupled three singular points

## 6 Phase trajectory portraits in the noHlinear dynamics of a class of rolling pendulums and structural analysis: Bifurcations, layering of phase trajectories, trigger of coupled singularities

In this part of our paper, using the equation of a curvilinear trajectory in a rotating vertical plane, around a vertical axis at a constant angular velocity $\Omega: y=f(x)$ in general, for in particular cases $y=f(x)=k x^{2}$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2} \quad$ or $\quad y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right) \quad$ or $f(x)=-k x^{2}\left(x^{2}-a^{2}\right)\left\langle c^{4}-\left(x^{2}-b^{2}\right)^{2}\right\rfloor$, where $a, b, c$ and $k$ are known constants, and with the following relation $a<b$, we will analyze the structure of phase portraits using the equation of phase trajectories in the form (16), in the phase plan - the relative angular velocity $\omega_{P}(x, \dot{x})$ of thin rigid disk relative rolling, without slipping, along a curved line route $y=f(x)$ and independent generalized coordinate $x$, i.e. in the form (15), in the phase plane a derivative $\dot{x}=\dot{x}(x)$ of the independent generalized coordinate and the independent generalized coordinate $x$.

The following Figures 11-15 show the characteristic phase portraits of the nonlinear rolling dynamics of a heavy thin, rigid disk along curvilinear paths of curvilinear path shape, $y=f(x)=k x^{2} \quad$ or $\quad y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2} \quad$ or $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$ or $f(x)=-k x^{2}\left(x^{2}-a^{2}\right)\left[c^{4}-\left(x^{2}-b^{2}\right)^{2}\right]$, with corresponding extreme points, minimum and maximum. We use the findings from the analysis of the number of zeros and the existence of triggers of coupled singular points, which we conducted in the previous chapter of this paper, by analyzing the number of zeros of the characteristic equation $h(x)=K(x)$ for a certain shape of the curvilinear trajectory, and for the corresponding value of the bifurcation parameter - the angular velocity $\Omega$ of rotation of the vertical plane around the vertical axis.

In order to obtain one of the phase portraits, it is necessary to draw a series of phase trajectories for different values of the initial conditions, using the
equation of phase trajectories in the form (16), in the phase plane- the relative angular velocity $\omega_{P}(x, \dot{x})$ of thin rigid disk relative rolling, without slipping, along a curved line route $y=f(x)$ and independent generalized coordinates $x$, i.e. in the form (15), in the phase plane a derivative $\dot{x}=\dot{x}(x)$ of the independent generalized coordinate $x$ and the independent generalized coordinate $x$. From an infinite set of phase trajectories, we choose characteristic series, as well as separatrix phase trajectories-hmoclinic phase trajectories, which separated individual series of phase trajectories, which describe similar properties of the dynamics of the studied nonlinear dynamics.



Figure 11. A phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a parabola in the form $y=f(x)=k x^{2}$, where $k$, is a known constant, in a stationary vertical plane, $\Omega=0$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ and a phase trajectory in phase plane $\dot{x}=\dot{x}(x), x$

In Figure 11 a phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a parabola in the form $y=f(x)=k x^{2}$, where $k$, is a known constant, in a stationary vertical plane, $\Omega=0$, and in Earth's field of gravity, in phase
coordinates $\omega_{P}(x), x$ and a phase trajectory in phase plane $\dot{x}=\dot{x}(x), x$ are presented.


Figure 12. A phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the four degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$, where $a$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$

In Figure 12 a phase portrait of the nonlinear dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the four degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$, as a be-quadratic parabola, where $a$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ is presented.

The same Figure 12, also, shows the curvilinear path of the shape of the square parabola of the equation $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$ along which the disk rolls. This path has two minimums and one maximum. At the selected value of the angular velocity $\Omega$ of rotation around the vertical axis of the vertical plane in which the curvilinear trajectory is, bifurcation and trigger of
coupled singular points occur around each minimum trajectory, as we determined by analyzing the zeros of the characteristic equation $h(x)=K(x)$.

From the structure of the phase portrait from Figure 12, we see that it contains two types of separatrix trajectories - homoclinic orbits in the shape of the number "eight". One of these homoclinic orbits self-intersects at a singular point of the unstable saddle type, which corresponds to the maximum of the curvilinear trajectory and exists in the phase portrait and when the disk rolls along a stationary path and then surrounds two singular points of stable cevtar type on each side of the singular point of stabile center cprrespomding to minima od trace of rplling.

When the curvilinear trajectory is in a rotating vertical plane, around the vertical axis at a constant angular velocity, and when bifurcation of each of the singular points of the stable center type occurs at the minimum of the curvilinear trajectory, then this phase trajectory surrounds on each side of the self-intersection unstable saddle type point. Each of these triggers was created by bifucation and contains two singular points of the stable center type and one singular point between them of the unstable saddle type.

The other two separatrix phase trajectories in the shape of the number "eight" intersect at singular points of the unstable saddle type of each of the formed triggers of coupled singularities, about two minimum curvilinear paths along which a thin disk rolls. They mean that in the observed phase portrait, two types of triggers of conjugated singularities appeared with bifurcation.

One homoclinic orbit in the form of number "eight" contains two coupled triggers of coupled singular points, and two homoblinic orbits in the form of number "eight" contains each ine trigger of coupled singular points.

The term trigger of coupled singular points contains three singular points, one type of unstable saddle and two types of stable centers, and was created by bifurcation of a singular point of the stable center type. The term trigger of coupled singularities includes a trigger of coupled singular points and a trajectory separatrix-homoclinic orbit in the form of number "eight" with a self-intersect at a no stable seddle type singular point and sournd teo singular points stable centre types.

We can conclude that this phase portrait contains two triggers of coupled singularities and one trigger of coupled triggers of coupled singularities, because it surrounds two triggers of coupled singularities.

In Figure 13 a phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the four degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$, bequadratic parabola, where $a$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ and a phase trajectory in phase plane $\dot{x}=\dot{x}(x), x$ are presented.


Figure 13. A phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the four degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)$, be-quadratic parabola, where $a$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ and a phase trajectory in phase plane $\dot{x}=\dot{x}(x), x$

In Figure 14 a phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the eighth degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, where $\mathrm{w} a, b$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ is presented.

We can conclude that, in the observed case, in the phase trajectory portrait, from Figure 14, three types of separatist phase trajectories - homoclinic orbits in the shape of the number "eight" are observed:

The first type of separatrix phase trajectories surrounds only three coupled singular points, two types of stable center and one type of unstable saddle, which is intersect, and all these elements represent a trigger of first order coupled singularities. There are as four triggers of coupled singularities as there are four minimum of the rolling paths and in this case.


Figure 14. A phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the eighth degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, where $\mathrm{w} a, b$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$

The second type of separatrix phase trajectories surrounds two triggers of coupled singularities and only intersects at one singular point of the unstable saddle type between them. There are two such separatrix phase trajectories in the studied phase portrait.

The third type of third-order homoclinic orbits surrounds one secondorder homoclinic orbit, as well as two first-order homoclinic orbits. Here, in the observed case, in the phase trajectory portrait, in Figure 14, there is only one such homoclinic orbit - the separatrix phase trajectory, and it surrounds all four triggers of coupled singularities, each of which is about one of the four minimum positions on the generalized rolling pendulum rolling path.

Between these separatrix phase trajectories in phase trajectory portrait, are regular closed phase trajectories corresponding to periodic rolls of the
generalized rolling pendulum with corresponding periods of oscillatory rolling which depend on the initial conditions and the value of total mechanical energy which achieves conservative nonlinear dynamics, and the number of equilibrium positions on the path through which the body passes for a period of one rolling oscillation.


Figure 15. A phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the eighth degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, where $\mathrm{w} a, b$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$

In Figure 15 a phase portrait of the dynamics of a generalized rolling pendulum, which rolls, without slipping, along a curvilinear path defined by a polynomial of the eighth degree in the form $y=f(x)=k x^{2}\left(x^{2}-a^{2}\right)^{2}\left(x^{2}-b^{2}\right)$, where $\mathrm{w} a, b$ and $k$, are known constants, in a rotate vertical plane around the vertical axis at a constant angular
velocity $\Omega$, and in Earth's field of gravity, in phase coordinates $\omega_{P}(x), x$ is presented/

By analyzing the shape of the paths along which the body of the generalized rolling pendulum rolls, without slipping, as well as by analyzing a series of phase portraits and the structure of sangular points in them, as well as structural stability and sensitivity to changes in the system's bifurcation parameters, bused on series published author's References [11, 12, 15, 29] , as well as a large number of numerical experiments and obtained different graphs of nonlinear phenomena in nonlinear dynamics of generalized rolling pendulum, a new theorem of bifurcation and of trigger of coupled singularities can be defined in the following formulation:

Theorem on bifurcation and on the trigger of coupled singularities in the dynamics of generalized rolling pendulums along curvilinear routes in a rotating vertical plane around a vertical axis at a constant angular velocity: Let the curved line, given with $y=f(x)$, for which it is valid $f(x)=f(-x)$, and which has at the points for extreme values $E X_{s}\left(x_{s}, y_{s}=f\left(x_{s}\right)\right)$ for $f^{\prime}\left(x_{s}\right)=0$, the minimums $C_{s}\left(x_{s}, y_{s}=f\left(x_{s}\right)\right)$ for $f^{\prime}\left(x_{s}\right)=0, f^{\prime \prime}\left(x_{s}\right)>0$, and the maxima $S_{s}\left(x_{s}, y_{s}=f\left(x_{s}\right)\right)$ for $f^{\prime}\left(x_{s}\right)=0, f^{\prime \prime}\left(x_{s}\right)<0$, the curvilinear route, along which rolls, without slipping, a heavy homogeneous thin disk, of radius $r>0$, and let it located in the Earth's gravitational field, and in the vertical plane, which rotates around the vertical axis, at a constant angular velocity $\Omega>0$. The characteristic equation for determining the singular points, as well as the position of the relative equilibrium of the disk on the curvilinear path, in the vertical rotating plane around the vertical axis at a constant angular velocity $\Omega>0$, is of the form:

$$
\begin{align*}
& h(x)=K(x)=f^{\prime}(x)\left\{1-r \frac{f^{\prime \prime}(x)}{\left[1+\left[f^{\prime}(x)\right]^{2}\right]^{\frac{3}{2}}}\right\}-  \tag{18}\\
& -\frac{2 \kappa}{3 g} \Omega^{2}\left\langle x-\frac{r f^{\prime}(x)}{\sqrt{1+\left[f^{\prime}(x)\right]^{2}}}\right\rangle\left\langle 1-\frac{r f^{\prime \prime}(x)}{\left[1+\left[f^{\prime}(x)\right]^{2}\right] \sqrt{1+\left[f^{\prime}(x)\right]^{2}}}\right\rangle=0
\end{align*}
$$

in which it is $\kappa=\frac{\mathbf{J}_{P}}{M r^{2}}=\frac{\mathbf{i}_{P}^{2}}{r^{2}}=\frac{\mathbf{i}_{C}^{2}}{r^{2}}+1=\kappa$, that is $\kappa=\frac{3}{2}$, the rolling coefficient of the disk, because is $\mathbf{J}_{\mathrm{C} z}=\sigma \frac{r^{4}}{4} \pi=M \frac{r^{2}}{4}$ and $\mathbf{J}_{P}=\mathbf{J}_{\mathbf{C}}+M r^{2}=\frac{3}{2} M r^{2}$, and $g$ the acceleration of the Earth is heavier. Around each extremum of the curvilinear trajectory, which is the minimum defined by $C_{s}\left(x_{s}, y_{s}=f\left(x_{s}\right)\right)$ for $f^{\prime}\left(x_{s}\right)=0, f^{\prime \prime}\left(x_{s}\right)>0$, in the dynamics of thin dick rolling, triggers of conjoined singularities appear, and around each extremum, which is maximum defined with $S_{s}\left(x_{s}, y_{s}=f\left(x_{s}\right)\right)$ for $f^{\prime}\left(x_{s}\right)=0, f^{\prime \prime}\left(x_{s}\right)<0$, there is no trigger of coupled singularities (see Figures 11, 12, 13, and 14).

## 6 Concluding remarks

The paper presents an analogy [5,28-30] of the nonlinear dynamics of a heavy material point along curvilinear paths in a rotating vertical plane, at a constant angular velocity, around a vertical axis and the nonlinear dynamics of a generalized rolling thin heavy disk pendulum along the same curvilinear paths in both these cases. One theorem are presented and additionally graphically proofed. The theorem describes the process of bifurcation and occurrence and disappearance of triggers of coupled three singular points in the local area of the minimum of curvilinear paths in rotating vertical planes, at a constant angular velocity around the vertical axis, caused by the angular velocity of rotation as a vifurcation parameter.

Based on a numerical experiment with various curvilinear rolling routes, a large number of graphs of the characteristic function of nonlinear dynamics of generalized rolling thin heavy disk pendulum, were obtained, such as phase trajectory portraits of nonlinear dynamics of a generalized rolling thin heavy disk pendulum along curvilinear paths in a rotating vertical plane at different values of constant angular velocity about a vertical axis, also are presented.

From a large number of obtained graphics, the most characteristic examples were selected and presented in the paper. The results of previous published author's references for particular examples of the shape of curvilinear paths along which the thin heavy disk of a generalized rolling pendulum rolls were also usedas initial ideas fpr research continuations.

The observed bifurcation and triggers of coupled singularities is a property of the nonlinear dynamics of generalized rolling thin heavy disk pendulums along rotating curvilinear route about vertical axis at a constant angular velocity. Identification of the triggers of coupled singularitues in the cuples rotations in system dynamics is very important for explanation of numerous phenomena in real engineering system dynamics.

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# A quantum dynamical map in the creation of optimized chaotic S-boxes 

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#### Abstract

The substitution boxes are an open challenge due to not meeting the theoretical criteria of a good S-box. Recently, the use of chaos in the design of efficient S-boxes was proposed. In this article, after introducing a new quantum system, we examine its effect on the formation of chaotic S-boxes. We compare the proposed quantum chaotic map with previous results. Also, in the previous work, the PSO algorithm was improved with the help of the classical map and then used in the optimization of chaotic S-boxes. We are using and improving the performance of PSO in generating the S-box, by the introduced quantum chaotic map. Then, by changing the type of optimization, we examine its effects. For the first time, the harmony search algorithm is improved by the said quantum map, and then we use it to optimize the produced chaotic S-box. By examining the performance of generated S-boxes by common attacks such as nonlinearity, BIC, SAC, LP, and DP. The results for the improved harmony search algorithm is better. Keywords: Quantum dynamical map, Substitution box(S-box), Harmony search algorithm, Particle swarm optimization(PSO), Nonlinearity.


## 1 Introduction

Many researchers in recent decades, to achieve higher security, have combined the two fields of chaos and cryptography under the heading of chaotic-based cryptography [1-4]. Due to their many applications, quantum dots are one of the favorite topics of researchers. So far, quantum dots have been used in solar cells [5], diodes [6], medical imaging [7], and quantum computing [8]. When quantum dots are paired with other quantum dots or external fields, They have a long periodicity, making them suitable for use in cryptography. The National Institute of Standards and Technology (NIST) proposed the Data Encryption Standard (DES) for the encryption and decryption process in 1977 [9], which was replaced by the AES symmetric-key algorithm in 2001 [10]. Sbox, which performs confusion, has been widely employed in traditional cryptographic standards such as DES and AES. Making efficient boxes is a major issue for security experts. Recently, some S-box algorithms based on the chaotic map have been proposed [11-14]. Then optimization algorithms are used to improve the performance of chaotic S-boxes $[11,15,16]$. All optimizers require a fitness function, which ref. [11] shown to use nonlinearity fitness for better
results. In this reference, classical maps are proposed to improve the performance of the PSO algorithm. Considering the theoretical criteria of a good S-box, there is a need to form new S-boxes.
In this work, a quantum map is replaced by a classical map. Also, the harmony search algorithm is replaced with the PSO algorithm to investigate the effect of the type of optimization.
The paper continues as follows: In Section 2, preliminary is proposed that includes the introduction of quantum dots and the study of their behavior. In section 3, S-box criteria are presented. Sections 4 includes the algorithm for creating improved PSO and optimized S-box. Improved HS and optimized Sbox is offered in Section 5. Section 6 provides an analysis of the performance of the S-boxes. Finally, a conclusion is proposed.

## 2 Preliminary

We introduce a generalized Dicke model presenting a new quantum chaotic map. It also investigates the chaos of this created system.

### 2.1 The maps of generalized Dicke model

The dynamical system governed by a generalized Dick Hamiltonian form is constructed as follows:

$$
\begin{gather*}
H=a^{\dagger} a+\omega_{A} J_{z}+\frac{\gamma}{\sqrt{N}}\left(a^{\dagger}+a\right)\left(J_{-}+J_{+}\right)+\frac{\gamma}{\sqrt{N}} V\left(J_{-}, J_{+}\right) \\
\Sigma_{n} \delta(t-n T) \tag{1}
\end{gather*}
$$

In fact, we consider delta function added to Dicke Hamiltonian. where, $a$ and $a^{\dagger}$ are respectively bosonic annihilation and creation operators. The parameter $\hbar \tilde{\omega}_{A}$ denotes the energy separation of N two-level atoms [17]. We assume that $\hbar=1, \omega_{A}=\tilde{\omega}_{A} / \tilde{\omega}_{f} \geq 0$, and $\tilde{\omega}_{f}$ is the field of frequency. $\gamma=\tilde{\gamma} / \tilde{\omega}_{f}$ is the coupling parameter. Furthermore, $J_{z}$ and $J_{ \pm}$are the atomic relative population operator and the atomic transition operator, respectively [18]. In reference [12], we introduced a chaotic mapping based on this Hamiltonian.

$$
\begin{equation*}
<J_{+(n+1)}>=\alpha\left(<J_{+(n)}>-<J_{-(n)} J_{+(n)}>\right) . \tag{2}
\end{equation*}
$$

According to previous studies for quantum systems [19], here in the same way map with quantum corrections for a system of coupled quantum dots is extracted. To appear the effect of the quantum correlations using $J_{+}=<J_{+}>$ $+\delta J_{+}$and $J_{-}=<J_{-}>+\delta J_{-}$, we have:

$$
\begin{gather*}
<J_{+(n+1)}>=r\left(<J_{+(n)}>-<J_{-(n)}><J_{+(n)}>\right)- \\
r<\delta J_{-} \delta J_{+}> \tag{3}
\end{gather*}
$$

Taking time derivation of $\left(\delta J_{+} \delta J_{-}\right)$implies

$$
\begin{equation*}
\frac{d}{d t}\left(\delta J_{+} \delta J_{-}\right)=\delta \dot{J}_{+} \delta J_{-}+\delta J_{+} \delta \dot{J}_{-} \tag{4}
\end{equation*}
$$

Next, by applying $<\delta J_{+}(n T) \delta J_{-}(n T)>=y_{n},<\delta J_{+} \delta J_{+}>=z_{n},<J_{+}(n T)>=x_{n}$, we obtain (See Appendix A):

$$
\left\{\begin{array}{l}
X_{n+1}=r\left(x_{n}-x_{n}^{2}\right)-r y_{n}  \tag{5}\\
Y_{n+1}=-y_{n}+r e^{-\beta}\left(\left(1-x_{n}+e^{2 \beta}-x_{n} e^{2 \beta}\right) y_{n}-z_{n} x_{n}-e^{2 \beta} z_{n} x_{n}\right) \\
Z_{n+1}=-z_{n} e^{2 \beta}+r e^{\beta}\left(2 z_{n}-2 x_{n} z_{n}-x_{n} y_{n}-y_{n} x_{n}\right)
\end{array}\right.
$$

The equations (5) give the lowest-order quantum corrections. For convenience, we consider that $\beta=i w_{A} T$ [20]. The sensitivity of the map to its initial values are shown in Fig. 1.1. We plotted Fig. 1.1 for constant parameter $r=9, b=0.5, y_{0}=0.435$ and $z_{0}=0.777$ as well as variable initial condition $x_{0}=0.423$ and $x_{0}=0.424$. Lyapunov exponent curve are seen in Fig. 1.2.


Fig. 1. (1) The sensitivity of the chaotic map to initial values the maps of generalized Dicke model for $x_{0}=0.423$ and $x_{0}=0.424$ where the control parameter $r=9, b=0.5$, $y_{0}=0.435$ and $z_{0}=0.777$. (2) The variation of the Lyapunov exponent the maps of generalized Dicke model in term of parameters $r$.

## 3 S-box criteria

An $n * m$ S-box is a nonlinear mapping $S: V_{n} \rightarrow V_{m}$, where $V_{n}$ and $V_{m}$ represent the vector spaces of $\mathrm{n}, \mathrm{m}$ elements from $G F(2)$. Important tests to check the performance of S-box are nonlinearity (NL), strict avalanche criterion (SAC), bit independence criterion (BIC), linear approximation probability (LP), and differential approximation probability (DP).

### 3.1 Nonlinearity

The nonlinearity value is calculated from the following equation:

$$
N=2^{n-1}-\frac{1}{2} \max _{a \in B^{n}}\left|\sum_{x \in B^{n}}(-1)^{f(x)+a \cdot x}\right|
$$

where $B=\{0,1\}, f: B^{n} \rightarrow B, a \in B^{n}$ and $a . x$ is the dot product between a and x (see [21] for example). Since the affine functions are weak in terms of cryptography, the similarity of the Boolean function variable of S-box is measured with the affine variable.

### 3.2 Strict avalanche criterion (SAC)

Webster and Tavares introduced SAC. If one bit in the input of Boolean function changed, half of the output bits should be changed [22]. The value of $\mathrm{SAC}=0.5$ is necessary for passing this test.

### 3.3 Bit independence criterion (BIC)

BIC, which calculate the independence of the avalanche vectors sets, is a desirable feature for any encryption transformation for S-box analysis (Webster and Tavares defined this test in [22]). If one changes the inverse of input single bits, these sets are created [23].

### 3.4 Linear approximation probability (LP)

LP [24] is:

$$
L P=\max _{a, b \neq 0}\left|\frac{\#\{x \mid x \cdot a=f(x) \cdot b\}}{2^{n}}-0.5\right|
$$

where $a, b$ are the input and output masks, and the set x contains all the possible inputs, and $2^{n}$ is the number of its elements. The maximum value of imbalance in the event between input and output bits is called LP. Low LP is necessary for resistance against linear attacks.

### 3.5 Differential approximation probability (DP)

DP is:

$$
D P=\max _{\Delta_{x} \neq 0, \Delta_{y}}\left(\# x \in X, f_{x} \oplus f\left(x+\Delta_{x}\right)=\Delta_{y} / 2^{n}\right)
$$

where X shows the set of all possible input values, and $2^{n}$ is the number of its elements. DP which calculate XOR the distribution between input and output bits of S-Box is introduced by Biham and Shamir [25]. The closer distribution between the input and output bits is necessary for resistance against differential attacks.

## 4 Improved PSO and optimized S-box

In PSO, the swarm consists of particles, each one representing a potential solution in the optimization problem. These particles have position and velocity. The PSO algorithm uses the unified function for the initial population and
the rand function to update the speed and position. In this paper, we use a quantum map for the initial population. Instead of the rand function, once we use the quantum map and for the second time, the classical hierarchy of rational-order chaotic maps (the best result of ref. [11]) (See Fig. 1). As can be seen, the best results are obtained for improved PSO with quantum maps and the hierarchy of rational-order chaotic maps(See Fig.2). Now we use this optimization algorithm to get the best S-box based on the highest nonlinear value (see Appendix B). The best S-box is seen in Table 2. The highest obtained nonlinearity value is 106 .

## 5 Improved HS and optimized S-box

Zong Woo Geem et al. in 2001 developed Harmony search which is a musicbased metaheuristic algorithm [26]. It used to solve many optimization problems such as function optimization, engineering optimization [27], water distribution networks [28]. To enhance the global convergence and to prevent to stick on a local solution, different HS methods based on chaotic maps have been proposed [29]. The improved HS(See Fig.3) steps and its application for optimizing the designed chaotic S-box are discussed. The steps of the algorithm are as follows:
step 1 Enter improved HS parameters (number of decision variables, decision variables matrix size, Maximum number of iterations, Harmony Memory size, number of new Harmonies, Harmony Memory consideration rate, Pitch Adjustment rate, Fret width(Band width), Fret width Damp ratio) and $r=5.5, \beta=0.5$ for Eq. 5.
step 2 Initialize Harmony Memory using liana function(liana function produce random number between 100 and 120 by using Eq. 5).
step 3 Creation of S-box based on quantum map Eq. 5:
1- Enter $r=5.5, \beta=0.5$ for Eq. 5 (consider Fig. 1).
2- Pass the transition state by repeating the map Eq. 5 .
3 - We create empty $16 * 16$ box.
4- Repeat the map Eq. 5 and select $x(f)$.
5 - The S-box numbers are obtained:

$$
S(i)=x(f) * 10^{5} \bmod 256
$$

6 - The process continues from 4 and select different $S(i)$.
step 4 Calculation of nonlinearity for all Harmony Memory positions.
step 5 Sort Harmony Memory from MAX to MIN.
step 6 Update Best solution ever found.
step 7 Create new Harmony position using liana function.
step 8 Pitch Adjustment using nafis function(nafis function produce random number between -1 and 1 by using Eq. 5).
step 9 If Nonlinearity(new position) ¿best solution save S-box.
step 10 Merge Harmony memory and new Harmonies.
step 11 Sort Harmony Memory from MAX to MIN.
step 12 Update Best solution ever found.
step 13 Save Best Nonlinearity.
step 14 If iteration finished, print Best Nonlinearity.
Optimized S-box creation algorithm using improved HS algorithm with quantum maps is presented in Fig. 4. The created S-box are seen in Table 3. Fig. 5 shows best nonlinearity of optimized S-box with nonlinearity fitness function for improved PSO algorithm with quantum maps and hierarchy of rational-order chaotic maps and improved HS algorithm with quantum maps.


Fig. 2. The variation of the cost function (sphere) for (1) PSO algorithm with unifrnd and rand functions (2) Improved PSO algorithm with quantum maps (3) Improved PSO algorithm with quantum maps and hierarchy of rational-order chaotic maps.


Fig. 3. The variation of the cost function (sphere) for (1) HS algorithm with unifrnd (2) Improved HS algorithm with quantum maps.


Fig. 4. Optimized S-box creation algorithm using improved HS algorithm with quantum maps.


Fig. 5. Best nonlinearity of optimized S-box for (1) Improved PSO algorithm with quantum maps and hierarchy of rational-order chaotic maps (2) Improved HS algorithm with quantum maps.

## 6 Security analysis

The security of any encryption is measured by its key(the keyspace size more than $\left.2^{100}[30,31]\right)$. We prob the keyspace of a quantum map to create the S-box. The order of complexity for decoding is:

$$
T\left(r, \beta, x_{0}, y_{0}, z_{0}\right)=\theta\left(r \times \beta \times x_{0} \times y_{0} \times z_{0}\right)
$$

If the computer's analysis power is $10^{16}$ decimal, the accuracy for each variable is $10^{-16}$. The number of these parameters for the quantum map in Eq. 5, is 5. So the keyspace for each is $10^{80}\left(2^{265}\right)$. These spaces could resist all types of
brute-force attacks.
Table 4 represent nonlinearity, SAC, BIC, LP and DP results for new S-boxes and compares with the other results.

## 7 Conclusion

We are using the introduced quantum map based on quantum dots to generate chaotic S-boxes. The proposed map results, improving in performance of introduced PSO and HS optimization algorithms. In comparing the with classic ones, it is effectively acting on generation the S-box. The obtained results show the importance of optimization algorithms in generating the S-box. The Harmony search algorithm for the known sphere function has a weaker answer than the PSO algorithm. In optimizing chaotic S-boxes, the use of Harmony search algorithms produces better results. The introduced S-boxes can be used in all image encryption, steganography, watermarking, and quantum digital signatures to increase security.


#### Abstract

$\begin{array}{lllllllllllllll}99 & 206 & 2 & 73 & 228 & 88 & 191 & 176 & 6 & 101 & 211 & 98 & 231 & 153 & 62\end{array} 207$ $\begin{array}{lllllllllllllll}164 & 179 & 49 & 195 & 108 & 31 & 141 & 8 & 185 & 57 & 27 & 249 & 91 & 128 & 209 \\ 154\end{array}$ $\begin{array}{llllllllllllll}252 & 201 & 138 & 205 & 247 & 76 & 60 & 165 & 14 & 55 & 5 & 56 & 12 & 238 \\ 139 & 240\end{array}$ $\begin{array}{lllllllllllllll}149 & 125 & 192 & 54 & 188 & 183 & 39 & 229 & 193 & 117 & 180 & 13 & 233 & 146 & 30\end{array} 150$ $\begin{array}{lllllllllllllll}214 & 97 & 106 & 82 & 35 & 109 & 131 & 230 & 173 & 152 & 127 & 182 & 41 & 25 & 47 \\ 236\end{array}$ $\begin{array}{lllllllllllllllll}92 & 196 & 160 & 122 & 242 & 111 & 34 & 220 & 212 & 81 & 175 & 170 & 77 & 118 & 132 & 4\end{array}$ $\begin{array}{lllllllllllllll}26 & 145 & 119 & 168 & 15 & 187 & 63 & 136 & 7 & 148 & 181 & 123 & 17 & 221 & 241\end{array}$ $\begin{array}{lllllllllllllll}254 & 250 & 255 & 67 & 1 & 239 & 93 & 103 & 46 & 226 & 157 & 90 & 167 & 51 & 184\end{array} 105$ $\begin{array}{lllllllllllllll}72 & 219 & 140 & 133 & 194 & 203 & 59 & 115 & 232 & 70 & 246 & 243 & 199 & 112 & 142\end{array} 224$ $\begin{array}{lllllllllllllll}19 & 42 & 213 & 186 & 177 & 66 & 94 & 68 & 129 & 79 & 21 & 256 & 234 & 80 & 172\end{array} 223$ $\begin{array}{llllllllllllllll}171 & 58 & 74 & 156 & 126 & 38 & 16 & 33 & 48 & 178 & 78 & 52 & 114 & 143 & 104 & 23\end{array}$ $\begin{array}{llllllllllllllll}200 & 32 & 251 & 151 & 216 & 237 & 65 & 89 & 28 & 190 & 75 & 202 & 83 & 159 & 69 & 245\end{array}$ $\begin{array}{llllllllllllllll}20 & 96 & 45 & 225 & 9 & 50 & 174 & 113 & 137 & 95 & 198 & 44 & 162 & 244 & 18 & 87\end{array}$ $\begin{array}{lllllllllllll}210 & 130 & 102 & 61 & 107 & 85 & 215 & 147 & 248 & 43 & 71 & 29 & 64 \\ 24 & 121 & 100\end{array}$ $\begin{array}{llllllllllllll}116 & 134 & 22 & 155 & 124 & 135 & 217 & 235 & 189 & 163 & 11 & 253 & 144 & 3\end{array}$ $\begin{array}{lllllllllllllll}204 & 110 & 86 & 208 & 158 & 10 & 197 & 161 & 120 & 222 & 37 & 169 & 40 & 36 & 227 \\ 166\end{array}$


Table 1. New S-box for the map of the Eq. 5

$$
\begin{array}{llllllllllllllll}
185 & 241 & 245 & 54 & 115 & 154 & 198 & 63 & 190 & 228 & 29 & 94 & 177 & 213 & 186 & 240 \\
192 & 191 & 28 & 200 & 208 & 193 & 194 & 238 & 34 & 244 & 188 & 132 & 254 & 164 & 107 & 2 \\
151 & 239 & 125 & 128 & 171 & 231 & 181 & 96 & 220 & 71 & 21 & 204 & 43 & 101 & 39 & 95 \\
256 & 33 & 41 & 218 & 127 & 141 & 137 & 230 & 207 & 201 & 44 & 4 & 102 & 124 & 70 & 10 \\
248 & 153 & 212 & 13 & 158 & 119 & 69 & 1 & 143 & 167 & 14 & 3 & 195 & 121 & 206 & 6 \\
81 & 17 & 152 & 82 & 111 & 210 & 109 & 113 & 199 & 27 & 140 & 211 & 131 & 148 & 233 & 112 \\
48 & 221 & 92 & 253 & 187 & 57 & 243 & 60 & 217 & 78 & 234 & 130 & 116 & 173 & 216 & 120 \\
31 & 227 & 246 & 179 & 83 & 7 & 162 & 196 & 232 & 23 & 182 & 47 & 45 & 126 & 72 & 91 \\
90 & 76 & 62 & 215 & 30 & 169 & 88 & 222 & 99 & 172 & 176 & 237 & 136 & 189 & 139 & 100 \\
197 & 235 & 64 & 156 & 229 & 77 & 87 & 142 & 157 & 98 & 166 & 105 & 51 & 183 & 61 & 59 \\
106 & 38 & 68 & 67 & 144 & 155 & 202 & 247 & 40 & 123 & 104 & 174 & 147 & 122 & 163 & 117 \\
79 & 36 & 255 & 22 & 37 & 236 & 20 & 74 & 32 & 138 & 223 & 165 & 35 & 86 & 97 & 226 \\
58 & 19 & 110 & 209 & 108 & 114 & 103 & 118 & 25 & 9 & 50 & 5 & 160 & 12 & 129 & 252 \\
65 & 24 & 149 & 16 & 249 & 52 & 224 & 184 & 55 & 66 & 178 & 225 & 219 & 150 & 242 & 93 \\
11 & 53 & 49 & 84 & 175 & 146 & 205 & 15 & 26 & 56 & 89 & 18 & 250 & 159 & 180 & 8 \\
170 & 214 & 42 & 133 & 46 & 161 & 75 & 145 & 134 & 85 & 203 & 80 & 251 & 73 & 168 & 135
\end{array}
$$

Table 2. New optimized S-box for the map of the Eq. 5 with improved PSO algorithm

$$
\begin{array}{llllllllllllllll}
206 & 4 & 51 & 105 & 57 & 121 & 73 & 247 & 36 & 152 & 101 & 109 & 18 & 134 & 119 & 173 \\
25 & 222 & 43 & 122 & 78 & 242 & 30 & 110 & 83 & 114 & 12 & 65 & 23 & 185 & 58 & 138 \\
141 & 96 & 1 & 64 & 209 & 135 & 116 & 126 & 156 & 226 & 212 & 84 & 237 & 238 & 160 & 128 \\
47 & 255 & 103 & 253 & 40 & 67 & 98 & 229 & 153 & 225 & 14 & 8 & 66 & 29 & 99 & 217 \\
21 & 155 & 146 & 219 & 37 & 246 & 181 & 227 & 108 & 17 & 171 & 220 & 7 & 52 & 256 & 94 \\
89 & 130 & 211 & 20 & 77 & 133 & 82 & 190 & 24 & 10 & 50 & 44 & 62 & 120 & 136 & 234 \\
224 & 208 & 80 & 3 & 163 & 251 & 245 & 195 & 148 & 143 & 203 & 235 & 113 & 72 & 216 & 117 \\
144 & 115 & 16 & 142 & 162 & 111 & 70 & 193 & 191 & 38 & 177 & 174 & 213 & 165 & 194 & 86 \\
145 & 42 & 34 & 45 & 202 & 204 & 22 & 158 & 139 & 31 & 157 & 75 & 92 & 180 & 241 & 198 \\
11 & 188 & 61 & 26 & 151 & 132 & 197 & 39 & 233 & 207 & 97 & 170 & 184 & 68 & 214 & 104 \\
149 & 182 & 35 & 49 & 112 & 189 & 60 & 140 & 107 & 239 & 56 & 100 & 199 & 150 & 87 & 186 \\
250 & 231 & 196 & 187 & 33 & 19 & 168 & 161 & 46 & 183 & 249 & 76 & 221 & 2 & 93 & 95 \\
9 & 201 & 240 & 91 & 13 & 90 & 192 & 236 & 223 & 125 & 28 & 5 & 147 & 131 & 244 & 129 \\
230 & 41 & 71 & 210 & 254 & 167 & 69 & 200 & 27 & 205 & 48 & 54 & 228 & 85 & 172 & 218 \\
166 & 176 & 248 & 55 & 159 & 106 & 88 & 102 & 15 & 243 & 59 & 164 & 6 & 53 & 124 & 179 \\
81 & 178 & 252 & 169 & 154 & 32 & 123 & 118 & 63 & 74 & 79 & 232 & 137 & 175 & 127 & 215
\end{array}
$$

Table 3. New optimized S-box for the map of the Eq. 5 with improved HS algorithm

| - | Nonlinearity SAC | BIC-Nonlinearity BIC-SAC LP | DP |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| New S-box | 105.5 | 0.512939 | 103.714 |  | 0.140625 | 10 |
| New optimized S-box with PSO | 106 | 0.499512 | 103.5 | 0.500837 | 0.132813 | 10 |
| New optimized S-box with HS | 106.5 | 0.501465 | 104.071 | 0.498047 | 0.132813 | 10 |
| $[11]$ | 106.5 | 0.503662 | 102.857 | 0.499512 | 0.140625 | 10 |
| $[12]$ | 105.25 | 0.495605 | 104.571 | 0.504325 | 0.140625 | 12 |
| $[13]$ | 104.2 | 0.4931 | 103.3 | 0.4988 | 0.1563 | 12 |
| $[14]$ | 106 | 0.52881 | 100 | - | - | 10 |
| $[10]$ | 112 | 0.5048 | 112 | - | - | 4 |

Table 4. Nonlinearity, SAC, BIC, LP, and DP results for new S-boxes and compare with the other results.

## A

In this appendix we derive the Eq. (5). By inserting expressions $J_{+}=<J_{+}>$ $+\delta J_{+}$and $J_{-}=<J_{-}>+\delta J_{-}$into force equation([12]) we get

$$
\begin{gather*}
f\left(J_{+}, J_{-}\right)=-<J_{+}>-\delta J_{+} \\
+e^{-i w_{A} T} r\left(<J_{+}>+\delta J_{+}-<J_{-}><J_{+}>-\delta J_{-} \delta J_{+}-\right. \\
\left.<J_{-}>\delta J_{+}-\delta J_{-}<J_{+}>\right) \tag{6}
\end{gather*}
$$

By considering $\dot{J}_{+}=\delta \dot{J}_{+}, \dot{J}_{-}=\delta \dot{J}_{-}$, and due to

$$
\begin{equation*}
\delta \dot{J}_{-}=\delta \dot{J}_{+}^{\dagger} \tag{7}
\end{equation*}
$$

we use reference [12] for obtaining:

$$
\begin{gather*}
\frac{d}{d t}\left(\delta J_{+} \delta J_{-}\right)=\left[i w_{A}\left(<J_{+}>+\delta J_{+}\right)-i \frac{\gamma}{\sqrt{N}} a^{\dagger}(0) e^{i t}\left(\delta J_{+}\right.\right. \\
\left.\delta J_{-}-\delta J_{-} \delta J_{+}\right)-i \frac{\gamma}{\sqrt{N}} a(0) e^{-i t}\left(\delta J_{+} \delta J_{-}-\delta J_{-} \delta J_{+}\right)+[- \\
<J_{+}>-\delta J_{+}+e^{i w_{A} T} r\left(<J_{+}>+\delta J_{+}-<J_{-}><J_{+}>\right. \\
\left.\left.\left.-\delta J_{-} \delta J_{+}-<J_{-}>\delta J_{+}-\delta J_{-}<J_{+}>\right)\right] \Sigma_{n} \delta(t-n T)\right] \delta J_{-} \\
+\delta J_{+}\left(-i w_{A}\left(<J_{-}>+\delta J_{-}\right)+i \frac{\gamma}{\sqrt{N}} a(0) e^{-i t}\left[\delta J_{-} \delta J_{+}^{-}\right.\right. \\
\left.\delta J_{+} \delta J_{-}\right]+i \frac{\gamma}{\sqrt{N}} a^{\dagger}(0) e^{i t}\left[\delta J_{-} \delta J_{+}-\delta J_{+} \delta J_{-}\right]+\left[-<J_{-}>\right. \\
-\delta J_{-}+e^{i w_{A} T} r\left(<J_{-}>+\delta J_{-}<J_{+}><J_{-}>-\delta J_{+}\right. \\
\left.\left.\left.\delta J_{-}<J_{+}>\delta J_{-}-\delta J_{+}<J_{-}>\right)\right] \Sigma_{n} \delta(t-n T)\right] . \tag{8}
\end{gather*}
$$

By integrating Eq. (A), from $n T$ to $(n+1) T$, and take the expectation value, by taking into account $<\delta J_{-}(n T)>=<\delta J_{+}(n T)>=0,<a^{\dagger}(0)>=<$ $a(0)>=0$ we obtain:

$$
\begin{gather*}
<\delta J_{+}((n+1) T) \delta J_{-}((n+1) T)>=-<\delta J_{+}(n T) \delta J_{-}(n T) \\
>+r e^{-i w_{A} T}<\delta J_{+}(n T) \delta J_{-}(n T)>-r e^{-i w_{A} T}<J_{-}> \\
<\delta J_{+}(n T) \delta J_{-}(n T)>-r e^{-i w_{A} T} r<\delta J_{-}(n T) \delta J_{-}(n T)> \\
<J_{+}>+r e^{i w_{A} T}<\delta J_{+}(n T) \delta J_{-}(n T)>-r e^{i w_{A} T} \\
<\delta J_{+}(n T) \delta J_{-}(n T)><J_{+}>-r e^{i w_{A} T}<\delta J_{+}(n T) \\
\delta J_{+}(n T)><J_{-}> \tag{9}
\end{gather*}
$$

The calculation of $<\delta J_{+} \delta J_{+}>$goes as follows:

$$
\begin{equation*}
\frac{d}{d t}\left(\delta J_{+} \delta J_{+}\right)=\delta \dot{J}_{+} \delta J_{+}+\delta J_{+} \delta \dot{J}_{+} \tag{10}
\end{equation*}
$$

We end up with:

$$
\begin{gathered}
\frac{d}{d t}\left(\delta J_{+} \delta J_{+}\right)=\left[i w_{A}\left(<J_{+}>+\delta J_{+}\right)-i \frac{\gamma}{\sqrt{N}} a^{\dagger}(0) e^{i t}\left[\delta J_{+}\right.\right. \\
\left.\delta J_{-}-\delta J_{-} \delta J_{+}\right]-i \frac{\gamma}{\sqrt{N}} a(0) e^{-i t}\left[\delta J_{+} \delta J_{-}-\delta J_{-} \delta J_{+}\right]+[- \\
<J_{+}>-\delta J_{+}+e^{-i w_{A} T} r\left(<J_{+}>+\delta J_{+}-<J_{-}><J_{+}>-\right. \\
\left.\left.\left.\delta J_{-} \delta J_{+}-<J_{-}>\delta J_{+}-\delta J_{-}<J_{+}>\right)\right] \times \Sigma_{n} \delta(t-n T)\right] \\
\delta J_{+}+\delta J_{+}\left[i w_{A}\left(<J_{+}>+\delta J_{+}\right)-i \frac{\gamma}{\sqrt{N}} a^{\dagger}(0) e^{i t}\left[\delta J_{+} \delta J_{-}\right.\right. \\
\left.-\delta J_{-} \delta J_{+}\right]-i \frac{\gamma}{\sqrt{N}} a(0) e^{-i t}\left[\delta J_{+} \delta J_{-}-\delta J_{-} \delta J_{+}\right]+\left[-<J_{+}>\right. \\
-\delta J_{+}+e^{-i w_{A} T} r\left(<J_{+}>+\delta J_{+}-<J_{-}><J_{+}>-\delta J_{-} \delta J_{+}\right.
\end{gathered}
$$

$$
\begin{equation*}
\left.\left.\left.-<J_{-}>\delta J_{+}-\delta J_{-}<J_{+}>\right)\right] \Sigma_{n} \delta(t-n T)\right] \tag{11}
\end{equation*}
$$

By integrating form Eq. (11), from $n T$ to $(n+1) T$, and by assuming $<$ $\delta J_{-}(n T)>=<\delta J_{+}(n T)>=0,<a^{\dagger}(0)>=<a(0)>=0$ we obtain:

$$
\begin{gather*}
<\delta J_{+}((n+1) T) \delta J_{+}((n+1) T)>e^{-2 i \omega_{A}(n+1) T}-<\delta J_{+}(n T) \\
\delta J_{+}(n T)>e^{-2 i \omega_{A} n T}=e^{-2 i \omega_{A} n T}\left(-<\delta J_{+}(n T) \delta J_{+}(n T)>\right. \\
\quad+e^{-i \omega_{A} T} r\left(<\delta J_{+}(n T) \delta J_{+}(n T)>-<J_{-}(n T)>\right. \\
<\delta J_{+}(n T) \delta J_{+}(n T)>-<J_{+}(n T)><\delta J_{-}(n T) \delta J_{+}(n T)> \\
))+e^{2 i \omega_{A} n T}\left(-<\delta J_{+}(n T) \delta J_{+}(n T)>+e^{-i \omega_{A} T} r( \right. \\
<\delta J_{+}(n T) \delta J_{+}(n T)>-<J_{-}(n T)><\delta J_{+}(n T) \delta J_{+}(n T)> \\
\left.\left.\quad-\delta J_{+}(n T) \delta J_{-}(n T)><J_{+}(n T)>\right)\right) \tag{12}
\end{gather*}
$$

## B

This appendix describes the improved PSO steps and its application for optimizing the designed chaotic S-box. The steps of the algorithm are as follows:
step 1 Enter improved PSO parameters (number of decision variables, size of decision variables matrix, Maximum number of iterations, population size, inertia weight, inertia weight damping ratio, personal learning coefficient, global learning coefficient) and $a_{1}=2.61, a_{2}=3.168$ for the Hierarchy of rational order chaotic maps ref. [11].
step 2 Initial population production using chaotic map Eq. 5.
step 3 Creation of S-box based on quantum map Eq. 5:
1- Enter $r=5.5, \beta=0.5$ for Eq. 5 (consider Fig. 1).
2- Pass the transition state by repeating the map Eq. 5.
3 - We create empty $16 * 16$ box.
4- Repeat the map Eq. 5 and select $x(f)$.
5 - The S-box numbers are obtained:

$$
S(i)=x(f) * 10^{5} \bmod 256
$$

6 - The process continues from 4 and select different $S(i)$.
step 4 Calculate nonlinearity of all primary particles and search personal and global best for this population.
step 5 Update the speed and position(consider jth dimension at iteration $t$ of each particle i):

$$
\begin{align*}
V_{i, j}(t+1)= & w V_{i, j}(t)+(c 1)(r 1)\left(\operatorname{Best} X_{i, j}(t)-X_{i, j}(t)\right)+ \\
& (c 2)(r 2)\left(\operatorname{GlobalBest}(t)-X_{i, j}(t)\right)  \tag{13}\\
& X_{i, j}(t+1)=X_{i, j}(t)+V_{i, j}(t+1) \tag{14}
\end{align*}
$$

where $V_{i, j}(t)$ is a velocity of particle i at iteration $\mathrm{t} ; X_{i, j}(t)$ it is a position of i particle at iteration t ; r 1 and r 2 are two random number between $(0,1)$ provided by the Hierarchy of rational order chaotic maps ref. [11]; $\operatorname{Best} X_{i, j}(t)$ is the local best particle i in all swarm and GlobalBest $(t)$ is the leader of the swarm or global best position of all population.
step 6 Local and global search and save the best nonlinearity and related S-box.

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# The interaction of memristor in cellular nonlinear network for image and signal processing 

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#### Abstract

In this paper, we describe the application of memristor in the neighborhood connections of 2D cellular nonlinear networks (CNN) essentially for image and signal processing. We focused particularly on the interaction of memristors between two cells allowing us to study the contribution of the memristor qualitatively and quantitatively. The dynamics and the steady state response of each cell is described. The resistance of a memristor is not fixed, hence the study takes into account the initial state of the memristance characterized by the previous amount of charge passed through the memristor. We show that the system transition and the steady state response depend strongly on the history of the memristor.


Keywords: Memristor, 2D cellular nonlinear networks, system of two cells, system dynamic, steady state response.

## 1 Introduction

Memristor is a nanoscale two terminals solid state device whose conductivity is controlled by the time-integral of the current flowing through it or the time-integral of the voltage across its terminals $[1,2]$. The dynamic conductance modulation and connection compatibility with complementary metaloxide semiconductor neurons, are essential features of memristor affirming its potentiality as synaptic function and memristive gird network [3-5].

This paper describes a memristor based 2D cellular nonlinear network using memristor in the coupling mode. The network is essentially for signal and image processing applications. Figure 1 shows the conventional 2D cellular nonlinear network with each cell constituting one linear capacitor in parallel with one nonlinear resistance, and a series resistance coupling [6]. Figure 2 shows the equivalent network using memristors in the coupling mode.

For any cell at a node $n$ and voltage $V_{n}$, the nonlinear current function through the nonlinear resistance is given by:

$$
\begin{equation*}
I_{N L_{n}}=\frac{V_{n}\left(V_{n}-V_{a}\right)\left(V_{n}-V_{b}\right)}{R_{o} V_{a} V_{b}} \tag{1}
\end{equation*}
$$



Fig. 1. 2D CNN using series resistance coupling [6].


Fig. 2. 2D CNN using memristors in the coupling mode.
and the nonlinear resistance, $R_{N L_{n}}$ :

$$
R_{N L_{n}}=\frac{R_{o} V_{a} V_{b}}{V_{n}\left(V_{n}-V_{a}\right)\left(V_{n}-V_{b}\right)}
$$

The characteristic roots of the cubic resistance are $0, V_{a}$ and $V_{b}$, meanwhile $R_{o}$ is its linear approximation. The corresponding potential energy $W\left(V_{n}\right)$ is obtained from equation (1) as:

$$
W\left(V_{n}\right)=\frac{1}{4} V_{n}^{4}-\frac{V_{a}+V_{b}}{3} V_{n}^{3}+\frac{V_{a} V_{b}}{2} V_{n}^{2}+\kappa
$$

where $\kappa$ is a constant. Figure 3 shows the response of the cubic resistance and the corresponding potential energy showing the possible equilibrium state. The lower potential energy state are at 0 and $V_{b}$ marked by numbers 1 and 2 respectively, meanwhile $V_{a}$ is the unstable state.


Fig. 3. Response of the nonlinear resistance in the cells. $V_{a}=0.7 \mathrm{~V}, V_{b}=1.2 \mathrm{~V}$ and $R_{o}=1023 \Omega$. (a) $I-V$ characteristic and (b) the corresponding potential energy. Labels 1 and 2 show the two possible equilibrium states corresponding to $V_{n}=0$ and $V_{n}=V_{b}$.

We focus on the study of memristor response based on the interaction of one cell with its neighbouring cells. Therefore, using the system of two cells coupled by a memristor, allows us to observe the quantitative and qualitative interaction of memristors in the network.

## 2 System of two cells

Figure 4 shows the network of two cells coupled by a memristor, thus the cells communicate together bidirectionally. One of the cells acts as the master while the other one as slave so that the direction of the flowing charge through the memristor becomes specific. The switchs $s_{1}$ and $s_{2}$ are closed simultaneously and the network gives the following set of bidirectional coupled equations:

$$
\begin{align*}
\frac{d q}{d t} & =-C \frac{d V_{m}}{d t}-\frac{V_{m}\left(V_{m}-V_{a}\right)\left(V_{m}-V_{b}\right)}{R_{o} V_{a} V_{b}}  \tag{2a}\\
\frac{d q}{d t} & =C \frac{d V_{s}}{d t}+\frac{V_{s}\left(V_{s}-V_{a}\right)\left(V_{s}-V_{b}\right)}{R_{o} V_{a} V_{b}}  \tag{2b}\\
\frac{d q}{d t} & =\frac{V_{m}-V_{s}}{M(q)} \tag{2c}
\end{align*}
$$

where: $I(t)=\frac{d q}{d t}$ is the flowing current through the memristor and $M(q)$ is the memristance function that has a desirable continuity for all the flowing


Fig. 4. System of two cells coupled by a memristor. The master and slave cells with their elements given by the subscripts letters $m$ and $s$ respectively.
charge [8]:

$$
M(q)= \begin{cases}R_{o f f}, & \text { if } q(t) \leq 0  \tag{3}\\ R_{o f f}-\frac{3 \delta R}{q_{d}^{2}} q^{2}+\frac{2 \delta R}{q_{d}^{3}} q^{3}, & \text { if } 0 \leq q(t) \leq q_{d} \\ R_{o n}, & \text { if } q(t) \geq q_{d}\end{cases}
$$

$q_{d}=\frac{D^{2}}{\mu_{v} R_{o n}}$ is a charge scaling factor depending on the technology parameters $[2,7]$ and $\delta R=R_{o f f}-R_{o n}$ is the difference between the two limiting resistance values of the memristor, that is, the ON and OFF states, represented here by $R_{o n}$ and $R_{o f f}$ respectively. Charge flows from the master cell to the slave one through the memristor until $V_{m}(t)=V_{s}(t)$ and that is when the network is stabilized. Equation (2) is reformulated as:

$$
\begin{align*}
\frac{d V_{m}}{d t} & =-\frac{V_{m}-V_{s}}{C \cdot M(q)}-\frac{V_{m}\left(V_{m}-V_{a}\right)\left(V_{m}-V_{b}\right)}{R_{o} V_{a} V_{b} C}  \tag{4a}\\
\frac{d V_{s}}{d t} & =\frac{V_{m}-V_{s}}{C \cdot M(q)}-\frac{V_{s}\left(V_{s}-V_{a}\right)\left(V_{s}-V_{b}\right)}{R_{o} V_{a} V_{b} C}  \tag{4b}\\
\frac{d q}{d t} & =\frac{V_{m}-V_{s}}{M(q)} \tag{4c}
\end{align*}
$$

As illustrated in Fig. 3b, 0 and $V_{b}$ are the only two possible equilibrium states. The stability of the cells at 0 or $V_{b}$ is determined by $V_{a}$. It can be seen that if $V_{b}-2 V_{a}>0$ the cell stabilizes at $V_{b}$ and if $V_{b}-2 V_{a}<0$ the cell stabilizes at 0 .

The initial conditions of $V_{m}, V_{s}$ and $q$ are $V_{m_{0}}, V_{s_{0}}$ and $q_{0}$ respectively. Figure 5 shows the time evolution of the 2 cells network for $V_{m_{0}}=2 V, V_{s_{0}}=$ $0 V, V_{b}=1.5 V, V_{a}=0.7 V, R_{o}=10 K \Omega$ and $C_{m}=C_{s}=1 \mu F$, hence $V_{b}-2 V_{a}>$ 0 . Initially, the voltage $V_{m}(t)$ and $V_{s}(t)$ evolve in the differential mode and thereafter the common mode when $V_{m}(t)=V_{s}(t)$ which continues to evolve until the steady state $V_{b}$. The charge $q(t)$ flows through the memristor until $V_{m}(t)=V_{s}(t)$. When $V_{m}(t)=V_{s}(t)$, the voltage across the memristor is 0 (i.e $\left.V_{d}(t)=0 V\right)$.


Fig. 5. The evolution of $V_{m}(t), V_{s}(t), V_{d}(t)=V_{m}(t)-V_{s}(t)$ and $q(t)$, for $V_{m_{0}}=2 V$, $V_{s_{0}}=0 V, V_{b}=1.5 \mathrm{~V}, V_{a}=0.7 \mathrm{~V}, R_{o}=10 \mathrm{~K} \Omega$ and $C_{m}=C_{s}=1 \mu F$. The charge $q(t)$ flows through the memristor until $V_{d}(t)=0$ and at this time, the combined evolution of $V_{m}(t)$ and $V_{s}(t)$ is the common mode which evolves further to stabilizes at the steady state $V_{b}$.

## 3 Discussion

Variation of the system parameters, such as $V_{a}, R_{o}$ and $q_{0}$ affects the steady state response of the system. The results of Fig. 6 show the variation of $V_{a}$ with respect to $V_{b}$, for example $V_{a}=\Upsilon V_{b}$. The results are obtained for $R_{0}=$ $2833 \Omega, V_{b}=1.5 V, V_{m_{0}}=2 V, V_{s_{0}}=0 V$ and $C_{m}=C_{s}=1 \mu F$. Hence, $V_{a}$ varies according to $\Upsilon=[0.25,0.45,0.49,0.5,0.51,0.55,0.75,0.9]$ with the corresponding results given by Figs. $6 \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}$ and $\mathbf{h}$, respectively. Furthermore, the difference $V_{b}-2 V_{a}$ is calculated and tabulated in Table 1. The results show two noticeable effects on the evolution of $V_{m}(t)$ and $V_{s}(t)$ based on the variation of $V_{a}$. The results show different timing at which $V_{m}(t)=V_{s}(t)$ and the change in the steady state at $V_{b}$ or 0 for $V_{a}<\frac{V_{b}}{2}$ or $V_{a}>\frac{V_{b}}{2}$ respectively.

| Figure 6 | $\mathbf{a}$ | $\mathbf{b}$ | $\mathbf{c}$ | $\mathbf{d}$ | $\mathbf{e}$ | $\mathbf{f}$ | $\mathbf{g}$ | $\mathbf{h}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\Upsilon$ | 0.25 | 0.45 | 0.49 | 0.5 | 0.51 | 0.55 | 0.75 | 0.9 |
| $V_{b}-2 V_{a}(V)$ | 0.75 | 0.15 | 0.03 | 0 | -0.03 | -0.15 | -0.75 | -1.20 |

Table 1. Table of $V_{b}-2 V_{a}$ for Fig. 6. $V_{s_{0}}=0 V, V_{a}=\Upsilon V_{b}, V_{b}=1.5 V$ and $V_{m_{0}}=2 V$.

The initial charge $q_{0}$ characterizes the initial memristance of a memristor. The initial condition of a memristor has strong effect on its circuit functionality [9]. Figure 7 shows the effect of changing initial condition of the memristor on the system evolution and stability. It also takes into account the variations of $R_{0}$. The initial memristance of the memristor is given by the initial charge $q_{0}$.


Fig. 6. Evolution of $V_{m}(t)$ and $V_{s}(t)$ showing the variations of $V_{a} \in\left[0, V_{b}\right]: R_{0}=$ $2833 \Omega, V_{s_{0}}=0 V, V_{m_{0}}=2 V, V_{b}=1.5 V$ and $\Upsilon=[0.25,0.45,0.49,0.5,0.51,0.55$, $0.75,0.9]$ as shown respectively by figures $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}$ and $\mathbf{h}$. Variation of $V_{a}$ affects the time at which $V_{m}(t)=V_{s}(t)$ and the steady state at $V_{b}$ or 0 depending on whether $V_{a}<\frac{V_{b}}{2}$ or $V_{a}>\frac{V_{b}}{2}$ respectively.

Four different initial charges are considered as: $q_{0_{1}}=20 \mu C, q_{0_{2}}=40 \mu C, q_{0_{3}}=$ $60 \mu C$ and $q_{0_{4}}=80 \mu C$, as indicated respectively, by the subscripts numbers 1-4 in Fig. 7A and B. Notice that only one parameter is varied at a time. Figure 7A: $R_{0}=2833 \Omega$ while $q_{0}$ varied and Fig. 7B: $R_{0}=10 K \Omega$ while $q_{0}$ varied. In each case, $V_{a}=0.7 V, V_{b}=1.3 V, V_{m_{0}}=1.5 V$ and $V_{s_{0}}=0 V$. Even though $V_{a}$ is the main parameter that plays significant role on the system steady state, the results show that other parameters (e.g. $R_{o}$ and $q_{0}$ ) affect the dynamics and steady state of the system.

## 4 Conclusion

Memristor based 2D cellular nonlinear network is introduced using memristors in the coupling mode. The cells correspond to pixels in image processing applications. Each elemental cell consists of one linear capacitor in parallel with one nonlinear resistance such as Fitzhugh Nagumo. Using the system of two cells coupled by a memristor, the dynamics and the steady state of each cell are observed, with mainly a competition between the role of cubic resistance


Fig. 7. Results showing the variation effect of $q_{0}$ and $R_{o}$ on the system evolution and the steady state. Four different initial charges are considered as: $q_{0_{1}}=20 \mu C$, $q_{0_{2}}=40 \mu C, q_{0_{3}}=60 \mu C$ and $q_{0_{4}}=80 \mu C$, as indicated respectively by the subscripts numbers 1-4 in figures a and $\mathbf{b}$. In each case, $V_{a}=0.7 \mathrm{~V}, V_{b}=1.3 \mathrm{~V}, V_{m_{0}}=1.5 \mathrm{~V}$, $V_{s_{0}}=0 V$ and $C_{m}=C_{s}=1 \mu F$. (A) $R_{0}=2833 \Omega$ and (B) $R_{0}=10 K \Omega$. It shows that values of $q_{0}$ and $R_{0}$ have an effect on the evolution and steady state of the system.
on one hand, and the role of memristor on the other hand. The parameter $V_{a}$ predominantly decides the system steady state, however other parameters (e.g $R_{o}, q_{0}$ etc...) affect the system steady state. The results show that the network can be used to realize a binarization circuit, for example, to generate different gray scaling. The ongoing study focuses on the implementation of the generalized 2D network for processing any number cells.

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# The Turing Model and Discrete Limit Cycles with Eddy and Convection 

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#### Abstract

The Turing model and discrete limit cycles are considered, from the viewpoint of chaos functions. Firstly, the Turing model is explained as a reaction-diffusion system of chemical substances, and a three-dimensional (3-D) time-dependent solvable chaos map corresponding to the model is derived on the basis of chaos function solutions. Secondly, a 2-D chaos map for the 2-D Turing model is proposed for simplicity, in order to present chaotic dynamics, and the 2-D map is shown to have symmetric bifurcation diagrams, a ring of cells and limit cycles with different patterns, depending on the system parameter. In particular, the limit cycles appear in pairs, and are discussed on left-handed and right-handed eddies, which generate convections, as nonlinear dynamics of nonequilibrium open systems.


Keywords: Turing model, Turing pattern, Chaos function, Bifurcation diagram, Symmetry, Fixed point, Limit cycle, Eddy, Convection, Non-equilibrium open system.

## 1 Introduction

For the study of nonlinear phenomena, nonlinear dynamics such as soliton, chaos and fractals, have been considered in the field of physics, chemistry, biology and engineering, as nonlinear science [1-4]. In particular, it is well known that one-dimensional (1-D) nonlinear difference equations possess a rich spectrum of dynamical behavior as chaos, and the chaotic modeling and simulation have been extended to medicine, optics, living systems, neuro science and life science, as interdisciplinary fields of science [5-8]. At the same time, large scale systems, such as atmosphere, climate, human brain, power grid, information system and communication network have been studied as nonequilibrium open systems and/or complex systems [9-14].
On the other hand, the Fisher equation has been proposed as a model for the propagation of gene [15]. After that, travelling wave solutions to reactiondiffusion systems are discussed widely, and the effect of boundaries on convection in a shallow layer of fluid heated from below has been considered in the field of fluid mechanics [16, 17]. Then, the propagation of waves observed in a chemical reaction system is reported, and has been considered as a nonequilibrium open system [18, 19]. Later, the data obtained in the experiment on the Belousov-Zhabotinsky (BZ) reaction have been analyzed [20, 21]. Moreover, a 2-D model of nonlinear differential equations is derived for the interaction between local reaction and diffusion as chaotic dynamics [22, 23]. In the meantime, a reaction-diffusion system called the Turing model has been presented as a chemical basis of morphogenesis [24]. The chemical pattern is considered as the Turing pattern, and stripe patterns on the skin of marine
angelfish have been discussed for understanding biological pattern formation [25-27].
Recently, 1-D, 2-D and 3-D time-dependent solvable chaos maps and a nonlinear time series expansion method have been proposed [28, 29]. Then, the 2-D maps corresponding to the FitzHugh-Nagumo (FHN) model, the BZ reaction and reaction-diffusion systems are derived, and the bifurcation diagrams and the discrete limit cycles have been studied for population growth, neural cells and chemical cells, respectively [30-33]. In addition, a limit cycle analysis and the interaction of limit cycles are presented for the 2-D logistic maps, as non-equilibrium open systems [34-36].
In this paper, firstly the Turing model is explained in Section 2 as a reactiondiffusion system of chemical substances, and a 3-D time-dependent solvable chaos map, which corresponds to the model, is derived on the basis of chaos function solutions. Then, in Section 3, a 2-D solvable chaos map for the 2-D Turing model is proposed for simplicity to find chaotic properties. The 2-D map is shown numerically to possess symmetric bifurcation diagrams, a ring of cells and discrete limit cycles with different patterns, depending on the system parameter. In particular, the limit cycles appear in pairs, and are discussed on left-handed and right-handed eddies of cells, which generate convections, as nonlinear dynamics of non-equilibrium open systems. Finally, Conclusions are summarized in the last section.

## 2 The Turing Model and 3-D Chaotic Maps

As is known, a reaction-diffusion system called the Turing model has been presented as a chemical basis of morphogenesis [24], in where a mathematical model of the growing embryo in biology is described, and the chemical reaction and diffusion are explained, under the assumption of reaction rates. In addition, the spherical symmetry of embryo is introduced, and the system is assumed to be far from homogeneous, in left-handed and right-handed organisms. Finally, from the mathematics of the ring, a set of nonlinear differential equations are formulated as the model equations;

$$
\begin{align*}
& \frac{d z_{0}}{d t}=p_{0} z_{0}+A z_{1}^{2}+B z_{2}^{2}  \tag{1}\\
& \frac{d z_{1}}{d t}=p_{1} z_{1}+C z_{1} z_{2}+D z_{0} z_{1}  \tag{2}\\
& \frac{d z_{2}}{d t}=p_{2} z_{2}+E z_{1}^{2}+F z_{0} z_{2} \tag{3}
\end{align*}
$$

with system parameters $\left\{p_{0}, p_{1}, p_{2}, A, B, C, D, E, F\right\}$, where $z_{0}(t), z_{1}(t)$ and $z_{2}(t)$ are dimensionless variables [24]. Then, the model equations can be rewritten into the following nonlinear difference equations by the difference method for numerical calculation as

$$
\begin{align*}
& z_{0, n+1}=\left(1+p_{0}(\Delta t)\right) z_{0, n}+(\Delta t)\left(A z_{1, n}^{2}+B z_{2, n}^{2}\right)  \tag{4}\\
& z_{1, n+1}=\left(1+p_{1}(\Delta t)\right) z_{1, n}+(\Delta t)\left(C z_{1, n} z_{2, n}+D z_{0, n} z_{1, n}\right)  \tag{5}\\
& z_{2, n+1}=\left(1+p_{2}(\Delta t)\right) z_{2, n}+(\Delta t)\left(E z_{1, n}^{2}+F z_{0, n} z_{2, n}\right) \tag{6}
\end{align*}
$$

where the passage from a point $\left(z_{0, n} \equiv z_{0, n}\left(t_{i}\right), z_{1, n} \equiv z_{1, n}\left(t_{i}\right), z_{2, n} \equiv z_{2, n}\left(t_{i}\right)\right)$ to the next one $\left(z_{0, n+1} \equiv z_{0, n+1}\left(t_{i+1}\right), z_{1, n+1} \equiv z_{1, n+1}\left(t_{i+1}\right), z_{2, n+1} \equiv z_{2, n+1}\left(t_{i+1}\right)\right)$ is considered as a 3-D mapping with the discrete time $t_{i}$ and the time step $\Delta t=t_{i+1}-t_{i}$ [30].
On the other hand, we introduce time-dependent chaos functions;

$$
\begin{align*}
& x_{n}\left(t_{i}\right)=a_{1} \cos \left(2^{n} t_{i}\right)+b_{1},  \tag{7}\\
& y_{n}\left(t_{i}\right)=a_{2} \sin \left(2^{n} t_{i}\right),  \tag{8}\\
& z_{n}\left(t_{i}\right)=a_{3} \sin \left(2^{n} t_{i}\right) \tag{9}
\end{align*}
$$

with real parameters $\left\{a_{1}, a_{2}, a_{3}, b_{1}\right\}$, and find a 3-D solvable chaos map with system parameters $\left\{\beta_{1}, \beta_{2}, \beta_{3}\right\}$ [28], corresponding to the 3-D map (4)-(6), as

$$
\begin{align*}
& x_{n+1}=\left(\frac{1}{a_{1}}\right)\left(x_{n}^{2}-2 b_{1} x_{n}\right)-a_{1}\left(1-\beta_{1}\right)\left(\frac{1}{a_{2}}\right)^{2} y_{n}^{2}-a_{1} \beta_{1}\left(\frac{1}{a_{3}}\right)^{2} z_{n}^{2}+\left(1+\frac{b_{1}}{a_{1}}\right) b_{1},  \tag{10}\\
& y_{n+1}=-2\left(\frac{b_{1}}{a_{1}}\right) y_{n}+2\left(\frac{a_{2}}{a_{1} a_{3}}\right) \beta_{2} x_{n} z_{n}+2\left(\frac{1}{a_{1}}\right)\left(1-\beta_{2}\right) x_{n} y_{n},  \tag{11}\\
& z_{n+1}=-2\left(\frac{b_{1}}{a_{1}}\right) z_{n}+2\left(\frac{a_{3}}{a_{1} a_{2}}\right) \beta_{3} x_{n} y_{n}+2\left(\frac{1}{a_{1}}\right)\left(1-\beta_{3}\right) x_{n} z_{n}, \tag{12}
\end{align*}
$$

where $x_{n} \equiv x_{n}\left(t_{i}\right), y_{n} \equiv y_{n}\left(t_{i}\right)$ and $z_{n} \equiv z_{n}\left(t_{i}\right)$. It is notable that the time step $\Delta t, 0<\Delta t \ll 1.0$ is not included in the coefficients of the 3-D map (10)-(12) [31].

## 3 2-D Chaotic Maps and Discrete Limit Cycles

In this Section, we find a 2-D Turing map for simplicity, by setting $z_{2, n}=0$ in the 3-D Turing map (4)-(6), as

$$
\begin{align*}
& z_{0, n+1}=\left(1+p_{0}(\Delta t)\right) z_{0, n}+(\Delta t) A z_{1, n}^{2}  \tag{13}\\
& z_{1, n+1}=\left(1+p_{1}(\Delta t)\right) z_{1, n}+(\Delta t) D z_{0, n} z_{1, n} \tag{14}
\end{align*}
$$

and then, from the chaos function solutions (7), (8) and the condition given by

$$
\begin{equation*}
\left(\frac{1}{a_{1}}\right)^{2}\left(x_{n}-b_{1}\right)^{2}+\left(\frac{1}{a_{2}}\right)^{2} y_{n}^{2}=1, \tag{15}
\end{equation*}
$$

we have a 2-D chaotic map;

$$
\begin{align*}
& x_{n+1}=\left(a_{1}+b_{1}\right)-2 a_{1}\left(\frac{1}{a_{2}}\right)^{2} y_{n}^{2},  \tag{16}\\
& y_{n+1}=\alpha\left(\frac{1}{a_{1}}\right)\left(x_{n}-b_{1}\right) y_{n} \tag{17}
\end{align*}
$$

with a system parameter $\alpha, 0<\alpha \leq 2.0$, where the 2-D chaotic map (16) and (17) has chaos function solutions at $\alpha=2.0$. Here, it is important to note that the time step $\Delta t, 0<\Delta t \ll 1.0$ is not included in the 2-D map (16) and (17), and the solutions $x_{n}\left(t_{i}\right)$ and $y_{n}\left(t_{i}\right)$ are restricted by the condition (15) for the generation of discrete limit cycles [34].
Bifurcation diagrams for the 2-D map (16) and (17) are firstly illustrated in Fig. 1 (a) and (b), where we carry out 200 iterations of the 2-D map at each value of $\alpha, \Delta \alpha=0.01$ with an initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$ for Fig. 1 (a) and $\left(x_{0}, y_{0}\right)=(0.2,-0.1)$ for Fig. 1 (b), respectively. Here, the bifurcation diagram depends on the initial point $\left(x_{0}, y_{0}\right)$ for iterations at each value of $\alpha$. Then, bifurcation diagrams of Fig. 1 (a) and (b) are presented in Fig. 1 (c), and it is found that the diagrams $y_{n 1}$ and $y_{n 2}$ are symmetric with respect to the $\alpha$-axis on the $\alpha-x_{n}, y_{n 1}, y_{n 2}$ plane. Thus, the bifurcation diagrams are enlarged for the interval $1.4 \leq \alpha \leq 1.6$ with $\Delta \alpha=0.005$, and are illustrated in Fig. 1 (d) to show the pitchfork bifurcation [37]. In addition, the 2-D map (16) and (17) has fixed points, which are defined by $x_{n}^{*}=F\left(x_{n}^{*}, y_{n}^{*}\right)$ and $y_{n}^{*}=G\left(x_{n}^{*}, y_{n}^{*} ; \boldsymbol{\alpha}\right)$, and are given by

$$
\left(x_{n}^{*}, y_{n}^{*}\right)=\left\{\begin{array}{l}
\left(a_{1}+b_{1}, 0\right), 0<\alpha \leq 2.0  \tag{18}\\
\left(\frac{a_{1}}{\alpha}+b_{1}, \pm a_{2} \sqrt{\frac{\alpha-1}{2 \alpha}}\right), 1.0<\alpha \leq 2.0
\end{array}\right.
$$

where the 2-D map is found to have pitchfork bifurcations with three fixed points in the interval $1.0 \leq \alpha \leq 2.0$. Thus, the numerical result at $a_{1}=1.0, a_{2}=1.0$ and $b_{1}=0.0$ of the 2-D map (16) and (17) is presented in Fig. 1 (a)-(d). The MATLAB program for Fig. 1 (c) is shown in Appendix-A1.
On the other hand, the 2-D chaotic map (16) and (17) corresponding the 2-D Turing map (13) and (14) is derived from the chaos function solutions (7) and (8), and as the numerical result, the chaotic solutions $x_{n}\left(t_{i}\right)$ and $y_{n}\left(t_{i}\right)$,
(a) The $\alpha-x_{\mathrm{n}}, y_{\mathrm{n} 1}, y_{\mathrm{n} 2}$ plane with an initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$
(b) The $\alpha-x_{\mathrm{n}}, y_{\mathrm{n} 1}, y_{\mathrm{n} 2}$ plane with an initial point $\left(x_{0}, y_{0}\right)=(0.2,-0.1)$

(c) The $\alpha-x_{\mathrm{n}}, y_{\mathrm{n} 1}, y_{\mathrm{n} 2}$ plane with an initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$ or (0.2, -0.1)


(d) Zoomed on the $\alpha-x_{\mathrm{n}}, y_{\mathrm{n} 1}, y_{\mathrm{n} 2}$ plane with an initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$ or (0.2, -0.1)


Fig. 1. Bifurcation diagrams:
(a) An initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$,
(b) $\left(x_{0}, y_{0}\right)=(0.2,-0.1)$ and $\triangle \alpha=0.01$,
(c) An initial point $\left(x_{0}, y_{0}\right)=(0.2,0.1)$ or $(0.2,-0.1)$, and
(d) A zooming diagram of (c) and $\triangle \alpha=0.005$,
at $a_{1}=1.0, a_{2}=1.0$ and $b_{1}=0.0$ for the 2-D map (16) and (17).
orbit solutions on the $x_{n}-y_{n}$ plane and a ring of sequential points are illustrated in Fig. 2 (a)-(d), on the basis of the solutions (7) and (8), at $a_{1}=1.0, a_{2}=1.0$ and $b_{1}=0.0$ with an initial point $\left(x_{0}, y_{0}\right)=(1.0,0.0)$. The chaotic time series shown in Fig. 2 (a)-(d) are calculated without the accumulation of round-off error caused by numerical iterations of nonlinear equations [30]. Then, the numerical result of the 2-D map (16) and (17) is presented in Fig. 3 (a)-(d): (a) Symmetric orbit solutions on the $x_{n}-y_{n}$ plane, (b) A ring of sequential points at $\alpha=2.0$ with initial points $\quad\left(x_{0}, y_{0}\right)=(1.0, \pm 0.000001)$, (c) Symmetric orbit solutions and (d) Symmetric limit cycles in pairs at $\alpha=1.69$ with inside initial points $\left(x_{0}, y_{0}\right)=(0.5, \pm 0.5)$, as a numerical example of 'dappled' pattern shown for the Turing model in [24].
Moreover, we find symmetric limit cycles in pairs with inside initial points $\left(x_{0}, y_{0}\right)=(0.5, \pm 0.5)$ for the 2-D map (16) and (17) as shown in Fig. 4 (a)-(d). Then, it is found that inside initial points (blue) converge to the limit cycles and form left-handed and right-handed eddies of cells, and one of outside initial points (red) converges to the opposite limit cycle, that is, the red initial points generate convections, as shown for $\alpha=1.60$ and $\alpha=1.55$ in Fig. 4 (a) and (c), respectively. The MATLAB program for Fig. 4 (a) and (b) is presented in Appendix-A2.
Here, it is interesting to emphasize that the 2-D logistic maps derived from chaos function solutions $x_{n}\left(t_{i}\right)=\sin ^{2}\left(2^{n} t_{i}\right)$ and $y_{n}\left(t_{i}\right)=\cos \left(2^{n} t_{i}\right)$ have discrete limit cycles, corresponding to the FHN model, the F-KPP equation, the BZ reaction and the reaction-diffusion systems, as presented in [30-33], and the 2-D chaotic map (16) and (17) derived as a 2-D Turing map (13) and (14) has chaos function solutions $x_{n}\left(t_{i}\right)=\cos \left(2^{n} t_{i}\right)$ and $y_{n}\left(t_{i}\right)=\sin \left(2^{n} t_{i}\right)$, which corresponds to the Lorenz system for atmospheric convection [28], the reaction-diffusion systems for fluid convections $[16,17]$ and the equation of motion derived from the Hénon-Heiles Hamiltonian for the third integral with chaotic properties [38].

## Conclusions

The 3-D solvable chaos map corresponding to the Turing model was firstly derived in Section 2, and it is explained that the Turing model has nonlinear dynamics, such as spherical symmetry, ring of cells and numerical patterns on the $x-y$ plane. In Section 3, the 2-D chaotic map for the 2-D Turing model is proposed in order to show the nonlinear dynamics, and the 2-D map (16) and (17) is shown to have symmetric bifurcation diagrams, the ring of cells and limit cycles in pairs with different patterns. From the numerical result, the limit cycles are presented to generate discrete eddies and convections, as chaotic dynamics of non-equilibrium open systems. Therefore, nonlinear dynamics of the Turing model may correspond essentially to fluid dynamics with chaotic properties of the Lorenz system for atmospheric convection and the reactiondiffusion systems for convection cells.
(a) Chaotic solution $x_{\mathrm{n}}\left(t_{\mathrm{i}}\right)$

(b) Chaotic solution $y_{\mathrm{n}}\left(t_{\mathrm{i}}\right)$

(c) Orbit solutions $\left(x_{\mathrm{n}}\left(t_{\mathrm{i}}\right), y_{\mathrm{n}}\left(t_{\mathrm{i}}\right)\right)$ on the $x_{\mathrm{n}}-y_{\mathrm{n}}$ plane
(d) A ring of sequential points on the $x_{\mathrm{n}}-y_{\mathrm{n}}$ plane



Fig. 2. Chaotic time series:
(a) Chaotic solution $x_{\mathrm{n}}\left(t_{\mathrm{i}}\right)$ (7),
(b) Chaotic solution $y_{\mathrm{n}}\left(t_{\mathrm{i}}\right)(8)$,
(c) Orbit solutions $\left(x_{\mathrm{n}}\left(t_{\mathrm{i}}\right), y_{\mathrm{n}}\left(t_{\mathrm{i}}\right)\right)$ on the $x_{\mathrm{n}}-y_{\mathrm{n}}$ plane,
(d) A ring of sequential points on the $x_{\mathrm{n}}-y_{\mathrm{n}}$ plane,
at $a_{1}=1.0, a_{2}=1.0$ and $b_{1}=0.0$ with an initial point $\left(x_{0}, y_{0}\right)=(1.0,0.0)$.


Fig. 3. Symmetric orbits and limit cycles of the 2-D map (16) and (17):
(a) Symmetric orbit solutions and
(b) Sequential points at $\alpha=2.0$ with $\left(x_{0}, y_{0}\right)=(1.0, \pm 0.000001)$,
(c) Symmetric orbit solutions and
(d) Symmetric limit cycles at $\alpha=1.69$ with $\left(x_{0}, y_{0}\right)=(0.5, \pm 0.5)$,
at $a_{1}=1.0, a_{2}=1.0$ and $b_{1}=0.0$.


Fig. 4. Limit cycles in pairs of the 2-D map (16) and (17):
(a) Orbit solutions with left-handed and right-handed eddies and convections,
(b) Sequential points at $\alpha=1.60$,
(c) Orbit solutions with eddies and convections,
(d) Sequential points at $\alpha=1.55$,
with inside (blue) and outside (red) initial points.

## Appendix

## A1

\% MATLAB program for Fig. 1 (c) by S. Kawamoto
\% initial conditions
ALFA=zeros(1, 400);
$\mathrm{X} 1=\mathrm{zeros}(1,200)$;
$\mathrm{Y} 1=$ zeros $(1,200)$;
XX1=zeros(1, 200);
$\mathrm{YY} 1=\mathrm{zeros}(1,200)$;
$\mathrm{X} 2=z \operatorname{eros}(1,200)$;
$\mathrm{Y} 2=z \operatorname{eros}(1,200)$;
XX2=zeros $(1,200)$;
YY2=zeros(1, 200);
$\mathrm{X} 10=0.2$;
Y10=0.1;
$\mathrm{X} 20=0.2$;
Y20=-0.1;
$\mathrm{A} 1=1.0$;
$\mathrm{A} 2=1.0$;
$\mathrm{B} 1=0.0$;
\% system parameter ALFA and bifurcation diagrams figure('Position',[100 100350 200])
for $\mathrm{I}=1$ : 400
ALFA(I) $=0.005^{*}$ I;
for $\mathrm{J}=1$
$\mathrm{X} 1(\mathrm{I}, \mathrm{J})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left((\mathrm{Y} 10)^{\wedge} 2\right) / \mathrm{A} 2^{\wedge} 2 ;$
$\mathrm{Y} 1(\mathrm{I}, \mathrm{J})=\mathrm{ALFA}(\mathrm{I}) *(\mathrm{X} 10-\mathrm{B} 1) * \mathrm{Y} 10 / \mathrm{A} 1$;
$\mathrm{X} 2(\mathrm{I}, \mathrm{J})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left((\mathrm{Y} 20)^{\wedge} 2\right) / \mathrm{A} 2^{\wedge} 2$;
$\mathrm{Y} 2(\mathrm{I}, \mathrm{J})=\mathrm{ALFA}(\mathrm{I}) *(\mathrm{X} 20-\mathrm{B} 1) *$ Y20/A1;
end
for J=2:200
$\mathrm{X} 1(\mathrm{I}, \mathrm{J})=(\mathrm{A} 1+\mathrm{B} 1)-2 * \mathrm{~A} 1^{*}\left((\mathrm{Y} 1(\mathrm{I}, \mathrm{J}-1))^{\wedge} 2\right) / \mathrm{A} 2^{\wedge} 2 ;$
$\mathrm{Y} 1(\mathrm{I}, \mathrm{J})=\mathrm{ALFA}(\mathrm{I}) *(\mathrm{X} 1(\mathrm{I}, \mathrm{J}-1)-\mathrm{B} 1) * \mathrm{Y} 1(\mathrm{I}, \mathrm{J}-1) / \mathrm{A} 1$; $\mathrm{X} 2(\mathrm{I}, \mathrm{J})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 2^{*}\left((\mathrm{Y} 2(\mathrm{I}, \mathrm{J}-1))^{\wedge} 2\right) / \mathrm{A} 2^{\wedge} 2$; $\mathrm{Y} 2(\mathrm{I}, \mathrm{J})=\mathrm{ALFA}(\mathrm{I}) *(\mathrm{X} 2(\mathrm{I}, \mathrm{J}-1)-\mathrm{B} 1) * \mathrm{Y} 2(\mathrm{I}, \mathrm{J}-1) / \mathrm{A} 1$;
end
for $\mathrm{J}=150$ :200
XX1(J)=X1(I, J);
YY1(J)=Y1(I, J); XX2(J)=X2(I, J); YY2(J)=Y2(I, J);
plot(ALFA(I), XX1(J), 'k.','MarkerFaceColor','k','MakerSize',4); hold on plot(ALFA(I), YY1(J), 'b.','MarkerFaceColor','b','MakerSize',4); hold on plot(ALFA(I), YY2(J), 'r.','MarkerFaceColor','r','MakerSize',4); hold on end
end
xlabel('Alfa'); ylabel('Xn, Yn1, Yn2')

## A2

\% MATLAB program for Fig. 4 (a)-(b) by S. Kawamoto
\% initial conditions
$\mathrm{T}=\mathrm{zeros}(1,200)$;
TT=zeros(1, 200);
$\mathrm{X} 1=z \operatorname{eros}(200,200)$;
Y $1=z e r o s(200,200)$;
XX1=zeros(1, 200);
YY1=zeros(1, 200);
X2=zeros(200, 200);
$\mathrm{Y} 2=$ zeros $(200,200)$;
XX2=zeros(1, 200);
YY2=zeros(1, 200);
X3=zeros(200, 200);
$\mathrm{Y} 3=z \operatorname{eros}(200,200)$;
XX3=zeros(1, 200);
YY3=zeros(1, 200);
$\mathrm{X} 4=z e r o s(200,200)$;
Y4=zeros(200, 200);
XX4=zeros(1, 200);
YY4=zeros(1, 200);
L0=1;
$\mathrm{PR}=431$;
$\mathrm{T} 0=0.0$;
$\mathrm{X} 01=0.5$;
Y01 $=0.5$;
$\mathrm{X} 02=0.8$;
$\mathrm{Y} 02=0.65$;
$\mathrm{X} 03=0.8$;
Y03 $=0.55$;
$\mathrm{X} 04=0.5$;
$\mathrm{Y} 04=-0.5$;
ALFA=1.6;
$\mathrm{A} 1=1.0$;
A2 $=1.0$;
$\mathrm{B} 1=0.0$;
\% limit cycles in pairs with initial points for $\mathrm{I}=1: 200, \mathrm{~T}(\mathrm{I})=\mathrm{T} 0+\mathrm{I} * \mathrm{~L} 0 * \mathrm{pi} / \mathrm{PR}$; end for $\mathrm{I}=1: 200$
for $\mathrm{N}=1$
$\mathrm{X} 1(\mathrm{I}, \mathrm{N})=(\mathrm{A} 1+\mathrm{B} 1)-2 * \mathrm{~A} 1 *\left(\mathrm{Y} 01^{\wedge} 2\right) /\left(\mathrm{A} 2^{\wedge} 2\right)$;
Y1(I, N)=ALFA*(X01-B1)*Y01/A1;
$\mathrm{X} 2(\mathrm{I}, \mathrm{N})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left(\mathrm{Y} 02^{\wedge} 2\right) /\left(\mathrm{A} 2^{\wedge} 2\right)$;
$\mathrm{Y} 2(\mathrm{I}, \mathrm{N})=A L F A *(X 02-\mathrm{B} 1) * \mathrm{Y} 02 / \mathrm{A} 1$;
$\mathrm{X} 3(\mathrm{I}, \mathrm{N})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left(\mathrm{Y} 03^{\wedge} 2\right) /\left(\mathrm{A} 2^{\wedge} 2\right)$;
Y3(I, N) $=A L F A *(X 03-B 1) * Y 03 / A 1$;
$\mathrm{X} 4(\mathrm{I}, \mathrm{N})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left(\mathrm{Y} 04^{\wedge} 2\right) /\left(\mathrm{A} 2^{\wedge} 2\right)$;
$\mathrm{Y} 4(\mathrm{I}, \mathrm{N})=\mathrm{ALFA} *(\mathrm{X} 04-\mathrm{B} 1) * \mathrm{Y} 04 / \mathrm{A} 1$;
end
for $\mathrm{N}=2$ : I
$\mathrm{X} 1(\mathrm{I}, \mathrm{N})=(\mathrm{A} 1+\mathrm{B} 1)-2^{*} \mathrm{~A} 1^{*}\left(\mathrm{Y} 1(\mathrm{I}, \mathrm{N}-1)^{\wedge} 2\right) /\left(\mathrm{A} 2^{\wedge} 2\right)$;

```
            Y1(I, N)=ALFA*(X1(I, N-1)-B1)*Y1(I, N-1)/A1;
            X2(I,N)=(A1+B1)-2*A1*(Y2(I, N-1)^2)/(A2^2);
            Y2(I, N)=ALFA*(X2(I, N-1)-B1)*Y2(I, N-1)/A1;
            X3(I,N)=(A1+B1)-2*A1*(Y3(I, N-1)^2)/(A2^2);
            Y3(I, N)=ALFA*(X3(I, N-1)-B1)*Y3(I, N-1)/A1;
            X4(I,N)=(A1+B1)-2*A1*(Y4(I, N-1)^2)/(A2^2);
            Y4(I,N)=ALFA*(X4(I, N-1)-B1)*Y4(I, N-1)/A1;
    end
end
for I=1
    TT(I)=T0;
end
for I=2:200
    TT(I)=T(I-1);
end
for I=1
    XX1(I)=X01;
    YY1(I)=Y01;
    XX2(I)=X02;
    YY2(I)=Y02;
    XX3(I)=X03;
    YY3(I)=Y03;
    XX4(I)=X04;
    YY4(I)=Y04;
end
for I=2:200
    XX1(I)=X1(I-1, I-1);
    YY1(I)=Y1(I-1, I-1);
    XX2(I)=X2(I-1, I-1);
    YY2(I)=Y2(I-1, I-1);
    XX3(I)=X3(I-1, I-1);
    YY3(I)=Y3(I-1, I-1);
    XX4(I)=X4(I-1, I-1);
    YY4(I)=Y4(I-1,I-1);
end
% figures (a)-(b)
figure('Position', [100 100 350 350])
plot(XX4, YY4, '-b.','MarkerFaceColor','b','MarkerSize', 7); hold on
plot(XX3, YY3, '-r.','MarkerFaceColor','r','MarkerSize', 7); hold on
plot(XX2, YY2, '-r.','MarkerFaceColor','r','MarkerSize', 7); hold on
plot(XX1, YY1, '-b.','MarkerFaceColor','b','MarkerSize', 7); hold off
xlabel('xn(ti)'); ylabel('yn(ti)')
figure('Position', [100 100350 350])
plot(XX4, YY4, 'b.','MarkerFaceColor','b','MarkerSize', 7); hold on plot(XX3, YY3, 'r.','MarkerFaceColor','r','MarkerSize', 7); hold on plot(XX2, YY2, 'r.','MarkerFaceColor','r','MarkerSize', 7); hold on plot(XX1, YY1, 'b.','MarkerFaceColor','b','MarkerSize', 7); hold off xlabel('xn(ti)'); ylabel('yn(ti)')
```


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# Optimization Approaches when Calculating the "Massif - Innovative Fastening Parameters" Spatial System 

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#### Abstract

Efficiency and safety are the main components of the requirements to the solid fossil fuels production industry. Currently, the combined use of alternative energy sources and innovative resource-saving technologies is relevant. Optimization approaches ensuring the stability of the "massif - innovative fastening parameters" spatial system elements are studied. The principles and methods of resource-saving increase in stability on the basis of minimizing the intensity of the rock pressure manifestation by adjustable operating modes of innovative fastening systems have been scientifically substantiated. The resulting developments make it possible to achieve the technological processes intensification during the minerals extraction and to develop a method for calculating a function that describes the rational deformation-strength characteristic depending on mining-and-geological conditions.


Keywords: Optimization approaches, Stability, Resource saving, Minimization, Spatial system.

## 1 Introduction

The use of a mathematical experiment in the geomechanics problems makes possible to avoid unnecessary labour-intensive full-scale experiments, the setting of which not only requires significant financial costs, but also leads to significant losses in production time (Krukovskyi et al.[1]; Małkowski et al.[2]; Bondarenko et al.[3]). In addition, a computational experiment, in contrast to full-scale experimental setups, gives an ability to accumulate the results obtained during the study of a certain range of problems, and then quickly and flexibly apply them for solving similar problems (Kovalevska et al.[4]; Bondarenko et al.[5], [6]; Majcherczyk et al.[7]).

To solve the problems of geomechanics, the following information should be obtained:

- texture with account of the existing geological disturbances and physical and mechanical properties of the studied stratified or homogeneous rock mass;
- types and values of force impacts applied to certain areas of the rock mass and mining facilities;
- type of problems to be studied numerically: distribution of stresses, deformations, displacements and weakening (of varying degree) of certain areas of the rock mass, underground structures, and the like;
- geometric, mechanical and force parameters of the studied mining facilities.

Based on the collected data, a computational scheme is formed, the type of which determines the choice of a method for solving a specific geomechanics problem. Thus, a system of mathematical equations is developed that expresses the ratio of the specified and sought values, which must be solved until the final values are obtained.

When solving the geomechanics problems, one constantly has to face the problem of calculating the systems with a complex geometric configuration and an irregular physical structure. The rock mass and its constituent rocks have a large number of characteristics, which can be taken into account in mathematical modeling only when using finite-difference calculation schemes (Kovalevs'ka et al.[8]).

A review of existing research provides strong evidence that the use of the finite element method (FEM) to solve the problems of geomechanics is becoming more widely used (Maghous et al.[9]; Kovalevska et al.[10]):

- the methodology for performing calculations using the FEM is constantly being improved and complicated;
- solving a volumetric problem in an elastic-plastic formulation has already become the norm in geomechanics;
- in many cases, researchers take into account a large number of factors and their combinations, which have characteristics of local disturbances of the system;
- the development of mathematical models is aimed at finding an obtainable accuracy of calculation, which is performed according to classical methods, but taking into account the maximum possible number of significantly influencing parameters.

Geomechanical systems, studied by mankind for several centuries, are characterized by a complex and multi-factor structure, which, from a dialectical point of view, cannot be "ideally" modelled once and for all. New knowledge about the properties and behaviour of the rock mass is constantly emerging, and the development of industry, including mining sector, extends the variety of practical tasks.

The factors and stages of research using the FEM method have acquired a certain refinement on the example of modeling the behaviour of the geomechanical system "rock mass - mine working suppor", where the procedure for conducting a computational experiment is consistently being substantiated.

## 2 Search algorithm for the rational modes of interaction and optimization

The work (Kovalevska et al.[11]) substantiates the main provisions of the process for loading the fastening systems of extraction mine workings in the zone of stope operations influence and gives schematic representations of the mutual influence of the deformation-strength characteristic of the lowering roof rocks (with disturbances in their structure) and fastening structures with different modes of resistance. This paper studies the changeover from the general qualitative pattern of such interaction to quantitative assessments of the parameters with the final objective of determining the rational deformation-strength characteristic of the fastening system as a whole and its main constituent elements, depending on the geomechanical factors of the mine working maintenance.

The search algorithm for rational parameters of the fastening systems is as follows. The basis for optimizing the operating mode of the fastening system is the deformation-strength characteristics of the weakening rock mass; it is necessary to determine the deformation-strength characteristic $P(u)$ of the applied fastening system. This is a complex task of calculating the structure as a whole and each main fastening element with regard to their force interaction with each other. Based on such a strength calculation, by varying the fastening system parameters, we select the deformation-strength characteristic $P(u)$ of the fastening system as a whole so that it corresponds to the rational function determined at the previous stage.

When selecting a rational deformation-strength characteristic $P(u)$, the equal strength condition should be taken into account, the fulfilment of which imposes its own restrictions on the variation range and discrete values of a number of the fastening system parameters.

Numerous analytical studies (Li[12]; Skipochka et al.[13]; Kaiser and Cai[14]; Bondarenko et al.[15]) indicate a relatively weak influence of the fastening system reaction on the restricted rock contour displacements in the drift. A review of these works shows the level of influence of the order of 3 $15 \%$ with various combinations of geomechanical factors and the value of the support repulse. An approximately equivalent degree of influence is also noted in studies of the stress-strain state (SSS) of geomechanical systems using the FEM method. Thus, the deformation-strength characteristic $q_{1}(u)$ of the weakening mass is exposed to the restricted influence of the deformationstrength characteristic $P(u)$ of the fastening system. The peculiarity is that the reaction is very sensitive to the value of the yielding property of the fastening system and, with its relatively small variation, the value $P$ can change up to several times. This fact is described in sufficient detail on previously developed schemes (Bondarenko et al.[15]). Consequently, with a relatively small transformation of the SSS of the weakening mass with a change in the reaction of the fastening system, a very significant transformation of the fastening
system SSS is observed (with varying the value of yielding property), which provides an effective tool for optimizing the fastening system and its deformation-strength characteristic.

The second peculiarity relates to the deformation-strength characteristic $q_{2}(u)$ of the rocks in the dome of natural equilibrium. Here, the experience of solving elastic-plastic problems using the FEM method also indicates the restricted influence of the fastening system reaction on the change in the SSS of the adjacent rock mass in the areas where the formation of the dome of natural equilibrium is predicted (Salcher and Bertuzzi[16]; Malkowski et al.[17]). In our opinion, this is conditioned by the use in the computational experiment of the model of a coupled medium, which does not adequately describe the state of weakening and loosening rocks inside the dome, losing stability and creating a load on the fastening system with their weight. It is advisable to model the behaviour of this rock volume by the full diagram of their deformation, including the superlimiting stages of the state (Skipochka et al.[13]; Brodny[18]).

Summarizing the methodological aspects, the search algorithm for the fastening system rational parameters includes the following positions:

- interaction of the deformation-strength characteristics of the weakening rock mass and the fastening system is studied using the FEM method in an elastic-plastic formulation;
- the search for a rational equilibrium state is based on a joint consideration of the deformation-strength characteristic of the weakening mass $q_{1}(u)$, determined by the FEM method, and the deformation-strength characteristic of rocks in the dome of natural equilibrium $q_{2}(u)$, determined by the normative methodology (Instruction[19]; SNiP II-94-80[20]);
- optimization of the deformation-strength characteristic of the fastening system is performed on the basis of FEM calculations taking into account the function $q_{2}(u)$;

The parameters of fastening system elements, under the condition of their equal strength, are optimized based on the study of their SSS by the FEM method.

## 3 Methodology for minimizing the load on the support

In accordance with the search algorithm for rational operating modes of the fastening system, the determination of a critically important point characterizing the choice of such a yielding property value $u_{A}$ of the support at which the load $P_{A}$ on it decreases to the minimum possible value in the given mining-andgeological conditions of maintaining the extraction mine working is specified. To solve this problem, a methodology has been developed to minimize the load on the fastening system.

It has been proved earlier (Kovalevska et al.[21]) that the criterion for minimizing the load is the achievement of the condition of loads equality by the factor of displacement of the adjacent mass being weakened $q_{1}$ and by the factor of formation of the dome of natural equilibrium $q_{2}$.

To increase the adequacy of the geomechanical model SSS calculations, they are performed in an elastic-plastic formulation using a bilinear deformation diagram of both rocks and fastening materials. This makes it possible to take into account the occurrence of the limiting state in arbitrary areas of the model, which is accompanied by plastic flow of steel fastening elements and quasiplastic deformation of rocks, provided that their volume is constant.

An algorithm has been developed for determining the deformation-strength characteristic $q_{1}(u)$ of a weakening rock mass, depending on the main influencing geomechanical factors.

The second component of the optimization scheme is the deformationstrength characteristic $q_{2}(u)$ of rocks in the dome of natural equilibrium.

When performing mathematical transformations, a linear function of the deformation-strength characteristic $q_{2}(u)$ of rocks in the dome of natural equilibrium has been determined:

$$
\begin{equation*}
q_{2}(u)=K_{d} B \gamma \frac{\left(1-\alpha_{1}\right)^{2}}{0.15+0.03 \alpha_{2}-0.18 \alpha_{1}} u \tag{1}
\end{equation*}
$$

where $K_{d}$ is coefficient of dynamics, which takes into account possible conventionally instantaneous displacements of the rock mass around the extraction mine working and is determined according to recommendations (Instruction[19]); $B$ is mine working width when driving; $\gamma$ is weight-average unit specific gravity of rocks in the dome of natural equilibrium; $\alpha_{1}$ and $\alpha_{2}$ are parameters that set the ratio between the lowering of the mine working roof in the areas: outside the zone of stope works influence; in the zone of front bearing pressure of the approaching longwall face; behind the stope face in the zone of stabilization of rock pressure manifestations.

The parameters $\alpha_{1}$ and $\alpha_{2}$ are obtained on the basis of calculated expressions (Instruction[19]) by transforming them for the problem to be solved for determining the function $q_{2}(u)$ :

$$
\begin{align*}
& \alpha_{1}=\frac{1.5 R_{1}^{b}\left(R_{3}^{r}+R_{3}^{b}\right)}{(3.0+2 m) R_{3}^{b}\left(R_{1}^{r}+R_{1}^{b}\right)}  \tag{2}\\
& \alpha_{2}=\frac{3.9 R_{2}^{b}\left(R_{3}^{r}+R_{3}^{b}\right)}{(3.9+2 m) R_{3}^{b}\left(R_{2}^{r}+R_{2}^{b}\right)}, \tag{3}
\end{align*}
$$

where $m$ is coal seam extraction thickness; $R_{1,2,3}^{r, b}$ is calculated values of the compressive resistance of the roof and bottom rocks of the coal seam in the corresponding areas: 1 is outside the zone of stope works influence; 2 is in the zone of front bearing pressure; 3 is behind the longwall face.

When determining the deformation-strength characteristic $q_{2}(u)$ of the rocks in the dome of natural equilibrium, the basic methodological provisions of the normative geomechanical phenomenon, such as limiting the dimensions of the dome due to the fastening system reaction, have been adopted (Krukovskyi et al.[1]; Małkowski et al.[2]). The theoretical essence of this phenomenon is substantiated in the work (Skipochka et al.[13]), where it has been proved that through the so-called "small impacts" (the level of the support reaction in comparison with the acting stresses) it is possible to restore rock volumes from an unstable state to a stable state; then these volumes are excluded from the process of forming the load on the fastening system of the extraction mine working.

In practical terms, to determine the limitation degree of load and displacements of the rock contour, the methodology and results of research (Bondarenko et al.[22]) on the optimization of interaction modes between the support of mine workings and the rock mass are used. A set of calculations has been performed according to existing methodologies and a database has been obtained, which is summarized in Table 1.

Table 1. Values of the coefficient $K_{r}$ of the fastening system reaction influence, \%

| Weight-average <br> compressive resistance <br> of rocks in the dome <br> $R_{\text {dome }}, \mathrm{MPa}$ | Fastening system reaction $P, \mathrm{kPa}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 50 | 100 | 150 | 200 | 250 | 300 |
| 5 | 4.2 | 9.6 | 15.6 | 22.1 | 28.8 | 35.9 |
| 10 | 3.0 | 6.8 | 11.1 | 15.6 | 20.4 | 25.4 |
| 15 | 2.5 | 5.6 | 9.1 | 12.9 | 16.8 | 20.9 |
| 20 | 2.1 | 4.8 | 7.8 | 11.0 | 14.4 | 18.0 |
| 30 | 1.7 | 3.9 | 6.4 | 9.0 | 11.7 | 14.6 |
| 40 | 1.5 | 3.4 | 5.5 | 7.8 | 10.2 | 12.7 |

Taking into account the influence of the fastening system reaction $P$, the expression for calculating the deformation-strength characteristic of the rocks in the dome of natural equilibrium is transformed as follows

$$
\begin{equation*}
q_{2}(u)=K_{d} B \gamma \frac{\left(1-\alpha_{1}\right)^{2}\left(1-\frac{K_{p}}{100}\right)}{0.15+0.03 \alpha_{2}-0.18 \alpha_{1}} u, \tag{4}
\end{equation*}
$$

where $K_{r}$ is coefficient of influence of the fastening system reaction on the limitation of lowering roof rocks in the extraction mine working, \%; it is determined by Table 1 .

As a result, a methodology has been developed for determining the deformation-strength characteristic of a weakening rock mass $q_{1}(u)$ and rocks in the dome of natural equilibrium $q_{2}(u)$. They are key positions when optimizing the operating modes of the fastening system in the extraction mine workings.

## 4 Determining the patterns of the fastening system optimal parameters in view of geomechanical factors

To develop the technology of searching for the minimum load $q_{A}$ on the fastening system with appropriate $u_{A}$, the preliminary (test) calculations of functions $q_{1,2}(u)$ have been made in the areas of boundary values of geomechanical parameters characterizing favourable and difficult mining-andgeological conditions for maintaining extraction mine workings, which is shown in Fig. 1. Here, lines I and II represent the calculation results of the deformationstrength characteristic $q_{1}(u)$ of the weakening rock mass, performed with the use of multivariate computational experiments; therefore, the dependences are shown in the form of multilinear graphs, each fracture of which corresponds to one computational experiment with a specific thickness of the artificial yielding layer. The calculations are performed for the minimum (line I) and maximum (line II) values of the index $H / R$, which approach to the boundaries of the studied range of the specified mining-and-geological conditions; therefore, the area enclosed between lines I and II gives a fairly complete understanding of the family of functions $q_{1}(u)$.

The deformation-strength characteristic of the rocks in the dome of natural equilibrium is a "smooth" function $q_{2}(u)$, which is calculated by the formula (4). The variation range of $q_{2}(u)$ at a fixed value of the rock contour displacement $u$ is conditioned by mining-and-geological conditions of maintaining the extraction mine working, the standard size of its section and the degree of influence of the fastening system reaction on restricting the drift contour displacements. The presented variation range of the function $q_{2}(u)$ also represents as much as possible the list of probable situations for maintaining the extraction mine working.

Point $A$ is one of the key positions in the optimization scheme; it has been determined for favourable $\left(A_{b}\right)$ and difficult $\left(A_{c}\right)$ mining-and-geological conditions for maintaining the extraction mine workings.

As it can be seen, the deterioration of conditions for maintaining the extraction mine working leads to an increase in the minimum load by $51.8 \%$; in this case, the optimal value of the fastening system yielding property increases by 2 times.


Fig. 1. To the analysis of reliability and adequacy of the methodology for optimizing the modes of the fastening system interaction with the surrounding mass: $\mathrm{I}-H / R=9.6 \mathrm{~m} / \mathrm{MPa}$; II $-H / R=88.9 \mathrm{~m} / \mathrm{MPa}$; " $b$ " $-B=5.18 \mathrm{~m}$, $R_{\text {dome }}=30.2 \mathrm{MPa}$; " $c$ " $-B=4.50 \mathrm{~m}, R_{\text {dome }}=5.3 \mathrm{MPa}$; - - limits of the range of loads on yielding support according to normative documents

To assess the degree of reliability of the developed optimization methodology, the results are compared with normative methodologies. In a generalized sense, they take into consideration the yielding property of the support as a factor in reducing the load $P(u)$; Fig. 1 shows the variation range of the function $P(u)$ with dashed lines and shaded. The above-noted methodologies do not solve the problem of optimizing the modes of the support interaction with the surrounding mass, therefore, the load on the yielding support is significantly higher than the optimal values $P_{A_{b}}$ and $P_{A c}$, but to a certain extent corresponds to the studied examples of non-optimal operating modes of the support with increased yielding property. Thus, under favourable conditions, the optimal load $P_{A_{b}}$ is by $36.9-57.5$ lower; in difficult mining-and-geological conditions, this difference is $10.8-27.5 \%$. But, if to compare
the irrational increased yielding property of the fastening system, then the noted difference in the values of loads is reduced to $3.7-19.4 \%$ under favourable conditions and to $1.7-18.0 \%$ under difficult mining-and-geological conditions.

By assessing the degree of adequacy and reliability of the developed methodology for optimizing the support interaction modes with the surrounding rock mass, positive results have been obtained. In addition, the next stage of research has been substantiated, namely, the search for patterns in the relationship between coordinates $P_{A}$ and $u_{A}$ of the point $A$, depending on the index $H / R$ and structure of rocks in the coal-overlaying formation (Fig. 2). The essence of the revealed dependences is as follows.


Fig 2. Patterns of the relationship between the optimal parameters of reaction $P_{A}(-)$ and yielding property $u_{A}(---)$ of the fastening system depending on the index $H / R$ of mining-and-geological conditions and the type of the structure of coal-overlaying formation: 1 - group I; 2 - group II; 3 - group III

Firstly, there is a clear pattern of a decrease in the optimal load $P_{A}$ and yielding property $u_{A}$ of the fastening system with a decrease in the index $H / R$, regardless of the type of the rocks structure in the coal-overlaying formation.

Secondly, according to the results of a set of geomechanical calculations, it becomes obvious the relevance of the task for optimizing the interaction modes between the rock mass and the support in terms of a stable decrease in the required force parameters of the latter, regardless of the degree of the conditions complexity for maintaining the reused mine workings.

For convenient practical use of the determined patterns (Fig. 2), a system of regression equations has been obtained that set the ratio between the optimal parameters of the deformation-strength characteristic of the support with the geomechanical index $H / R$, as well as with the groups of generalized structures of the coal-bearing mass.

Group I

$$
\begin{equation*}
P_{A}=284(H / R)^{0.21}, \mathrm{kN} / \mathrm{m} ; \tag{5}
\end{equation*}
$$

$$
\begin{equation*}
u_{A}=321(H / R)^{0.21}, \mathrm{~mm} \tag{6}
\end{equation*}
$$

Group II

$$
\begin{equation*}
P_{A}=270(H / R)^{0.18}, \mathrm{kN} / \mathrm{m} \tag{7}
\end{equation*}
$$

Group III

$$
\begin{align*}
& P_{A}=260(H / R)^{0.15}, \mathrm{kN} / \mathrm{m}  \tag{9}\\
& u_{A}=104(H / R)^{0.38}, \mathrm{~mm} \tag{10}
\end{align*}
$$

$$
\begin{equation*}
u_{A}=172(H / R)^{0.32}, \mathrm{~mm} \tag{8}
\end{equation*}
$$

Correlation-dispersion analysis of the optimization data evidences a stable power-law relationship between the parameters $P_{A}$ and $u_{A}$ with the index $H / R$ regardless of the coal-bearing mass structure. Therefore, the obtained scientific result can be formulated as follows: the optimal parameters $P_{A}$ and $u_{A}$ of the fastening system deformation-strength characteristic of the reused extraction mine workings are in power-law relationship with the geomechanical index $H / R$, regardless of the type of the coal-bearing mass structure.

## 5 Substantiation and calculation of the rational deformationstrength characteristic of the fastening system

The performed optimization of the fastening system operating modes provides the condition $\left(P_{A}, u_{A}\right)$ of the equilibrium state of its interaction with the surrounding mass, which expresses only the final result in the form of the point $A$ coordinates on the line of the function $P(u)$ of the deformation-strength characteristic of support. It is meant here that the process of the rock mass interaction with the support (fastening system) develops in time and space, passing through many states with changing coordinates $P_{j}, u_{j}$, starting from the period of the fastening
system erection to the moment of the studied geomechanical process stabilization. In this regard, it is important to ensure the stability of mine working (using resource-saving methods) throughout the entire period of development of the support interaction with the rock mass. Consequently, the main task is to find the optimal function $P(u)$ of the deformation-strength characteristic of the fastening system, for which two boundary values are known: $P=0, u=0$ and $P=P_{A}$, $u=u_{A}$. To substantiate the principle of searching for a function $P(u)$, Fig. 3 shows its schematic representation.


Fig. 3. Scheme for calculating the rational deformation-strength characteristic of the fastening system: - deformation-strength characteristic of rocks in the dome of natural equilibrium $q_{2}(u)$; - - variants of the deformation-strength characteristic of fastening systems (lines "OP", 1, 2 and 3)

Provided that the support is sufficiently yielding (not less than the value $u_{A}$ ), the load $q_{1}$ becomes less than the load $q_{2}$. Then the determining factor is the deformation-strength characteristic of the rocks in the dome of natural equilibrium $q_{2}(u)$ : upon condition that:

$$
\begin{equation*}
P(u) \geq q_{2}(u) \tag{11}
\end{equation*}
$$

over the entire range $\left(0 \leq u \leq u_{A}\right)$ of development of the rock contour
displacements in the mine working, its stability is ensured.
Let us consider the most typical variants of condition (11) shown in Fig. 3. The optimal variant of the deformation-strength characteristic $P(u)$ of support is the fulfilment of the equality according to the condition (11); in this case, the support reaction is minimally sufficient.

Therefore, it is expedient to set for the optimal function $P(u)$ a certain margin factor $\left(K_{m}>1\right)$, which compensates for the possible negative factors effect (line 1 in Fig. 3):

- an increase in the required support reaction by the value of $\Delta P_{m}$ improves reliability of recommendations in the case of an unpredictable rock pressure increase;
- an increase in the required yielding property by the value of $\Delta u_{m}$ provides an "escape" from excessive load in the case of an unpredictable increase in the rock contour displacements of the mine working.

As can be seen from the scheme, the parameters $\Delta P_{m}$ and $\Delta u_{m}$ are interconnected and dependent on the value of the margin factor $K_{m}$. For miningengineering calculations, it is generally accepted to set an accuracy within 15 $20 \%$ to take into consideration the influence of various kinds of poorly predictable factors. Therefore, in the first approximation, $K_{m}=1.15-1.20$ can be taken, and the formula for calculating the rational deformation-strength characteristic of the support takes the form:

$$
\begin{equation*}
P(u)=K_{d} \cdot K_{m} B \gamma \frac{\left(1-\alpha_{1}\right)^{2}\left(1-\frac{K_{r}}{100}\right)}{0.15+0.003 \alpha_{2}-0.18 \alpha_{1}} u \tag{12}
\end{equation*}
$$

The function $P(u)$ that expresses the rational deformation-strength characteristic of the support is shown by line $l$ in the scheme (Fig. 3). As can be seen, it is located slightly above the optimal line "OP", but an "excess" in the stability margin of the support is relatively small and is determined by the shaded area. The margin value for the support reaction is:

$$
\begin{equation*}
\Delta P_{m}=\left(K_{m}-1\right) P_{A}, \tag{13}
\end{equation*}
$$

for yielding property of the support

$$
\begin{equation*}
\Delta u_{m}=\left(K_{m}-1\right) u_{A} \tag{14}
\end{equation*}
$$

Here, the optimal parameters $P_{A}$ and $u_{A}$ are calculated using the expressions (5) - (10).

Finally, the rational deformation-strength characteristic of the support (fastening system) is determined by the formula (12), its load-bearing capacity $P_{\text {max }}$ is calculated by the expression:

$$
\begin{equation*}
P_{\max }=K_{m} P_{A}, \tag{15}
\end{equation*}
$$

and the maximum yielding property $u_{\text {max }}-$ by the formula

$$
\begin{equation*}
u_{\max }=u_{A} \tag{16}
\end{equation*}
$$

with account of the equations (5) - (10).
There are other variants to select a rational deformation-strength characteristic of the support, the function of which is not similar to the function $q_{2}(u)$. Thus, the line 2 in Fig. 3 represents the well-known mode of the support constant resistance, which, not without reason, is considered by many experts as the most effective.

If the support reaction in the mode of constant resistance is equal to $P_{\max }$ (as shown in the scheme of Fig. 3), then such its deformation-strength characteristic is assigned to the group of rational ones, provided that the structural yielding property of the support is not less than the value $u_{\max }$ for a given mining-and-geological conditions of the mine working maintenance.

Another variant of the support deformation-strength characteristic is shown by line 3 and is quite widespread (Salcher and Bertuzzi[16]; Bondarenko et al.[22]) for various types of roof-bolt supports. Such a deformation-strength characteristic cannot be considered satisfactory, since at a certain point in time ( $u=u_{\text {str }}$ ) the fastening structure reaction becomes less than the optimal value $P_{A}$ and its rapid strengthening is required by the value of $P_{s t r}$ (see Fig. 3).

Summing up the performed research, it should be noted that a very accessible methodology has been developed for calculating the deformationstrength characteristic of the support fastening system, depending on the mining-and-geological conditions of the mine working maintenance.

## Conclusions

The search algorithm for rational modes of the fastening system interaction with the coal-bearing mass surrounding the extraction mine working has been substantiated; the algorithm involves performing a number of studies that are closely related to each other by the general parameters of the interaction process:

- formation of the minimum possible load in specific mining-and-geological conditions for the mine working maintenance;
- concordance between the deformation-strength characteristics of the elements included in the fastening system;
- optimization of design parameters of the fastening elements according to the criterion of their equal strength.

Methodological principles have been developed to minimize the load on the fastening system of the reused extraction mine workings, which are based on the use of a combination of studies by the FEM method and recommendations of normative documents for calculating the dimensions of the dome of natural equilibrium.

Based on the formulated principles, a methodology has been developed for obtaining the deformation-strength characteristic of a weakening mass, depending on the main influencing geomechanical factors.

The patterns of the geomechanical factors influence on the choice of the optimal parameters of the fastening system deformation-strength characteristic have been determined. A stable power-law relationship between the fastening system optimal parameters and the geomechanical index of the mining conditions, regardless of the coal-bearing mass structure, has been revealed; this makes possible to adopt a unified strategy of resource-saving improvement of the fastening systems in mine workings.

Based on the found optimal parameters of the operating modes of the fastening systems, a substantiation has been conducted and a methodology has been developed for calculating the function that describes its rational deformation-strength characteristic depending on the mining-and-geological conditions for the reused extraction mine workings maintenance. The methodology is distinguished by the simplicity and efficiency of the necessary calculations of rational parameters of the fastening system as a whole, for which the fastening elements constituting it are selected.

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# Nonideality of Parametric Systems as a Trigger of Chaos 

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#### Abstract

There are several characteristics common to all resonance vibrations occurring in systems with a circumferential coordinate (rings, cylindrical and spherical shells, etc.). The specifics of resonance phenomena make it possible mathematically to reduce the problem to the study of vibrations in distributed systems and examine an analogous problem with fewer dimensions. In the current theory of dynamical systems the coupling of can be ignored without altering the qualitative pattern of the phenomenon only when the coupling does not produce a bifurcative change in the stability of the process as a whole. After analyzing averaged equations for systems with a circumferential coordinate in cases where different types of resonance loads were applied, we were able to classify the resonances that occur and determine that chaos is possible only with forced resonance excited by loading without interaction with source of energy or it is possible at parametric resonance in the systems under nonideal excitation.


Keywords: Systems with a circumferential coordinate, Interaction with source of energy, Steady state regimes, Chaos.

## 1 Introduction

A spherical pendulum is the simplest example of an oscillator with two degrees of freedom of equal frequency. Many of the phenomena typical for the behavior of a spherical pendulum also show up in the dynamics of systems with distributed parameters with a periodic coordinate. Examples include rings, cylindrical and spherical shells, circular plates, and media inside cylindrical and spherical cavities, see in Kubenko et al.[1], Krasnopol'skaya and Shvets[2]. Therefore, knowledge of the properties of a spherical pendulum gives an understanding of oscillations in these other systems. In the present article, we classify different models that describe the vibrations of distributed systems and have few dimensions.

## 2 Parametric oscillations of a spherical pendulum with interaction with power source

In this part of the paper we consider the parametric oscillations of a kinematically driven spherical pendulum. In the case of an ideal (infinite power) driving mechanism the averaged equations describing parametric oscillations of a spherical pendulum have only regular solutions in the steady state as we would show in the paragraph 3 of this paper. The change in the type of dynamical system
(the pendulum) from deterministic (having only regular motion) to chaotic (capable of dynamical chaos) due to interaction with an energy source shows the importance of properly taking into account the interaction process, see Krasnopol'skaya and Shvets[3, 4, 5], Krasnopolskaya[6, 7], Balthazar et al.[8], Shvets[9]. This is particularly true for experimental studies of the properties of dynamical systems. Chaos in the system can be caused only by the finiteness of the power of the driving mechanism, and not by the properties of the system itself. Let's consider the system shown schematically in Fig. 1. A crank-shaft slide bar mechanism connects the rotor of a motor with the point of support of a physical pendulum. When the crank-shaft turns by an angle the slide bar together with the support (when the length of the connecting rod $b \gg a$ ) has a displacement $w(t)=-a \cos \psi$ along the vertical axis of a fixed coordinate system. In Cartesian coordinates $O x y z$ the kinetic energy can be written in the form Krasnopol'skaya and Shvets[4], Miles[10, 11] (in a fixed coordinate system)

$$
\begin{equation*}
T=0.5 I \dot{\psi}^{2}+0.5 m\left[\dot{x}^{2}+\dot{y}^{2}+(\dot{z}+\dot{w})^{2}\right], \tag{1}
\end{equation*}
$$

and the potential energy has the form

$$
\begin{equation*}
V=m g(l-z-w), \tag{2}
\end{equation*}
$$

where $x, y, z$ are the Cartesian coordinates of the center of mass of the pendulum; $I$ is the moment of inertia of the rotor of the electric motor; $m$ is the mass of the pendulum; $l$ is the reduced length of the pendulum; and $g$ is the acceleration of gravity. The masses of the slide bar and support are neglected.


Fig. 1. Schema of the system
Following Miles[10], we introduce the new variables $\alpha$ and $\beta$ defined as $x=l \sin \alpha, y=l \sin \beta$. Because for a pendulum we have the relation in a fixed coordinate system

$$
x^{2}+y^{2}+z^{2}=l^{2},
$$

it follows that $z=l \sqrt{1-\sin ^{2} \alpha-\sin ^{2} \beta}$. For small $\alpha$ and $\beta$ the Lagrangian of the system $L=T-V$ can be written in the form

$$
\begin{align*}
& L=0,5 I \dot{\psi}^{2}+0,5 m l^{2}\left[\dot{\alpha}^{2}+\dot{\beta}^{2}+\right. \\
& \left.+2 \alpha \beta \dot{\alpha} \dot{\beta}-2(\alpha \dot{\alpha}+\beta \dot{\beta}) \dot{\psi} \frac{a}{l} \sin \psi+\dot{\psi}^{2} \frac{a^{2}}{l^{2}} \sin ^{2} \psi\right]-  \tag{3}\\
& -g m l\left(\frac{\alpha^{2}}{2}-\frac{\alpha^{4}}{24}+\frac{\beta^{2}}{2}-\frac{\beta^{4}}{24}+\frac{\alpha^{2} \beta^{2}}{4}+\frac{a}{l} \cos \psi\right)
\end{align*}
$$

Lagrange's equations for the basic variables $\psi(t), \alpha(t)$, and $\beta(t)$ take the form

$$
\begin{align*}
& I \ddot{\psi}=H_{1}(\dot{\psi})-H_{2}(\dot{\psi})- \\
& -m l a\left[\ddot{\psi} \frac{a}{l} \sin ^{2} \psi+\dot{\psi} \frac{a}{l} \sin \psi \cos \psi+\frac{g}{l} \sin \psi-\right. \\
& \left.-\left(\dot{\alpha}^{2}+\dot{\beta}^{2}\right) \sin \psi-(\alpha \ddot{\alpha}+\beta \ddot{\alpha}) \sin \psi\right] ; \\
& \ddot{\alpha}+\omega_{0}^{2}\left(\alpha-\frac{\alpha^{3}}{6}+\frac{\alpha \beta^{2}}{2}\right)+\delta_{1} \dot{\alpha}+\alpha\left(\dot{\beta}^{2}+\beta \ddot{\beta}\right)-  \tag{4}\\
& -\frac{a}{l} \alpha\left(\dot{\psi}^{2} \cos \psi+\ddot{\psi} \sin \psi\right)=0 ; \\
& \ddot{\alpha}+\omega_{0}^{2}\left(\alpha-\frac{\alpha^{3}}{6}+\frac{\alpha \beta^{2}}{2}\right)+\delta_{1} \dot{\alpha}+\alpha\left(\dot{\beta}^{2}+\beta \ddot{\beta}\right)- \\
& -\frac{a}{l} \alpha\left(\dot{\psi}^{2} \cos \psi+\ddot{\psi} \sin \psi\right)=0 ; \\
& \ddot{\beta}+\omega_{0}^{2}\left(\beta-\frac{\beta^{3}}{6}+\frac{\alpha^{2} \beta}{2}\right)+\delta_{1} \dot{\beta}+\beta\left(\dot{\alpha}^{2}+\alpha \ddot{\alpha}\right)- \\
& -\frac{a}{l} \beta\left(\dot{\psi}^{2} \cos \psi+\ddot{\psi} \sin \psi\right)=0 .
\end{align*}
$$

Here $H_{1}(\dot{\psi})$ is the torque of the electric motor; $H_{2}(\dot{\psi})$ is the internal torque of resistance to the rotation of the rotor, see Sommerfeld[12], Kononenko[13], Ganiev and Krasnopolskaya[14]; $\omega_{0}=\sqrt{g / l}$ is the natural frequency of the pendulum; $\delta_{1}$ is the damping coefficient of the drag force of the medium in which the pendulum moves.
The above equations describe the complicated interaction between the rotation of the shaft of the motor (producing the driving force) and the spatial oscillations of the pendulum. The equations are essentially nonlinear and cannot be solved analytically. To simplify (4) we introduce the small parameter $\varepsilon=a / l$, where
we assume that $a \ll l$. In addition, we assume fundamental parametric resonance, where the velocity $\dot{\psi}$ is close to $2 \omega_{0}$ :

$$
\begin{equation*}
\dot{\psi}(t)=2 \omega_{0}+\varepsilon \omega_{0} v \tag{5}
\end{equation*}
$$

The resonant oscillations of the pendulum are studied using the relations

$$
\begin{align*}
& \alpha(t)=\varepsilon^{1 / 2}\left[p_{1}(t) \cos \frac{\psi(t)}{2}+q_{1}(t) \sin \frac{\psi(t)}{2}\right] \\
& \beta(t)=\varepsilon^{1 / 2}\left[p_{2}(t) \cos \frac{\psi(t)}{2}+q_{2}(t) \sin \frac{\psi(t)}{2}\right] . \tag{6}
\end{align*}
$$

Using (6) we transform to the new variable $p_{1}(\tau), q_{1}(\tau), p_{2}(\tau), q_{2}(\tau)$, where $\tau$ is the slow time

$$
\begin{equation*}
\tau=0.75 \psi(t) . \tag{7}
\end{equation*}
$$

We use the method of averaging, which simplifies the original system of equations somewhat and in some cases makes it possible to obtain analytical solutions. Without using the method of averaging it would be difficult to identify the main trends in the interaction process.
We substitute (6) into (4) and use the relations

$$
\dot{\alpha}(t)=\varepsilon^{1 / 2} \frac{\dot{\psi}}{2}\left(-p_{1} \sin \frac{\psi}{2}+q_{1} \cos \frac{\psi}{2}\right), \dot{\beta}(t)=\varepsilon^{1 / 2} \frac{\dot{\psi}}{2}\left(-p_{2} \sin \frac{\psi}{2}+q_{2} \cos \frac{\psi}{2}\right) .
$$

After averaging with respect to fast time $\psi(t)$ from 0 to $2 \pi$ we obtain

$$
\begin{align*}
& \frac{d v}{d \tau}=N_{2}-N_{1} v-\mu M \\
& \frac{d p_{1}}{d \tau}=-\delta p_{1}-(v+0,125 E) q_{1}-0,75 M p_{2}+2 q_{1} ; \\
& \frac{d q_{1}}{d \tau}=-\delta q_{1}+(v+0,125 E) p_{1}-0,75 M q_{2}+2 p_{1} ;  \tag{8}\\
& \frac{d p_{2}}{d \tau}=-\delta p_{2}-(v+0,125 E) q_{2}+0,75 M p_{1}+2 q_{2} ; \\
& \frac{d q_{2}}{d \tau_{1}}=-\delta q_{2}+(v+0,125 E) p_{2}+0,75 M q_{1}+2 p_{2} .
\end{align*}
$$

The quantities $E=p_{1}^{2}+q_{1}^{2}+p_{2}^{2}+q_{2}^{2}, \quad M=p_{1} q_{2}-p_{2} q_{1}$ are dimensionless kinetic energy and the momentum relative to the $O z$-axis of the pendulum.
Here we use the linear approximation to the static characteristic of the motor, where

$$
\begin{equation*}
\frac{H_{1}(\dot{\psi})-H_{2}(\dot{\psi})}{I+0,5 m a^{2}}=0,5 \varepsilon \omega_{0}\left(N_{0}-N_{1} \dot{\psi}\right)+\varepsilon^{2} . \tag{9}
\end{equation*}
$$

Therefore

$$
N_{2}=\frac{l}{a}\left(\frac{N_{0}}{\omega_{0}}-2 N_{1}\right), \text { and } \mu=\frac{2 m l^{2}}{I+0,5 m a^{2}}, \delta=\delta_{1} / \omega_{0} .
$$

The purpose of the present part of the paper is to show all possible classes of steady-state motion for the system of equations (8). In practice the different types of steady-state motion can be found for (8) only with the help of numerical methods of solution. As it was found in the works Krasnopol'skaya and Shvets[4], Shvets[9], the system under consideration has all types of steady-state when bifurcation parameters are changed. The most interesting results are connected to transition to chaos in such system. With the help of numerical experiments were determined the existence regions for steady-state chaotic motion in the system and analyzed the transition from regular motion to chaotic motion.

## 3 Parametric resonance under ideal excitation

It is interesting to compare our previous results with a system in which the interaction between the driving force and the vibrational loads is not taken into account. In this case the process is described by a system of equations obtained from (8) as follows. The first equation of (8) is dropped and the unknown function $v$ in the second, third, fourth, and fifth equations is taken as a constant parameter. Shown below is the system of evolutionary equations in amplitudes of resonance vibrations that was obtained by Miles[11] for parametric resonance on the basis of Hamilton's principle after averaging the Lagrangian function over fast time

$$
\begin{align*}
& \frac{d p_{1}}{d \tau}=-\delta p_{1}-(v+0,125 E) q_{1}-0,75 M p_{2}+2 q_{1} \\
& \frac{d q_{1}}{d \tau}=-\delta q_{1}+(v+0,125 E) p_{1}-0,75 M q_{2}+2 p_{1} \\
& \frac{d p_{2}}{d \tau}=-\delta p_{2}-(v+0,125 E) q_{2}+0,75 M p_{1}+2 q_{2}  \tag{10}\\
& \frac{d q_{2}}{d \tau}=-\delta q_{2}+(v+0,125 E) p_{2}+0,75 M q_{1}+2 p_{2}
\end{align*}
$$

The resulting system has unique properties: these properties following from the physical essence of parametric resonance. Such resonance is excited by vertical vibrations of the suspension point. Thus, the pendulum is insensitive to the orientation of the horizontal axes. This is reflected in Eqs. (10) by the fact that they are invariant relative to the replacement of $\left(p_{1}, q_{1}\right)$ by $\left(p_{2}, q_{2}\right)$.
When such excitation occurs, the moments of the exciting forces relative to the $O z$-axis will be equal to zero. Equation (10) allows us to obtain the equation $d M / d \tau=-\delta M$ for the dimensionless moment of momentum $M$, so that $M \rightarrow 0$ as $\tau \rightarrow 0$. Thus, $M=0$ for the asymptotic steady-state regimes being
examined here. In this case, it is easy to prove that $p_{2}=k p_{1}$ and $q_{2}=k q_{1}$, where $k$ is the proportionality factor that reduces system (14) to two equations

$$
\begin{align*}
& \frac{d p_{1}}{d \tau}=-\delta p_{1}-\left[v+0,25\left(1+k^{2}\right)\left(p_{1}^{2}+q_{1}^{2}\right)-2\right] q_{1} \\
& \frac{d q_{1}}{d \tau}=-\delta q_{1}+\left[v+0,25\left(1+k^{2}\right)\left(p_{1}^{2}+q_{1}^{2}\right)-2\right] p_{1} \tag{11}
\end{align*}
$$

System (11) has a dimensionality of two and, in accordance with the theory of dynamic systems Kuznetsov[15], has no chaotic steady-state regimes.
This rule also applies to the laws that govern the occurrence of parametric resonance in distributed systems with a circumferential coordinate, when only "coupled" modes with the same natural frequencies are excited. A system of equations of the form (11) can be obtained for the amplitudes of the resonance modes of distributed systems after using the procedure of averaging in slow time, see Krasnopol'skaya and Podchasov[16]. Thus, we may conclude that the interaction with energy source which we studied in previous part of this article (Eqs. (8)) is the trigger to chaos.

## 4 Forced resonance of pendulum oscillations under ideal excitation

Let's consider the case when the pendulum is kinematically excited by the pereodic motion of its suspension point without taking into account interaction with energy source. Let this point and the center of mass of the pendulum in a cartesian coordinate system $O x y z$ with the vertical axis $O z$ be $\left(x_{0}, y_{0}, z_{0}\right)$ and $(x, y, z)$, respectively. If the suspension point moves only along the $O x$-axis so that

$$
\begin{equation*}
x_{0}=a \cos \omega t, y_{0}=z_{0}=0 \tag{12}
\end{equation*}
$$

then the vibrations of the pendulum in the direction of the $O x$-axis will be directly excited, while the vibrations occurring in the direction of the $O y$-axis will be induced by coupling.
For the spherical pendulum being considered the Lagrangian function $L$ can be written in the form:

$$
\begin{equation*}
L=0,5 m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)-m \omega_{0}^{2} l(l-z) . \tag{13}
\end{equation*}
$$

Here, the coordinates of the spherical pendulum must satisfy the coupling equation $\left(x-x_{0}\right)^{2}+y^{2}+z^{2}=l^{2}$.
We will assume that conditions leading to forced resonance exist, i. e., we will assume that $\omega$ is close to $\omega_{0}$. Thus,

$$
\begin{equation*}
\omega^{2}=\omega_{0}^{2}+\varepsilon v_{1} \omega^{2} \tag{14}
\end{equation*}
$$

Where $\varepsilon=a / l$ is a small positive parameter; $v_{1}$ is the frequency difference parameter.
The slow of motion of the center of mass of the pendulum will be sought in the form Miles[10, 11]

$$
\begin{align*}
& x=\varepsilon^{1 / 2} l\left[p_{1}\left(\tau_{1}\right) \cos \omega t+q_{1}\left(\tau_{1}\right) \sin \omega t\right]  \tag{15}\\
& y=\varepsilon^{1 / 2} l\left[p_{2}\left(\tau_{1}\right) \cos \omega t+q_{2}\left(\tau_{1}\right) \sin \omega t\right]
\end{align*}
$$

using slow time

$$
\begin{equation*}
\tau_{1}=0,5 \varepsilon \omega t \tag{16}
\end{equation*}
$$

Inserting Eqs. (12), (15-16) into Eq. (13), averaging (13) over fast time $\omega t$, and using Miles' method (based on Hamilton's principle), we obtain the following system of differential equation:

$$
\begin{align*}
& \frac{d p_{1}}{d \tau_{1}}=-\delta p_{1}-\left(v_{1}+0,125 E\right) q_{1}-0,75 M p_{2} \\
& \frac{d q_{1}}{d \tau_{1}}=-\delta q_{1}+\left(v_{1}+0,125 E\right) p_{1}-0,75 M q_{2}+1 \\
& \frac{d p_{2}}{d \tau_{1}}=-\delta p_{2}-\left(v_{1}+0,125 E\right) q_{2}+0,75 M p_{1}  \tag{17}\\
& \frac{d q_{2}}{d \tau_{1}}=-\delta q_{2}+\left(v_{1}+0,125 E\right) p_{2}+0,75 M q_{1}
\end{align*}
$$

System (17) has the simplest form of the equations that describe the forced vibration of the pendulum, since it contains only four constants coefficients $\delta$, $v_{1}, 0,025$, and 0,75 .
Such a system of equations is used in problems concerning forced resonance vibration in distributed systems with a circumferential coordinate. When a system is subjected to excitation distributed in space with respect to one of the natural modes and when the exciting frequency is close to the corresponding natural frequency, the evolution of the amplitudes of the two coupled modes can be described by relations of the form (15). In this case, the system of averaged equations differs from system (17) only in the values of the constant coefficients with $E$ and $M$. Thus, the characteristic properties of system (17) may also be manifested in distributed systems with a circumferential coordinate. Let us mention the most important properties of the solution of system (17). There are three classes of solutions which are possible in the system: steady-state, periodic, and chaotic, as stated in Krasnopol'skaya and Podchasov[16]. Thus, the amplitudes of the forced resonance vibrations may be constant, periodic, or chaotic quantities for the system under ideal excitation. It should be noted that the fact that the vibrations of the pendulum are excited in the direction of the $O x$ axis is reflected in system (17) by the fact that the second equation of the system contains the term 1. Vibrations are excited in the oy direction only due to the nonlinear coupling of the displacements in both directions. However, the development of such vibrations helps destabilize the directly excited vibration
process. In fact, the appearance of the both components of motion is responsible for the existence of the periodic and chaotic solutions of system (17).
This effect can be manifest as follows in distributed oscillatory systems. The stability of vibrations in a resonance mode excited directly by an external load can be disturbed by the initiation of vibrations in a conjugate mode excited simply as a result of the nonlinear coupling of the modes Kubenko et al.[1]. The indirectly induced vibrations may in turn lead to chaotic vibrations in both resonance modes.

## 5 Forced resonance in ideal pendulum kinematically excited by motion about a circle

An interesting property of forced resonance which makes it similar to parametric resonance was observed in the study of vibrations of a pendulum with a suspension point rotating in the circumferential direction. This property is possessed by resonance vibrations of the free surface of a liquid in a cylindrical or spherical tank when the tank undergoes translatory motion and its center is displaced in the horizontal plane about a circle. The same property is also seen in the vibration of a liquid between two cylindrical shells when the vibrations are excited by a deformation travelling waves inside the inner shell in the circumferential direction, see Krasnopol'skaya and Podchasov[16], Krasnopolskaya and van Heijst[17]. After Galerkin's method is used, the equations for the amplitudes of the "coupled" resonance modes will be similar to the equations for the amplitudes of vibration of a pendulum in the direction of the $O x$ and $O y$ axes. The difference will be the values of the corresponding coefficients. As before, we will examine the derivation of the equations for such excitation of forced resonance by using a spherical pendulum as an example. Let the suspension point of the pendulum undergo motion over a circle of radius $a$ in the horizontal plane. Thus, its coordinates $x_{0}$ and $y_{0}$ change in accordance with the relations

$$
\begin{equation*}
x_{0}=a \cos \omega t, y_{0}=a \sin \omega t \tag{18}
\end{equation*}
$$

Proceeding on the basis of Lagrangian function (13), assuming that the exciting frequency is close to the natural frequency (15), and making a substitution of the time (16), we can obtain the following system of equations in slow time (16)

$$
\begin{align*}
& \frac{d p_{1}}{d \tau_{1}}=-\delta p_{1}-\left(v_{1}+0,125 E\right) q_{1}-0,75 M p_{2} ; \\
& \frac{d q_{1}}{d \tau_{1}}=-\delta q_{1}+\left(v_{1}+0,125 E\right) p_{1}-0,75 M q_{2}+1 ; \\
& \frac{d p_{2}}{d \tau_{1}}=-\delta p_{2}-\left(v_{1}+0,125 E\right) q_{2}+0,75 M p_{1}-1 ;  \tag{19}\\
& \frac{d q_{2}}{d \tau_{1}}=-\delta q_{2}+\left(v_{1}+0,125 E\right) p_{2}+0,75 M q_{1},
\end{align*}
$$

where $\delta$ is a coefficient expressing the damping forces.
This system differs from (17) in the presence of the term -1 in the third dimensionless equation, which shows that the vibrations of the pendulum in the oy direction are directly excited. We can use system (19) to obtain the equation

$$
\begin{equation*}
\frac{d(0,5 E-M)}{d t}=-2 \delta(0,5 E-M) \tag{20}
\end{equation*}
$$

the solution of which has the form

$$
\begin{equation*}
0,5 E-M=\text { const } \exp (-2 \delta \tau) \tag{21}
\end{equation*}
$$

Thus, for asymptotic steady-state regimes, $0,5 E-M=0$ or

$$
\begin{equation*}
\left(p_{1}-q_{2}\right)^{2}+\left(p_{1}+q_{2}\right)^{2}=0 . \tag{22}
\end{equation*}
$$

For the steady vibrations we are examining, it follows from (22) that

$$
\begin{equation*}
p_{1}=q_{2}, \quad p_{2}=-q_{1} . \tag{23}
\end{equation*}
$$

It can be seen that the evolution of the steady-state regimes in the given case is governed by Eqs. (23) and the following system of averaged equations

$$
\begin{align*}
& \frac{d p_{1}}{d \tau_{1}}=-\delta p_{1}-\left[v_{1}-0,5\left(p_{1}^{2}+q_{1}^{2}\right)\right] q_{1} \\
& \frac{d q_{1}}{d \tau_{1}}=-\delta q_{1}+\left[v_{1}-\left(p_{1}^{2}+q_{1}^{2}\right)\right] p_{1}+1 \tag{24}
\end{align*}
$$

for which only regular regimes are possible. Chaos is impossible in the secondorder system.
Thus, after analyzing averaged equations for systems with a circumferential coordinate in cases where different types of resonance loads were applied, we were able to classify the resonance that occur and determine that chaos is possible only with forced resonance excited by loading in a certain direction.

## Conclusions

It is shown that chaotic oscillations at the parametric resonance result from the interaction of the pendulum with a driving mechanism of finite power. Chaotic vibrations also are possible in the pendulum without interaction with exciting
mechanism when it is excited in one horizontal direction only due to the nonlinear coupling of the displacements in both directions. However, the development of vibrations in the second horizontal direction destabilizes the directly excited vibrations under ideal excitation, what can lead to chaotic regimes. In the study of vibrations of a pendulum with a suspension point rotating along a circle was determined that chaos is impossible under ideal excitation. Vibrations of a pendulum may have chaotic regimes under nonideal excitation. The system analogous to (19) would have an additional equation for the source of energy. Thus, nonideality is a trigger of transition to chaos.

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# Synergetic Control of the Separation of the Upper Stage and the Carrier Aircraft with non-simultaneous Breaking the Connections 

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#### Abstract

A synergetic approach to the synthesis of a control system for a nonlinear dynamic object is proposed on the basis of the well-known method of analytical design of aggregated regulators - ADAR. An applied problem of synthesis of control algorithms for the separation of the upper stage (US) and the carrier aircraft (CA) when the nose link is broken and the upper stage rotates at the main nodes is considered. The upper stage is located on the upper surface of the carrier aircraft and is connected to it by three nodes, the bow and two main ones. The carrier aircraft with the upper stage makes a flight to a given altitude, at which they are separated (air start of the upper stage). At the moment of separation, the nasal connection is broken. When the elevator is deflected and under the influence of the incoming flow, the upper stage rotates by a predetermined angle around the axis passing through the main attachment points. This angle is set so that the aft parts of the upper stage and the carrier aircraft do not collide. The obtained control algorithm provides stabilization of the upper stage rotation angle and compensation of external piecewise-constant disturbances acting on the system. The effectiveness of the synergetic approach is confirmed by the asymptotic stability of the closed-loop system "object of control - regulator", as well as its invariance to the influence of disturbances in the external environment, which is clearly demonstrated in numerical studies of the synthesized system.


Keywords: Synergetics, Synthesis of Control Algorithms, Piecewise Constant Disturbances, Carrier Aircraft, Upper Stage.

## 1 Introduction

In order to increase the fuel efficiency and payload of space systems, at the present stage of space exploration, scientists and engineers are increasingly inclined to use multistage aerospace systems (AS) for launching spacecraft into orbit, with upper-stage, located on a carrier aircrafts upper surface. With the upper location of the upper stage, solution of the problem of separating the aircraft presents certain difficulties: it is necessary to ensure the safe mutual movement of the separated aircraft after the moment of breaking the mechanical bonds. It is necessary to exclude the collision of aircraft, the negative impact on the structure of the aircraft carrier of the jet of heated gases from the engines of

[^2]the upper stage; to minimize errors in piloting, to ensure the invariance of the system to the effects of various kinds of disturbances. Solution to these problems is possible with help of flight control automation systems at the most critical stages. The aerospace complex considered in this study is a two-stage system. The first stage is a carrier aircraft; the second stage is an upper stage located on top of the carrier aircraft. The carrier aircraft and the upper stage included in the AS are complex nonlinear control objects. The dynamics of motion of such objects is described by a system of nonlinear differential equations. The task of controlling the entire complex as a whole at the stage of joint flight, as well as the carrier aircraft and the upper stage during their separation and autonomous flight is multidimensional. For the synthesis of control systems for the aerospace complex during launch at a given speed to the desired altitude (for air launch); and also for the separation of US and CA in case of a different-time breaking of bonds, in this work it is proposed to use the method of analytical design of aggregated regulators. ADAR method was invented by Professor A.A. Kolesnikov and developed in the works of his apprentices and followers, Kolesnikov[1], Veselov et al.[2], Kreerenko,[3]. This method makes it possible to work with a complete nonlinear model of the motion of an aircraft and to carry out coordinated control over all phase variables to transfer the control object to a given state. The work considered and solved the following tasks: development of a procedure for synergetic synthesis of control algorithms for the rise of the AS at a given speed to a given height for an air launch of the US; development of a procedure for synergistic synthesis of control algorithms for the separation of the US and the CA with nonsimultaneous breaking the connections.

## 2 Synthesis of AS control algorithms of reaching launch altitude

### 2.1 Mathematical model of AS

To synthesize autopilot control law of the AS at the stage of ascent to a given altitude, we will use the mathematical model of the longitudinal motion of the aircraft in projections on the axis of a semi-connected coordinate system, Bukov [4]. Taking into account expressions for aerodynamic forces and pitching moment, interference and reactions in the communication nodes between upper stage and carrier aircraft, mathematical model will take the form
$\dot{V}=\frac{P}{m} \cos \alpha-g \sin (\vartheta-\alpha)+\frac{N_{y} \sin \alpha+A_{x} \cos \alpha-q S\left(c_{x}+c_{x}^{\delta_{\theta}} \delta_{\theta}+k_{i n 1} \Delta c_{x i n}\right)}{m} ;$
$\dot{\alpha}=\omega_{z}-\frac{P \sin \alpha}{m V}+\frac{g \cos (\vartheta-\alpha)}{V}+\frac{N_{y} \cos \alpha-A_{x} \sin \alpha-q S\left(c_{y}+c_{y}^{\delta_{s}} \delta_{\varepsilon}+k_{i n 2} \Delta c_{y i n}\right)}{m V} ;$
$\dot{\omega}_{z}=\frac{q S b_{a}}{I_{z}}\left[m_{z}+\frac{m_{z}^{\overline{\omega_{z}}} b_{a} \vartheta}{V}+m_{z}^{\delta_{z}} \delta_{6}+k_{i n 3} \Delta m_{z i n}\right]+\frac{N_{y N} h_{x N}+N_{y M} h_{x M}-A_{x} h_{y M}}{I_{z}} ;$
$\dot{H}=V \sin (\vartheta-\alpha) ; \quad \dot{\vartheta}=\omega_{z} ; \quad \dot{x}=V \cos (\vartheta-\alpha)$,
where $V$ - airspeed; $\alpha$ - attack angle; $\omega_{z}$ - angular pitch velocity of the carrier aircraft; $H$ - flight altitude; $\vartheta$ - carrier aircraft pitch angle; $x$ - longitudinal displacement of the aircraft gravity center (c.g.); $m$ - mass of carrier aircraft; $I_{z}$ - the moment of inertia of the aircraft relative to the OZ axis; $g$ acceleration of gravity; $c_{x}, c_{y}, m_{z}$ - dimensionless coefficients of drag force, lift force and pitch moment; $S$ - aircraft wing area; $l$ - wingspan; $b_{a}$ - middle aerodynamic wing chord; $\Delta c_{x i n}, \Delta c_{y i n}, \Delta m_{z i n}$ - additions of the coefficients of the drag force, lift and pitching moment from the interference between the CA and the US; $N_{y}=N_{y N}+N_{y M} ; N_{y N}, N_{y M}$ - normal forces in nasal and in main node; $A_{x}$ - longitudinal force in strut of the main node
$A_{x}=\left(V_{y} \omega_{z}+\omega_{z}^{2} r_{x}+\frac{P-m g \sin \vartheta-q S\left(c_{x}+c_{x}^{\delta_{s}} \delta_{b}+k_{i n 1} \Delta c_{x i n}\right)}{m}-\frac{G_{x R B}-X_{R B}}{m_{R B}}\right) \frac{m_{R B} m}{m_{R B}+m} ;$
$N_{y N}=\frac{1}{r_{x_{-} M \_N}}\left(-G_{x R B} r_{1 y R B}+G_{y \times B} r_{1 x R B}+X_{R B} r_{1 y R B}+Y_{R B} r_{1 \times R B}\right) ;$
$N_{y M}=\left(\omega_{z}{ }^{2} r_{y}-V_{x} \omega_{z}+\frac{q S\left(c_{y}+c_{y}^{\delta_{s}} \delta_{\epsilon}+k_{i n 2} \Delta c_{y i n}\right)-m g \cos \vartheta}{m}-\frac{G_{y R B}+Y_{R B}}{m_{R B}}\right) \frac{m_{R B} m}{m_{R B}+m}-N_{y N}$,
$k_{\text {in1 }}, k_{\text {in2 }}, k_{\text {in } 3}$ - coefficients that take into account change in interference depending on the distance between CA and US; $m_{z}^{\bar{\omega}_{z}}$ - derivative of longitudinal moment coefficient with respect to relative angular pitch velocity $\bar{\omega}_{z}$, Byushgens and Studnev[5]; $q$ - velocity head; $r_{x_{-} M_{-} N}-$ distance between nose and main supports of US; $r_{1 y R B}, r_{1 x R B}$ - shoulders of gravity projections $G_{x R B}$ and $G_{y R B}$ relative to main support, respectively; $V_{x}, V_{y}$ - carrier aircraft linear velocity vector projection on the associated coordinate system axis; $X, Y-$ drag force and lift of CA; $r_{x}, r_{y}$ - coordinates of radius vector $\bar{r} ; \bar{r}-$ radius vector connecting c.g. AS and c.g. US (fig. 1, Mehta et al.[6]); $P$ engine thrust of CA; $\delta_{\sigma}$ - elevator deflection angle of the CA.


Fig. 1. Forces and moments acting on the US in free flight and reaction from CA

### 2.2 Synthesis of control algorithms

The control task is to lift aerospace system to a given altitude $H^{*}$ (at which the US will separate from carrier aircraft), as well as to move the AS at a given speed $V^{*}$ at this altitude, that is, to create such starting conditions for the upper stage so that after completion of the separation maneuver with a carrier aircraft, he could make an autonomous flight with a climb. Let us find in analytical form the control vector $u=\left[\delta_{6}, P\right]$, depending on the state variables of the system (2.1), which ensures the fulfillment of the given technological invariants

$$
\begin{equation*}
V=V^{*} ; \quad H=H^{*} . \tag{2.2}
\end{equation*}
$$

The synthesis uses standard ADAR procedure, Kolesnikov[7], Kolesnikov and Kobzev[8], Kreerenko[9]. For system (2.1), we introduce invariant manifolds:

$$
\begin{equation*}
\psi_{1}=V-V^{*}=0 ; \quad \psi_{2}=\omega_{z}-\varphi_{1}=0 \tag{2.3}
\end{equation*}
$$

where $V^{*}$ is the desired value of the variable corresponding to the set control goal (2.2); $\varphi_{1}$ - internal management.
Manifolds (2.3) must satisfy the solution of the system of functional equations

$$
\begin{equation*}
T_{1} \cdot \dot{\psi}_{1}+\psi_{1}=0 ; \quad T_{2} \cdot \dot{\psi}_{2}+\psi_{2}=0 \tag{2.4}
\end{equation*}
$$

where: $T_{1}, T_{2}$ - time constants affecting the quality of dynamic processes in the closed system "object of control - autopilot". Asymptotic stability in the large of system (2.4) with respect to manifolds $\psi_{1}=0, \psi_{2}=0$ is ensured at $T_{1}>0, T_{2}>0$. As a result of the dynamic "contraction" of the phase space at the intersection of invariant manifolds $\psi_{1}=0, \psi_{2}=0$, the decomposed system will take the following form:

$$
\begin{equation*}
\dot{H}(t)=V^{*} \sin (\vartheta-\alpha) ; \quad \dot{\vartheta}(t)=\varphi_{1} . \tag{2.5}
\end{equation*}
$$

For system (2.5), we introduce an invariant manifold $\psi_{3}$ :

$$
\begin{equation*}
\psi_{3}=V^{*} \sin (\vartheta-\alpha)+H-H^{*}=0, \tag{2.6}
\end{equation*}
$$

where $H^{*}$ is desired value of variable (2.2). The joint analytical solution of equations (2.5), (2.6) and the functional equation $T_{3} \cdot \dot{\psi}_{3}+\psi_{3}=0$, allows you to find an expression for "internal" control $\varphi_{1}$, in the form of a function of state variables $H, \alpha, \vartheta$, time constant $T_{3}$ and desired parameter values: $V^{*}, H^{*}$

$$
\begin{equation*}
\varphi_{1}=-\left(1+\frac{1}{T_{3}}\right) \operatorname{tg}(\vartheta-\alpha)-\frac{H-H^{*}}{T_{3} V^{*} \cos (\vartheta-\alpha)} . \tag{2.7}
\end{equation*}
$$

According to the procedure of the ADAR method, from the joint solution of (2.3), (2.4), the system of functional equations (2.4) and equations of the model (2.1), we obtain expressions for the control actions: the deflection angle of the elevator $\delta_{6}$ and the thrust of the engines $P$. These expressions are external controls and are functions that depend on the system state variables $\delta_{6}, P=f\left(V, H, \alpha, \omega_{z}, \vartheta, x\right)$. After substituting the expressions for the control actions into the model of the control object (2.1), we obtain a closed nonlinear motion control system of the aerospace complex. By setting the parameters of
the controller and choosing the invariants, we obtain a system that depends only on the state variables. Below are the results of numerical studies of the dynamic properties of the resulting closed-loop system.

### 2.3 Simulation

For the numerical solution of a closed nonlinear system, we will use RungeKutta method, Maple software package. Figures 2-3 show that aerospace complex achieves desired speed of $800 \mathrm{~km} / \mathrm{h}$ and a flight altitude of 10000 m ; figures 4-5 present changes of control actions with respect to integration time.
V, m/s

The simulation results show that the motion of the closed-loop system is asymptotically stable in the entire region of the phase space for various combinations of the initial values of the state coordinates. The exception is the points at which the considered mathematical model of the object is not defined (at an angle of attack and an angle of inclination of the trajectory equal to $90^{\circ}$ ).

## 3 Synthesis of control laws of US rotation at the main nodes

Consider the initial stage of separation of the upper stage and the carrier aircraft. At this moment, lock of nose attachment point is open and acceleration unit rotates about the axis passing through the main attachment points, Demeshkina et al.[10]. The main safety condition at this stage is to ensure that this maneuver is carried out without collision of the upper stage and the carrier aircraft. Rotation control of the upper stage is carried out by deflecting the upper stage
elevator. The US engines at this moment are not yet working due to the proximity of the carrier aircraft and the danger of the negative impact of a jet of heated gases on the structure of the CA. The rotation of the upper stage relative to the main attachment points leads to an improvement in the process of separation of the carrier aircraft and the upper stage in comparison with the case of simultaneous breaking of links, Leutin[11]. By the time the US is detached from the main attachment points, a significant angular velocity of the US develops for pitching. This leads to an increase in the angle of attack and vertical overload of the upper stage immediately after decoupling of the main units, Kreerenko, O. and Kreerenko, E.[12] . This, in turn, leads to a noticeable increase in the difference between the vertical overloads of the upper stage and the carrier aircraft. During the turning maneuver on the main attachment points, probability of collision between the US and the CA is reduced by fixing the aft part of the upper stage relative to the carrier aircraft, Decker and Wilhite[13], Moelyadi et al.[14], Decker and Gera [15].

### 3.1 Features of the mathematical model

Let us synthesize an autopilot law for controlling the angle of rotation of the upper stage relative to the carrier aircraft at the main attachment points. For the problem under consideration, we will assume that the movement of the upper stage occurs without roll and slip, the roll and yaw angles are equal to zero. We restrict ourselves to considering the equations of angular motion in the longitudinal plane. Then, to describe the rotation of the upper stage on two attachment points, we use the following differential equations:

$$
\begin{equation*}
\dot{\omega}_{z r 1}(t)=\frac{1}{I_{z R B}} \sum M_{R B_{-} \text {Main }} ; \quad \dot{\varphi}_{r 1}(t)=\omega_{z r 1}, \tag{3.1}
\end{equation*}
$$

where $\varphi_{r 1}$ - US rotation angle around the main body when maneuvering separation (fig. 1); $\omega_{z r 1}$ - the angular velocity of rotation of the US relative to the axis passing through the main attachment points; $I_{z R B}$ - moment of inertia US; $\sum M_{R B_{-} \text {Main }}$ - the total moment of all forces acting on the upper stage, relative to the main (rear) attachment point of the US to the CA.
Let us determine the sum of the moments acting on the US, relative to the main (rear) attachment point of the US to the CA:

$$
\begin{equation*}
\sum M_{R B_{-} \text {Main }}=M_{G_{-} \text {Main }}+M_{a_{-} \text {Main }}+M_{N_{n_{-} \text {Main }}} \tag{3.2}
\end{equation*}
$$

where $M_{G_{-} \text {Main }}$ - is the moment from the force of gravity of the US, $M_{a_{-} \text {Main }}$ is the aerodynamic moment, $M_{N n_{-} \text {Main }}$ is the moment from the reaction force in the nose attachment point. The aerodynamic moment acting relative to the center of gravity the US is brought to the attachment point of the main strut. Using expression (3.2), equating to zero reaction in nasal node, we obtain:

$$
\begin{equation*}
\dot{\omega}_{z r 1}(t)=\frac{1}{I_{z R B}}\left(M_{G_{-} M a i n}+M_{a_{-} M a i n}\right) ; \quad \dot{\varphi}_{r 1}(t)=\omega_{z r 1}, \tag{3.3}
\end{equation*}
$$

The moment from the force of gravity of the upper stage relative to the main attachment point of the upper stage to the carrier aircraft

$$
\begin{equation*}
M_{G_{-} M a i n}=G_{x R B} r_{1 y R B}-G_{y R B} r_{1 x R B}, \tag{3.4}
\end{equation*}
$$

where, $r_{1 y R B}, r_{1 \times R B}$ are the shoulders of the projections of the force of gravity $G_{x R B}$ and $G_{y R B}$ relative to the main support, respectively. Aerodynamic moment of the upper stage relative to the main attachment point of the US to the CA

$$
\begin{equation*}
M_{a_{-} \text {Main }}=-X_{R B} r_{1 y R B}-Y_{R B} r_{1 x R B}+M_{z a}^{R B}, \tag{3.5}
\end{equation*}
$$

where $M_{z a R B}$ is the pitching moment of the upper stage.
Substituting expressions (3.4), (3.5) into (3.3), we obtain:

$$
\begin{align*}
& \dot{\omega}_{z r 1}(t)=\frac{1}{I_{z R B}}\left(G_{x R B} r_{1 y R B}-G_{y R B} r_{1 x R B}-X_{R B} r_{1 y R B}-Y_{R B} r_{1 x R B}+M_{z a}^{R B}\right) ;  \tag{3.6}\\
& \dot{\varphi}_{r 1}(t)=\omega_{z r 1},
\end{align*}
$$

The pitching moment of US is determined by following expression:

$$
\begin{equation*}
M_{z a}^{R B}=\left(m_{z R B}+m_{z R B}^{\bar{\omega}_{z R}} \frac{b_{a}^{R B} \omega_{z R B}}{V_{R B}}+k_{\delta_{s R B}} m_{z R B}^{\delta_{\varepsilon R B}} \delta_{g}^{R B}+k_{i n 3}^{R B} \Delta m_{z i n}^{R B}\right) \frac{q_{R B} S_{R B} B_{a}^{R B}}{I_{z R B}}, \tag{3.7}
\end{equation*}
$$

where $b_{a R B}$ is the average aerodynamic chord of the US wing; $\alpha_{R B}$ - the angle of attack of the US; $\delta_{G R B}$ - the angle of deflection of the elevator of the US; $\bar{\omega}_{z R B}$ - is the relative angular velocity of the US pitch; $m_{z R B}$ - coefficient of the pitching moment of the US; $m_{z R B}^{\bar{\omega}_{R R}}$ - derivative of the longitudinal moment coefficient by $\bar{\omega}_{z R B} ; \bar{\omega}_{z R B}=b_{a R B} \omega_{z R B} / V_{R B} ; \omega_{z R B}$ - the angular velocity of the US pitch; $V_{R B}$ - linear speed of movement of the US; $m_{z R B}^{\delta_{\varepsilon R B}}$ - derivative of the longitudinal moment coefficient of the US by $\delta_{\sigma R B} ; k_{\delta_{\sigma R B}}$ is a coefficient that takes into account the decrease in the efficiency of the US due to the transfer of the center of rotation of the US from the central heating unit. into the attachment point of the equivalent main support; $\Delta m_{\text {zinRB }}$ - increment of the pitching moment of the US due to the influence of interference; $k_{\text {in } 3 R B}$ - coefficient of interference change when moving away from the CA. The projections upper stage gravity force on the associated coordinate system axis are:

$$
\begin{equation*}
G_{x R B}=-m_{R B} g \sin \vartheta_{R B} ; \quad G_{y R B}=-m_{R B} g \cos \vartheta_{R B}, \tag{3.8}
\end{equation*}
$$

where $\vartheta_{R B}$ is the pitch angle US.
Aerodynamic forces when the US rotates around the main attachment point:

$$
\begin{gather*}
Y_{R B}=Y_{R B}^{\prime} \cos \varphi_{r 1}-X_{R B}^{\prime} \sin \varphi_{r 1} ; X_{R B}=-X_{R B}^{\prime} \cos \varphi_{r 1}-Y_{R B}^{\prime} \sin \varphi_{r 1}, \\
Y_{R B}^{\prime}=c_{y R B} q_{R B} S_{R B} ; \quad X_{R B}^{\prime}=c_{x R B} q_{R B} S_{R B}, \tag{3.9}
\end{gather*}
$$

where $c_{x R B}, c_{y R B}$ are the dimensionless coefficients of US aerodynamic forces; $S_{R B}$ - US wing area, $q_{R B}=\rho\left|V_{R B}\right|^{2} / 2$ - velocity head; $V_{R B}$ - linear speed of US movement. Taking into account expressions (3.8-3.9) we get:

$$
\begin{aligned}
& \dot{\omega}_{z r 1}(t)=\frac{1}{I_{z R B}}\left(\left(-m_{R B} g \sin \vartheta_{R B}\right) r_{1 y R B}+\left(m_{R B} g \cos \vartheta_{R B}\right) r_{1 x R B}+q_{R B} S_{R B} \times\right. \\
& \left.\times\left[\left(c_{x R B} \cos \varphi_{r 1}+c_{y R B} \sin \varphi_{r 1}\right) r_{1 y R B}-\left(c_{y R B} \cos \varphi_{r 1}-c_{x R B} \sin \varphi_{r 1}\right) r_{1 x R B}\right]+M_{z a}^{R B}\right) \\
& \dot{\varphi}_{r 1}(t)=\omega_{z r 1},
\end{aligned}
$$

We write the expressions for, in the form

$$
\begin{equation*}
r_{1 x R B}=r_{1} \cos \left(\varphi_{0}+\varphi_{r 1}\right) ; \quad r_{1 y R B}=r_{1} \sin \left(\varphi_{0}+\varphi_{r 1}\right) \tag{3.11}
\end{equation*}
$$

where $r_{1}$ is the radius vector connecting the main attachment point and c.g. US; $\varphi_{0}$ - the angle formed by the radius vector $r_{1}$ and the axis $O_{1} X_{R B}$ (before the start of the US rotation maneuver relative to the main node) (Fig. 1).
Taking into account the expressions for, and the expressions for the pitch angle of the US, as well as the aerodynamic pitching moment of the upper stage (3.7), the system of equations (3.10) will take the form:

$$
\begin{align*}
\dot{\omega}_{z r 1}(t)= & \frac{1}{I_{z R B}}\left(\left(m _ { R B } g \left(-\sin \left(\vartheta+\varphi_{r 1}\right) r_{1} \sin \left(\varphi_{0}+\varphi_{r 1}\right)+\cos \left(\vartheta+\varphi_{r 1}\right) r_{1} \cos \left(\varphi_{0}+\right.\right.\right.\right. \\
& \left.\left.+\varphi_{r 1}\right)\right)+q_{R B} S_{R B}\left[\left(c_{x R B} \cos \varphi_{r 1}+c_{y R B} \sin \varphi_{r 1}\right) r_{1} \sin \left(\varphi_{0}+\varphi_{r 1}\right)-\right. \\
& \left.-\left(c_{y R B} \cos \varphi_{r 1}-c_{x R B} \sin \varphi_{r 1}\right) r_{1} \cos \left(\varphi_{0}+\varphi_{r 1}\right)\right]+\frac{q_{R B} S_{R B} b_{a R B}}{I_{z R B}}\left(m_{z R B}+\right.  \tag{3.12}\\
& \left.\left.+m_{z R B}^{\bar{\sigma}_{z R}} \frac{b_{a R B} \omega_{z R B}}{V_{R B}}+k_{\delta_{G R B}} m_{z R B}^{\delta_{s} R B} \delta_{6 R B}+k_{i n 3 R B} \Delta m_{z i n R B}\right)\right) ; \\
\dot{\varphi}_{r 1}(t)= & \omega_{z r 1},
\end{align*}
$$

where, $\omega_{z r 1}, \varphi_{r 1}$ are state variables.
The control of the rotation of the US on the main hinges is carried out by deflecting the elevator of the US, which leads to the appearance of an increase in the lifting force and a change in the value of the longitudinal moment. Thus, the control action is the deflection angle of the elevator of the US $\delta_{6 R B}$. The purpose of the control is to ensure the rotation of the upper stage relative to the axis passing through the main attachment points at a given angle:

$$
\begin{equation*}
\varphi_{r 1}=\varphi_{r 1}^{*} \tag{3.13}
\end{equation*}
$$

to create the most favorable conditions for separating upper stage and carrier aircraft. Formulation of the problem. It is required to find in analytical form a control law that stabilizes the angle of rotation of the upper stage $\varphi_{r 1}^{*}$ to create an increase in lift and ensure shockless separation of aircraft.

### 3.2 Synthesis of the control law

For the problem under consideration, the technological invariant is reduction to zero of difference between actual and given angle of rotation of the US:

$$
\begin{equation*}
\varphi_{r 1}-\varphi_{r 1}^{*}=0 \tag{3.14}
\end{equation*}
$$

In accordance with the procedure of the ADAR method for system (2.24), we introduce an invariant manifold of the following form:

$$
\begin{equation*}
\psi=\omega_{z r 1}+\beta_{1}\left(\varphi_{r 1}-\varphi_{r 1}^{*}\right)=0, \tag{3.15}
\end{equation*}
$$

where $\beta_{1}$ is a positive coefficient. According to the ADAR method, the macro variable must satisfy the functional equation

$$
\begin{equation*}
T \cdot \dot{\psi}+\psi=0, \tag{3.16}
\end{equation*}
$$

where: is the time constant that determines the time to transfer the system (3.16) to the finish manifold. The constant influences the quality of the dynamics of processes in the system "object of control - autopilot law" and is selected from the condition of obtaining a transient process of the desired type. Having solved together (3.12), (3.15) and (3.16) with respect to the deflection angle of the elevator of the upper stage, we obtain an expression for the control law

$$
\begin{equation*}
\delta_{\sigma R B}=f\left(\omega_{z r 1}, \varphi_{r 1}, \varphi_{r 1}^{*}, \beta_{1}, T\right) . \tag{3.17}
\end{equation*}
$$

### 3.3 Modeling

In the simulation, it was assumed that the CA is balanced in steady horizontal flight without roll and slip, the angular rates of pitch, roll and yaw are equal to zero; $V=800 \mathrm{~km} / \mathrm{h} ; \quad H=10000 \mathrm{~m}$. Invariant $\varphi_{r 1}^{*}=7^{\circ}$. Initial conditions $\omega_{z r 1}=0^{\circ} / \mathrm{s} ; \varphi_{r 1}=0^{\circ}$. Controller parameters: $T=0.1 \mathrm{~s} ; \beta_{1}=1$.

|  |  |
| :---: | :---: |
| Fig. 6. The angular rate of rotation of the RB at the main nodes | . Angle of rotation of US |
|  |  |
| Fig. 8. Deflection angle of elevator of US | Fig. 9. Invariant manifold |

Transient processes. For the numerical solution of a closed nonlinear system (3.12) and (3.17), the Maple software package, the Runge-Kutta integration method of the 4th order, was used. Figures 6-9 show the results of numerical studies of the dynamic properties of the resulting closed-loop system. Figures 6-

7 show the dependences of phase variables on time. The change in the control action depending on time is shown in figure 8 . Figure 7 shows that the upper stage reaches the required setting angle $\varphi_{r 1}^{*}=7^{\circ}$; the angular velocity of rotation of the US relative to the main nodes decays (fig. 6). The transition process shown in figure 9 for the introduced invariant attracting manifold shows that over time the manifold tends to zero. The simulation results show that the motion of the closed-loop system is asymptotically stable. The synthesized control law ensures the achievement of the set control goal: rotation of the upper stage on the main hinges at the desired angle with respect to the carrier aircraft.

## 4 Synthesis of astatic regulator for US rotation at main nodes

In some cases, to compensate for external disturbances acting on the aerospace complex, it is advisable to use astatic controllers. The synthesis of an astatic controller is based on the idea of expanding the state space of a controlled system by introducing additional integrating links into the structure of the controller. Integrators are introduced to reduce static error. The number of integrating links per unit exceeds the order of power-law perturbation function.

### 4.1 Extended Object Model

During the rotation of the upper stage on the hinges of the main attachment points to the carrier aircraft, various external disturbances can affect the system: wind shear, atmospheric fluctuations, etc. To suppress an external unmeasured piecewise constant perturbation, we synthesize an astatic control law. Suppose that the control object (3.12) is affected by an external unmeasurable piecewise constant perturbation $\operatorname{dist}(t)=$ const. Let us pose the problem of synthesizing a control law that ensures the fulfillment of invariant (3.13) $\varphi_{r 1}=\varphi_{r 1}^{*}$ and suppression of the external unmeasurable disturbance dist $(t)$. To solve the problem, according to the procedure of the ADAR method, we write down an extended model of the control object, Kreerenko, O. and Kreerenko, E.[16]

$$
\begin{align*}
\dot{\omega}_{z r 1}(t)= & \frac{1}{I_{z R B}}\left(\left(m _ { R B } g \left(-\sin \left(\vartheta+\varphi_{r 1}\right) r_{1} \sin \left(\varphi_{0}+\varphi_{r 1}\right)+\cos \left(\vartheta+\varphi_{r 1}\right) r_{1} \cos \left(\varphi_{0}+\right.\right.\right.\right. \\
& \left.\left.+\varphi_{r 1}\right)\right)+q_{R B} S_{R B}\left[\left(c_{x R B} \cos \varphi_{r 1}+c_{y R B} \sin \varphi_{r 1}\right) r_{1} \sin \left(\varphi_{0}+\varphi_{r 1}\right)-\right. \\
& \left.-\left(c_{y R B} \cos \varphi_{r 1}-c_{x R B} \sin \varphi_{r 1}\right) r_{1} \cos \left(\varphi_{0}+\varphi_{r 1}\right)\right]+\frac{q_{R B} S_{R B} b_{a R B}}{I_{z R B}}\left(m_{z R B}+\right.  \tag{4.1}\\
& \left.\left.+m_{z R B}^{\bar{\omega}_{z B}} \frac{b_{a R B} \omega_{z R B}}{V_{R B}}+k_{\delta_{s R B}} m_{z R B}^{\delta_{R R B}} \delta_{\sigma R B}+k_{i n 3 R B} \Delta m_{z i n R B}\right)\right)+z ; \\
\dot{\varphi}_{r 1}(t)= & \omega_{z r 1} ; \quad \dot{z}(t)=\eta\left(\varphi_{r 1}-\varphi_{r 1}^{*}\right),
\end{align*}
$$

where $z$ - is the dynamic variable (estimate of the external unmeasured disturbance performed by controller); $\eta>0-$ is a constant coefficient. The equation for $\dot{z}$ (4.1) is a dynamic perturbation model. When forming dynamic model, the requirement to fulfill set control goal (3.13) was taken into account.

### 4.2 Synthesis of the astatic regulator

In the synthesis of the astatic controller, the standard procedure of the ADAR method is used. For system (4.1), we introduce invariant manifold:

$$
\begin{equation*}
\psi=\omega_{z r 1}+\beta_{11}\left(\varphi_{r 1}-\varphi_{r 1}^{*}\right)+z=0, \tag{4.2}
\end{equation*}
$$

where $\beta_{11}$ - is a positive coefficient. From the joint solution (4.2), (4.3)

$$
\begin{equation*}
\dot{\psi}+(1 / T) \cdot \psi=0, \tag{4.3}
\end{equation*}
$$

and the equations (4.1), we obtain an expression for the control action $\delta_{6 R B}$

$$
\begin{equation*}
\delta_{\sigma R B}=f\left(\omega_{z r 1}, \varphi_{r 1}, \varphi_{r 1}^{*}, \beta_{11}, T, z\right) . \tag{4.4}
\end{equation*}
$$

Substituting obtained control law into the plant model (4.1), setting the controller parameters and technological invariant, we obtain a closed system.

### 4.3 Simulation

Instead of estimating an external unmeasurable disturbance $z$, we introduce a piecewise constant disturbance dist $=-4 / 57.3 \mathrm{rad} / \mathrm{s}^{2}$ into the extended model of system (4.1). Flight speed $V=800 \mathrm{~km} / \mathrm{h}$; altitude $H=10000 \mathrm{~m}$. Technological invariant: $\quad \varphi_{r 1}^{*}=7^{\circ}$. Initial conditions: $\omega_{z r 1}=0^{\circ} / \mathrm{s} ; \quad \varphi_{r 1}=0^{\circ}$. Controller parameters: $T=0.1 \mathrm{~s} ; \beta_{11}=2$. The modeling was carried out in the Maple software package, the integration method was Runge-Kutta. The simulation results taking into account the synthesis of the astatic controller are shown in fig. 6-11. As can be seen from the results of the analysis, the synthesized astatic controller ensures the achievement of the set goal by the control object, as well as compensation of the piecewise constant disturbance acting on the system, that is, the necessary adaptive properties of the closed-loop system.
( $\omega_{z r 1}$,


## Conclusions

Using the ADAR method in analytical form, a basic control law has been obtained, which provides stabilization of the swing angle of the upper stage when it rotates about an axis passing through the main attachment points; which makes it possible to create an increment in the lifting force on the wing of the booster block and helps to reduce the loads in the nodes of the mechanism for attaching the US to the CA; and also provides a separation process without collision of aircraft.
An astatic regulator of the upper stage rotation angle relative to the axis passing through the main attachment points has been developed when the nose link is broken and the US is delayed at the main nodes at the stage of separation from the carrier aircraft. The resulting controller ensures the asymptotic stability of the system with respect to the desired value of the US rotation angle at the main attachment points $\varphi_{r 1}^{*}$; and also invariance to external piecewise constant disturbance, for example, to wind shear.

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# Cloud electrification as a source of ignition for hydrogen lift-gas airships disasters 

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#### Abstract

The first half of the twentieth century became the Golden Age of the dirigible airship. After the Hindenburg disaster, (1937), the dirigible use fell into rapid decline leaving the non-rigid airships to serve in maritime roles until the mid 1960s. Throughout dirigible and non-rigid use, violent storm systems have been associated with in-flight airship disasters. In particular, the popular press at time instilled into the public perception that lightning strikes were the guilty ignition source of the disasters. Over the past 25 years, Saint Elmo's Fire has come forward as an alternative ignition source for in-flight airship disasters. Understanding the role of low energy discharges events is important for the emerging hydrogen economy that is intended to reduce the world's energy consumption and greenhouse emissions.


This paper reviews $2 \mathrm{H}_{2}+\mathrm{O}_{2}=2 \mathrm{H}_{2} \mathrm{O}$ combustion chemistry, the role of heterogeneous graupel chemistry within electrification of Cumulonimbus, and how the empirical mathematical construct of Peek's Law which attempts to identify the visual inception voltage in terms of the minimum electrical field stress required for the generation of Saint Elmo's Fire. Using this electrochemical knowledge, in-flight airship disasters associated with nearby cloud electrification, or violent storms systems, are correlated and reviewed. This study is supported by firsthand accounts (from survivors), including radio messages prior to an airship disaster, ground eyewitness accounts, along with the structural design of the airship. The hydrogen lift-gas airships reviewed here are four dirigibles (LZ-4 (L-10), SL-9, Dixmude and Hindenburg) and one non-rigid airship (NS.11). As a comparative control, this paper reviews the worst airship disaster, that of the helium lift-gas flying aircraft carrier, USS Akron (ZRS-4), which led to the loss of 73 lives. In addition to that of the sister airship, USS Macon (ZRS-5) disaster where two lives were also lost.

Keywords Saint Elmo's fire, plasmoid, dirigible, non-rigid airship, lightning

## 1 Introduction

In 2020, the authors of this paper presented two talks at the $13^{\text {th }}$ virtual CHAOS2020 conference Florence-Italy on how Peek's formula [1] may be used to estimate the visual inception voltage stress point on natural and artificial structures [2] and microwave oven plasma processing of nanomaterials [3]. In the former paper, the dirigible airship was one of the artificial structures examined. In the follow-up question and answer section, the main question was 'if not lightning, how does Saint Elmo Fire (SEF, also sometimes-called brush discharge, corona, or partial discharge) become a lethal threat to a hydrogen $\left(\mathrm{H}_{2}\right)$ lift-gas airship. The answer to this question involves a complexity of factors, including a detailed knowledge of the airship construction, the prevailing metrological conditions at the time of the disaster and if the airship's captain was 'Extraordinary Good', or 'Lucky'.

This paper consolidates published information regarding five $\mathrm{H}_{2}$ lift-gas airships (four dirigibles and one non-rigid airship) disasters that are associated with cloud electrification surrounding or near-by the airships. In these disasters, the airships are in-flight (not tied to a mooring-mast or their ground handling ropes secured to the ground, i.e. the airships obtain a quasi-equilibration to the local weather electrical field conditions. In this context amongst the factors and accounts considered are the prevailing metrological electrical and chemical environment, radio messages prior to the airship disaster, firsthand accounts (from survivors) and ground eyewitness accounts, along with the
structural design of the airship. References from the 'first rough draft of history' (newspapers and movie-reels) are used along with board of inquiries, contemporary and current aeronautical journals, metrological, physical chemistry and electrical engineering journals. The chronology of these articles reveals the complex processes (physical, commercial and political) were not inter-linked, but evolved overtime. To clarify these complex issues, the paper is organized as follows: Section 2 gives a historical view of SEF. Section 3 describes the process of airship disasters selection and classification of the selected dirigible and non-rigid airships (section 3.1). Section 4 looks at the airship construction, gasbag (section 4.1), non-rigid envelopes (section 4.2) and dirigible airframes (section 4.3). Section 5 provides an anatomy of $\mathrm{H}_{2}$ lift-gas fires. Section 6 revisits Peek's formula for a single metal electrode. Section 7 lists the airship under consideration in this paper. Finally, section 8 provides summary of this review.

## 2 Saint Elmo's fire (SEF)

Since classical Greek and Roman times Ermus of Fomia has been the patron Saint of Mediterranean sailors, to whom he appeared as SEF on the masts and spars of sailing ships as an electrical storm began to dissipate in electrical intensity. These good omens being manifest as characteristic cracking or hissing sound with a blue / violet flame-like glow. Between the years 1610-1611, art emulates real life when William Strachey's account of the ill-fated 'Sea adventure' voyage from the new world in 1610, is retold by William Shakespeare within the play 'The Tempest' [4]. In this play, SEF takes on a more sinister role as the spirit 'Arial' who manipulates the mariners off the ship. By 1886, this atmospheric phenomenon started to be systematically complied and reported as SEF, Ball lightning (BL) $[5-8]$ and fireball (FB) $[9,10]$. The latter two types proving to be more life threatening when compared to SEF. In addition, it has become clear that BL has the ability to interfere with radio broadcasts and to transfer part of its information through a glass windowpane with and without damage to the glass [5-7]. During world war one (WW1; 1914-1918) reports of non-lethal SEF encounters, both inside and outside of the airship airframe accumulates as airships flew though bad weather on their bombing raids and reconnaissance, due to the necessity of war, see for example Douglas W. Robinson's book 'The Zeppelin in Combat' [11]. Table 1 provides five examples of nonlethal SEF encounters, in each case the prevailing metrological conditions being a squall or thunderstorm containing lighting with a mixtures of rain, hail or snow.

Table 1. WW1 dirigible non-lethal airship-SEF encounters.

| Airship | Date | Location | Weather observations | Reference [11] |
| :--- | :--- | :--- | :--- | :--- |
| LZ-41 (L-11) | Aug 10, 1915 | Dogger bank | Thunderstorm, cloud-to- <br> cloud and cloud-to-sea <br> lighting. 1,000 to 4,600 feet | Page 121-122 |
| LZ 41 (L-11) | Mar 5, 1916 | Spurn head | Squall, cloud, snow and ice. <br> Wind speeds 55 m.p.h <br> Temperature + 3F | Page 148 |
| LZ-53 (L-17) | May 3, 1916 | North Sea | Thunderstorm, rain and hail | Page 160 |
| LZ-91 (L-42) | May 23-24, 1917 | Over London | Squall, hail, solid cloud | Page 244 |
| LZ-104 (L-59) | Nov 21, 1917 | Eastern Crete | Thunderstorm, black clouds <br> and flashes of lightning <br> close-by | Page 310 |

Since the early 1900s, metrology has shown that cloud-to-cloud and intra-cloud electrification has its origins in the Earth's troposphere ( 0 to 12 km ) [5-10]. Fairweather dc electric fields are modulated by ac and RF fields due to thunder and lightning activity. Moreover, the appearance of SEF around conducting tips and protrusions being due to the geometric field enhancement where equal-potential lines become bunched [1]. By 1928, the term for this electrical phenomenon began to be classified as 'plasma' (Greek: meaning mouldable substance), which considers an assembly of gas molecules that has some of its atoms or molecules temporally ionized or excited [12]. In 1952, Winston H. Bostick added the subclass 'plasmoid' that defines a separate plasma-magnetic entity that may be ejected from
the parent plasma [13].

## 3 Airship heuristic selection criteria and classification

This section lists the $\mathrm{H}_{2}$ lift-gas airship heuristic elimination criteria used to identify the airships destroyed by violent weather conditions. Figure 1, provides a chorological (1895 to 1960) time-stamp of forty airship accidents against the number of deaths per accident. In this period, two airship design generations appeared, the dirigible (1905 to 1937), and blimp (Russian (ca, 1920 to 1947) and USA (1930 to 1960)). Within this period, five-hundred and five lives were lost to airship disasters. In addition, during this period there were twenty-two nonlethal airships accidents (not shown) where airships were either lost or written-off. Note airships lost to enemy action are excluded from these tallies. The data in Figure 1, is given as a function disaster type (fire/explosion, midair and ground collision, pilot error, structural failure, lost, $\mathrm{H}_{2}$ and helium (He) lift-gas airships destroyed in storm conditions. Where multiple disasters occurred in one year (i.e. 1902, 1912, 1913, 1915...), the total loss of life is denoted with a + sign.


Fig 1. Lethal airships disasters between 1897 and 1960 as a function of related potential cause. The + symbol denotes the total number of in each year. For reference purposes only, ten airships are named here.

### 3.1 Heuristic elimination criteria

Using the forty airships disasters listed in figure 1 as a starting point, the heuristic elimination criteria (removal of airships from the list) is given as follows.

1. Remove all airship decommissioned (for example SL-8, 1918 [11], page 281)
2. Remove all airships set alight during inflation of gasbags within their hangar (for example, L-6 and L-9, Fuhlabüttel air field, 1916 [11], page 199)
3. Elimination of airships involved in a collision (for example, high voltage power lines (USS Roma TR-4, 1922 [14]), crash-landed on ice near the North Pole (Italia, 1928 [16] and grounded on a hillside (R101, 1930 [15], and SSSR-V6 1938 [17]).
4. Remove all airships destroyed whist flying in good weather (for example, LZ-104 (L-59), 1918 [11], page 315)
5. Remove due to pilot error (for example, SSSR-V10 1938 [17]
6. Remove all He lift-gas airships excluding the USS Akron and the USS Macon.

### 3.2 The airship classification

The airships disasters examined in this work are the LZ-40 (L-10) and SL-9 (Type E), the NS.11, Dixmude (formerly the LZ-72) and the Hindenburg (LZ-129). For comparison purpose, the He lift-gas airships, (USS Akron (ZRS-4) and USS Macon (ZRS-5)) are used. Note in the American airship number system, Z refers to Zeppelin mode of construction, R refers to ridged airframe and, S refers to flying aircraft carrier. As all of the airships have aero-engines as a means of propulsion, the airships are classified as either non-rigid or dirigible.
3.3 Zeppelin production number and tactical number classification

The Zeppelin company gave their airship a production number (LZ-xxx) whereas the German military gave their airships a tactical number ( $\mathrm{L}-\mathrm{xxx}$ ). This dual number system has led to some confusion. In this work, the Zeppelin production number is used. The Zeppelin tactical number is given in italics and is only used in sections 1 to 3 to provide a link between the airships, after which only the production number is used.

### 3.4 Non-rigid (pressure) airship

The non-rigid airship uses a $\mathrm{H}_{2}$, or He , lift-gas envelope that is pressurized with air-filled ballonets (air-filled compartments) to control lift and pitch, plus envelope shape and structural integrity. To achieve the weigh off (initial static-lift) the envelope is filled with $\mathrm{H}_{2}$ until the airship's volume equalizes with ground-level air volume. In equilibrium flight, the effect of slow varying updrafts, temperature changes and loss of fuel weight requires the airship to be maintained by blowing air into the ballonets or releasing air from the ballonets. Whereas dynamic lift is achieved by altering the elevator position with aero-engine power). When a rapid and violent updraft occurs, automatic springloaded lift-gas valves open to prevent the airship pressure ceiling being exceeded, resulting in a corresponding rapid loss of lift.

### 3.5 Dirigible airship

Unlike non-rigid airships, the dirigibles LZ-40 and SL-9, the NS.11, Dixmude, and the Hindenburg have multiple $\mathrm{H}_{2}$ lift gasbags located within a metal or wood airframe. In the case of the USS Akron and USS Macon, the lift gas is He. The Dirigible design ensues the airframe provides structural protection to the gasbags and greater shape protection from aerodynamic forces and moments. In the event of one of the gasbags is compromised, buoyancy is maintained by discharging ballast at the location of the compromised gasbag. See for example the USS Shenandoah (ZR-1) which was torn from its mooring-mast in 1924, and crashed in 1925 [18] and the R-33 30-hour unscheduled flight in 1925 [19]. Again when a rapid and violent updraft occurs, the automatic spring-loaded lift-gas valve opens, resulting in a rapid corresponding loss of lift.

## 4 Airship construction

This section reviews WW1, lighter-than-air flight. Section 4.1 looks the development of the gasbag (sometimes called cell), the non-rigid envelope (section 4.2), and the dirigible airframe structure (section 4.3). The airships in question are the LZ-40 and the SL-9, the Dixmude, the USS Akron and USS Macon and the Hindenburg, plus the non-rigid airship NS.11. Ladislas D'Orcy's 'International register and compendium of airships (built between 1873 and 1917)' [20], and Robinson's 'The Zeppelin in combat' [11] provides information on the techniques used in the manufacture of LZ-40, SL-9 and LZ-72 (latter to be named the Dixmude). In addition, written articles in the 'Journal Dirigible' are extensively used.

### 4.1. Gasbags

Since 1782 in Paris-France, 18-inch diameter balloons made from goldbeater skins filled with $\mathrm{H}_{2}$ were flown for recreational purposes [21]. The goldbeater skins originally obtained from cow intestines (cecum, or, caecum). This very lightweight material was found to exhibit a high inherent strength and is almost impervious to $\mathrm{H}_{2}$ gas. When cleaned and stretched having an approximate area of 20 cm in length and 25 cm in width. In 1883 move from making toy balloons to manufacturing for the 10,000 cubic foot balloon 'Heron' was performed by the Weinling family under direction of Captain Templar
at Chatham, England [22]. The Weinling family tried very hard to keep their gasbag manufacturing process at secret, but there is a suggestion of industrial espionage between Templar and the Italian government [23]. McKechnie's (Vickers. Ltd) 1919 patent details the manufacture of a lightweight and gastight 4-layer gasbag for airships and balloons [24]. The layers comprise; a single ply of linen coated with unvulcanized rubber followed by Goldbeater skin and vanishes. The patent, also states 'This represents about one ton increase of lift for a million cubic feet capacity' The quantity of Goldbeater skins for a standard WW1 German Navy dirigible airship is in the order of 20,000. It is no wonder that the Zeppelin Company had to recycle old gasbag material with greater outward $\left(\mathrm{H}_{2}\right)$ and inward (air) permeability properties that may lead to the loss of a dirigible and its crew [25]. After WW1, the number of skins used for dirigibles grew considerably. For example, the USS Shenandoah used over 750,000 goldbeater skins [18].
a

b


- Manoeuvring valve

Automatic spring-loaded 'blow-off' valve
Permeable outer skin
__ Non permeable outer skin

Fig 2. a) Schematic of early zeppelin lift-gas venting arrangement: b), Schütte-Lanz $\mathrm{H}_{2}$ and latter Zeppelin venting valve arrangement (schematic redrawn from A. Thomas (2014) [26]).

The early Zeppelins had a multitude of gasbags within a metal airframe covered with a waterproof, non-gas-tight skin. This construction allowed leakage of lift-gas to mix with the natural airflow up and round the gasbags and eventually permeate through the outer skin, see Figure 2a. However, within certain $\mathrm{H}_{2}$-air mixing ratios the gas mixture is flammable and liable to explode given a source of ignition. To counter act this problem, Schütte-Lanz airships improved on the design by adding forced ventilation which expelled the gas mixture via ducting from the bottom skin to the upper outer skin. In addition, a gas-tight coating to the bottom skin is added to prevent leaking lift-gas reaching the aero-engine exhausts. Both of these Schütte-Lanz designs were taken up by the Zeppelin Company during WW1, and ultimately, in a modified form for Hindenburg, see Figure 2b.

### 4.2. Non-rigid envelopes

Goldbeater skins, although having excellence gas-tightness, exhibited relatively low tensile strength and proved less than satisfactory against water. To cope with the stress encountered in non-rigid airship envelopes rubberized fabric of high tensile strength is used. Typical 2-3 layers are used where the threads of each layer is diagonal opposed. The envelope fabric, however, when subjected to an electrostatic fields may become electrified; and under certain conditions (such as when the envelope is
deflated (less taut) whilst the $\mathrm{H}_{2}$ lift-gas is being released) a fire may be ignited.

By 1917, Britain's answer to Germanys U-boat threat in the North Sea was the North Sea (NS) class $\mathrm{H}_{2}$ filled non-rigid airship. Using a tri-lobe lift-gas envelope based on the Astra-Torres design [27], fourteen of these airships were built. Within the envelope, there were six ballonets fitted with airblowers for buoyancy control: the control car and engine gondola being slung under the envelope. Initially designed for 24-hour flight endurance, on February 9 to 13, 1919 the NS. 11 smashed the nonrigid flight endurance specification by a record-breaking endurance flight of 400 miles in 100 hours and 50 minutes [28]. Table 2 lists the gasbag / envelope details of the six dirigibles and the one nonrigid airship discussed here.

Table 2. Airship gasbag and envelope information.

| Airship | Classification | Gasbag construction | Gas | Number of gasbags / ballonets | Gas capacity ( $\mathrm{m}^{3}$ ) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| LZ-40 | Dirigible | 3 layers of goldbeater on cotton | $\mathrm{H}_{2}$ | 12 | 31,900 |
| SL-9 | Dirigible | 3 layers of goldbeater on cotton | $\mathrm{H}_{2}$ | 12 | 38,750 |
| NS. 11 | Non-rigid | Rubberized cotton | $\mathrm{H}_{2}$ | 6 ballonets in one envelope | 10,194 |
| Dixmude | Dirigible | 3 layers of goldbeater on cotton | $\mathrm{H}_{2}$ | 16 | 68,470 |
| USS Akron | Dirigible | Rubberized cotton, \& cotton impregnated with gelatin-latex | $\mathrm{H}_{\mathrm{e}}$ | 12 | 194,000 |
| USS <br> Macon | Dirigible | cotton impregnated with gelatinlatex | $\mathrm{H}_{\mathrm{e}}$ | 12 | 194,000 |
| Hindenburg | Dirigible | 2 cotton fabric layer with celluloid in between which was the impregnated with a gelatinlatex applied | $\mathrm{H}_{2}$ | 16 | 200,000 |

### 4.3. Dirigible airframe structure.

The Pre-WW1 Zeppelin designs and Schütte-Lanz dirigible airframe designs are notable for their very different materials and methods of construction [11, 20]. The early Zeppelin designs on the Dave Schwarz of Zagreb patents using zinc aluminum alloy airframes. The general appearance of a Zeppelin is one of cigar shaped, with a parallel mid-section built from many transverse polygon rings of the same form. While the short (with respect to the mid-section) front and rear sections use similar reducing polygon rings apart from the aft section that has four tails fins built-in using a cruciform girder construction. The overall design allows mass production techniques to be used. By the start of WW1, aged-harden aluminum alloy (duralumin) containing copper ( 3.5 to $4 \%$ ) and manganese ( 0.5 to $1 \%$ ) began to be used for the airship airframes in Germany [29].

In the case of the twenty-four airships built by Schütte-Lanz, the airframe was one of the first successful geodesic latticework constructions. All, but two [SL-23 and SL-24] used wood and laminated wood all boned together with minimal metal fixings. Due to the large number of individual parts used, the construction time of the airframe was considerably greater than a comparable Zeppelin. However, the airframe tensgrity (tension and integrity) was flawed, as the laminated wood was prone to delaminate under moist conditions encounter in maritime roles leading the German Imperial Navy to mistrust these airships. Towards the end of WW1, a Schütte-Lanz engineering manager Hermann Müller, (Swiss by birth) defected to Britain and gave his knowledge of building wood airframe airships to the Short Brothers [30]. The outcome of which was R-31 and R-32 airships, which proved to have the same delaminating problem as the Schütte-Lanz airships. Later in 1928, Barnes Wallis patented the geodesic construction method using tubular metal for the contiguous transverse space frame design in the R101 [31].

During WW1 Britain, France and USA studied the construction of shot-down German airships, in particular the duralumin airframes. With final terms of the WW1 armistice signed on June 28, 1919 Germany was mandated to handover its airships (and High Sea Fleet) as war reparations. The political and revolutionary feelings within Germany at the time resulted in the scuttling of the High Sea Fleet at Scapa Flow and after seven airships were destroyed on the ground. Among the reparation demands following this act of destruction, Germany had to make-good the lost airships and handover all their airship technology. This forced reparation process meant that the allies received the 'Height Climber' class of Zeppelin that where designed for high-altitude (such as the Dixmude) at the expense of airframe structural integrality whist maneuvering at low altitude, a design feature that would plague the allied countries development of commercial airships for years to come.

In 1925, flying aircraft carrier proof-of-concept trails using the British R-33 airship that involved the launching and recapture of parasitic fighters. By 1929, the USA experimented (under land-based conditions) the concept of the flying aircraft carrier airship using the USS Los Angeles (ZR-3) as the mother ship. With completion of the British and American trials, the Goodyear-Zeppelin Corporation was formed for the design and construction of the first purpose built He lift-gas flying aircraft carrier: USS Akron and USS Macon. The hull design incorporated improvements in transverse frames for rigid airships as lay out by Richmond and Scott [32], which later appears to morph from the Barnes Wallis's space frame design, 1928 [31]. The airships used twelve He-lift gasbags using the Goodyear Tire and Rubber's rubberized cotton as the outer skin.

The original ship design used eight Maybach VL 11 aero-engines placed inside the hull (four each side) for driving propellers located in-line outside of the hull. In this configuration the engines disturbed air (wash) to the next inline propeller resulting sever airframe vibration and loss in available aero-engine power. To reduce vibration to the airframe, the propellers had to be operated in contrarotation to the next in-line aero-engine. This also provided greater engine thrust. In addition, during the design stage, the Navy requested for the bottom of the lower fin to be visible from the control car. To achieve this goal the goal car was moved 2.4 m aft and all the fins were shortened and deepened. The alteration meant that the leading edge root of the fins no longer coincided with an original main transverse frame fixing; instead, the attachment was now to a weaker intermediate traverses frame. The contra-rotating propeller preference combined with the weak tail fine attachment points have been the subject of much speculation of the USS Akron's many crashes and its final demise of the USS Akron (section 7.5) along her sister airship (USS Macon (section 7.6).

## 5 Anatomy of a $\mathbf{H}_{\mathbf{2}}$ lift-gas fire

To prevent an airship exceeding its safe pressure ceiling under rapid and violent updraft conditions, automatic spring-loaded lift-gas valves blow-off gas from the gasbags. Early Zeppelin airships (pre 1920s) the valves where located at the bottom of the gasbags to enable contaminated $\mathrm{H}_{2}$ gas to be blown off. Manually venting of $\mathrm{H}_{2}$ life-gas in storm conditions was prohibited in German airships from late 1915 (section 7.1). During this automatic process, the released $\mathrm{H}_{2}$ gas mixed with air within the airship's volume then diffuses through the outer airship fabric to mix with the airship's slipstream.

Unlike $\mathrm{H}_{2}$-air mixtures, pure $\mathrm{H}_{2}$ is difficult to ignite as many aircraft pilots firing solid metal bullets into the WWI Zeppelin and Schütte-Lanz airships found, see for example LZ-76 first and last raid on London [11]. When the metal bullets did hit the airship gasbags, they simply went through leaving small holes were $\mathrm{H}_{2}$ would slowly escaped and become quickly diluted by the surrounding air. It was not until the autumn of 1916 when the newly developed explosive bullet (Pomperoy, containing nitroglycerin) and the incendiary bullet (Brock, containing potassium chlorate) were fired in combination, the gasbags become blown apart when hit. As large quantities of released $\mathrm{H}_{2}$ mixes with atmospheric air, the incendiary bullet [33] ignites the flammable oxyhydrogen gas mixture.

Pure $\mathrm{H}_{2}$ gas burns with low radiant heat, almost without color, and becomes red-yellow depending
upon the amount and variety low molecular weight carbide and carbon monoxide impurities). In early 1900s, the German preferred method of $\mathrm{H}_{2}$ production was to pass steam over hot iron at high temperature to produce Knallgas (bang gas). Today's hydrogen economy the process is known as Steam Methane Reforming and the product $\mathrm{H}_{2}$ gas termed as gray $\mathrm{H}_{2}$ (a mixture $\mathrm{H}_{2}$ and $\mathrm{CO}_{2}$ ) or Blue $\mathrm{H}_{2}$ if the $\mathrm{CO}_{2}$ is removed. However, as Zeppelin warfare increased the production of blue $\mathrm{H}_{2}$ could not keep-up with lift-gas demand leading to greater $\mathrm{CO}_{2}$ impurities in the supplied life-gas. In addition, production accidents (Seddin gas plant, June 7, 1917) and train supply problems between the North Sea and Baltic bases [11], page 271-273 affected continuity of life-gas supply to the airships. In the inter war years, Britain faced a smaller but similar problem which was overcome by using mobile batch process units containing sodium hydroxide, ferrosilicon, and water that generated sodium metasilicate and $\mathrm{H}_{2}$ gas ( 99.3 to 99.6 pure), see equation 1 [34].
$2 \mathrm{NaOH}(\mathrm{s})+2 \mathrm{HO}_{2}(\mathrm{l})+\mathrm{Si}(\mathrm{s}) \xrightarrow{>150 \mathrm{C}} 2 \mathrm{NaSiO}_{3}(\mathrm{l})+2 \mathrm{H}_{2}(\mathrm{~g})$
Depending on pressure ( $p$ ) and temperature ( $T$ ), the flammability limit of $\mathrm{H}_{2}$ in air is generally between 4 to $75 \% \mathrm{H}_{2}$ by volume, and the explosive limit of $\mathrm{H}_{2}$ in air is 18.3 to $59 \%$ by volume. It only requires spark or electrical discharge of sufficient energy to crack both the $\mathrm{H}-\mathrm{H}$ bond ( $432 \mathrm{kJ.mol}^{-1}$ ) and the $\mathrm{O}-\mathrm{O}$ bond $\left(146 \mathrm{kJ.mol}^{-1}\right)$ to ignite the mixture and burn until the $\mathrm{H}_{2}$ fuel is consumed. Equation 2 depicts an almost physically impossible exothermic stoichiometric equation for these reactants to form water vapor $\left(\mathrm{H}_{2} \mathrm{O}\right)$ along with the associated $-482 \Delta \mathrm{H}$ value per two molecules of $\mathrm{H}_{2}$ fuel.
$2 \mathrm{H}_{2}(\mathrm{~g})+\mathrm{O}_{2}(\mathrm{~g}) \rightarrow 2 \mathrm{H}_{2} \mathrm{O}(\mathrm{g}), \Delta \mathrm{H}=-482 \mathrm{~kJ}$
At atmospheric pressure, the stoichiometric mixture autoignition temperature is in the order of $570^{\mathrm{C}}$ ( 843.15 K ) with a calculated minimum spark energy of the order of $0.02 \times 10^{-3} \mathrm{~J}$ [35]. However, this simple thermodynamic equation greatly misrepresents the electrical breakdown process of the gases, as both pressure and temperature; electric field stress, ignition frequency (dc, ac, or radio frequency), relative gas buoyancy, and the liquid-gas interface at the airship outer skin surface in storm conditions all have a role in the breakdown process.

## $5.1 \quad \mathrm{H}_{2}-\mathrm{O}_{2}$ gas vapor chain reaction mechanism

This section postulates a limited series of reaction steps within the $\mathrm{H}_{2}-\mathrm{O}_{2}$ gas vapor reaction. The steps proceed by initiation (3), branching $(4,5)$ and propagation (6).

Initiation: $\mathrm{H}_{2}+\mathrm{O}_{2} \rightarrow \mathrm{HO}_{2} \cdot+\mathrm{H}$.
Where, the initiation step (2) proceeds with the dissociation of some amount of molecular gas ( $\mathrm{H}_{2}$ and $\mathrm{O}_{2}$ ) by a spark, flame, or electric discharge.

The resulting hydrogen radical $(\mathrm{H} \cdot)$ attacks the reactants $\mathrm{O}_{2}$ through (4)
Branching: $\mathrm{H} \cdot+\mathrm{O}_{2} \rightarrow \mathrm{HO}+\mathrm{O}$.
Followed by the products above steps attack the $\mathrm{H}_{2}$ fuel $(5,6)$
Branching: $\mathrm{O} \cdot+\mathrm{H}_{2} \rightarrow \mathrm{HO} \cdot+\mathrm{H}$.
Propagation: $\mathrm{HO} \cdot+\mathrm{H}_{2}+\cdot \rightarrow \mathrm{H} \cdot+\mathrm{H}_{2} \mathrm{O}+$ heat
In these oxygen - hydrogen reactions, the chain cycle starts with one H - atom product (4), then the cycle generates additional H . atoms $(5,6)$. Steps (4 and 5) are named branching steps because they produce OH • radicals which further attacks the $\mathrm{H}_{2}$ fuel to generate two further radicals. The branching steps therefore promotes the rate of heat release which may increase exponentially, to the point that
the heat generated cannot be removed faster enough from the vapor at which an explosion occurs. In addition, in each chain cycle, the propagation step (6) produces a water molecule with an associated release of energy, which in turn promotes steps (4 and 5) along with the energy kick from the Pomperoy and Brock bullets. When all of the $\mathrm{H}_{2}$ is consumed, the cycle process is terminated. In this context, reaction steps 3 to 6 redefine the role of $\mathrm{H}_{2}$ from a lift-gas to an energy source.

In chemistry textbooks [36], the $\mathrm{H}_{2}-\mathrm{O}_{2}$ reaction is shown to have a complex dependence on pressure and temperature, specifically a zigzag curve that separates the non-explosive ( $p, T$ ) regimes from the explosive $(p, T)$ regimes. The free branches of the curve are called the first, the second and the third explosion limit. Early airships could gain altitudes of 2.5 km [20] and later WW1 version 'Height Climbers' reaching altitudes of 6 km [11]. These altitudes equate a standard pressure range of 1013 kPa to 47.1 kPa along with ambient temperature variation of approximately 15 to $-24^{\mathrm{C}}$ ( $\sim 288$ to $\sim 249 \mathrm{~K}$ ). This $p, T$ range places the airship flight altitude is well within the first and second explosive limits branching steps $(7,8)$ are explosively efficient.

On YouTube there are many slow motion photography sequences of balloon detonations filled with stoichiometric mixtures of $\mathrm{H}_{2}-\mathrm{O}_{2}$, see for example [37]. The slow motion films reveal that the initial shockwave ruptures the balloon, followed by the oxyhydrogen mixture burning with a typically yellow-orange that expands out from center of where the balloon once was. When a balloon filed with pure $\mathrm{H}_{2}$ is ignited, the reaction with the surrounding air is less rapid and the sound is less loud. From these demonstrations, the explosion is caused by a sudden pressure effect through the action of heat.

### 5.2. Fairweather electric field between cloud and ground / sea.

In this section, the heterogeneous chemistry within Cumulonimbus cloud is consisted as a source of ignition for the $\mathrm{H}_{2}$ filled airships. First consider the convection of warmed air (mainly a mixture of $\mathrm{N}_{2}$ and $\mathrm{O}_{2}$ ) from the earth as it expands adiabatically as it rises through the troposphere until it reaches the stratosphere, where the sun's energy reheats the circulated air. This natural convection process allows the cloud to capture positive charged particles resulting in a initial electrification of the cloud. With increasing electrification a negative charge begins to be formed on the upper cloud boundary which then flows down outside to the base of the cloud. The accumulation of negative charge at the base of the cloud, now by convention called 'Cumulonimbus' reinforces the cloud-ground/sea electric field. The electric field in this region is of the order of $1-3 \mathrm{kV} \mathrm{cm}^{-1}$ that is not sufficient to overcome the dielectric strength of air. To achieve the required field strength an inductive charge process within clouds has been considered by Saunders [38] and Prevenslik [39] where water moisture $\left(\mathrm{H}_{2} \mathrm{O}\right)$ is propelled to high altitudes by updroughts and cools to form graupel (a mixture of water and ice particles) that undergoes a continuous dissociation-recombination process forming hydronium ions $\left(\mathrm{H}_{3} \mathrm{O}^{+}\right)$and hydroxyl ions $\left(\mathrm{OH}^{-}\right)$intermediate products. This reversible reaction process is given in equation 7 where approximately $20 \%$ of the intermediate product ions are available for electrification.
$2 \mathrm{H}_{2} \mathrm{O} \stackrel{\text { graupel }}{\longleftrightarrow} \mathrm{H}_{3} \mathrm{O}^{+}+\mathrm{OH}^{-}$
Under natural background acidic conditions, charge separation of the available ions then follows, where the $\mathrm{H}_{3} \mathrm{O}^{+}$ions move into the vapor phase, and due to their buoyancy are lifted by updroughts to the top of the cloud leaving the larger and denser $\mathrm{OH}^{-}$charged graupel to fall under gravity to bottom of the cloud. This dynamic process generates a potential difference between the top and bottom cloud boundaries. With increasing gravitational separation, the negative charged graupel forms a negative space-charge that enhances the pre-existing fairweather electric field between the cloud and ground / sea. When the charge attraction between the cloud bottom boundary and ground strengthens, electrons and negative charged ions shoot down from the cloud as stepped leaders to meet upward positive charged streamers to produce a lighting channel. As the enhanced electric field subsides, sufficient energy remains to partially ionized nitrogen molecules $\left(\mathrm{N}_{2}\right)$ at the enhanced electrical fields at metal protrusions, at ground level, or in airships flying through, or near, the cloud [9] to produce the characteristic blue/violet. Westcox, using optical emission spectroscopy (OES) of SEF on aircraft measurement reveals a blue/violet emission that can be attributed to the $2^{\text {nd }}$ positive system of $\mathrm{N}_{2}$

$$
\left(\mathrm{C}^{3} \Pi_{u}^{+}-\mathrm{B}^{3} \Pi_{g}^{+}\right):<18 \mathrm{eV}[40]
$$

Prevenslik [39] proposed that where a cluster (10 or more) of charged graupel particles fall to the ground in the absence of a metal protrusion SEF does not occur but a collective discharge action occurs breaking down the surrounding atmospheric air causing the production of buoyant BL. The characteristic optical emission of which ranges from yellow, through orange, to red $(\lambda=550-780$ nm ) as indicated by [41]. The associated atomic and molecular ion spectra are: atomic-H-Balmer- $\alpha$ line $(\lambda=656 \mathrm{~nm})$, the $1^{\text {st }}$ positive system nitrogen $(\lambda=580$ and 654 nm$)$, the $\mathrm{O}\left(3 p^{5} \mathrm{P}-3 s^{5} \mathrm{~S}\right)(\lambda=777$ nm ) and the excited $\mathrm{NO}_{2}{ }^{*}$ molecule continuum ( $\lambda=450-800 \mathrm{~nm}$ ) [42]. Plus metastable neutral molecular oxygen $\left(\mathrm{O}_{2} ; \lambda=557.7 \mathrm{~nm}\right)$ [43]. The emission lines and bands quenching as the graupel finally melts.

### 5.3 Precipitation static

Wireless equipment having a range of 300 km started to be installed in airships as early as 1910 [11, 20] followed by their installation in aircraft. From the outset, the performance of the wireless communication degraded when flying through rain, mist and snow and it is thought that this precipitation caused an accumulation of electrical charge on the wireless antenna and other surfaces. To prevent electrical arcing and flashovers the standard approach was to bond all electrical equipment along with the airship's outer surface to central Earth point so that the airship has 'theoretically' an equal-potential throughout. Operationally it becomes standard practice to reel-in all wireless antennas when passing through a thunderstorm [9, 44], see section 7.4 (Dixmude)

Marriot reports one of the first investigations of electrostatic interference in 1914 [45]. By 1937, this electrostatic interference became known as precipitation static, or P-static [46]. With the advent of aircraft, high-speed flight the flux of charged particle due to increased antenna drag became a major problem and aerodynamic shielding measures where sought [47]. From the early 1990s it was shown, based on space born X-ray measurements that lightning produces high-energy radiation, in form of an abundance of electrons (up to 10 s MeV ) and gamma-ray glow / flash that drives the thundercloud electrostatic interference [48].

## 6 Peek's formula for a single metal electrode

It is well known that increasing the electrical stress around a single metal electrode tip (or protrusions) ultimately results in local air breakdown around the electrode. At this level of stress, the local air volume is weakly ionized followed by a rapid electron recombination back to the ground state discharge. On the milliseconds time scale the outer boundary of this volume, the ionization frequency $\left(v_{\mathrm{i}}\right)$ just balances the electron loss frequency $\left(v_{\mathrm{a}}\right)$ by attachment [49], see equation (8), and Figure 3. Under these condition a static corona discharge, or SEF, appears attached to the electrode with the visual inception voltage being higher than the visual extinction voltage because, once started there are always electrons to ionize gas molecules.
$V_{i}-V_{a} \approx 0$
Upon increasing the voltage stress level further $\left(\sim 5 \mathrm{kV} \mathrm{cm}^{-1}\right)$, the discharge extends outward to form multiple streamers flowing from the electrode, where breakdown is enhanced by the production of electrons at the head of streamer. If the voltage becomes large or a counter electrode is close by (110 cm ) a conducting trail or channel may form producing a flashover discharge. If the applied voltage is maintained sparks may be also formed. Further increasing the voltage stress creates bidirectional leaders are formed, which involve; space-charge and a gas heating ( $\geq 500 \mathrm{~K}$ ) mechanisms, rather than corona onset alone.

From this sequence of increasing discharge energy states, it is reasonable to assume that SEF influences streamer and leader production. Table 3 (adopted from Gibson [50]) provides a guide to the inception voltage for the three different discharge types. The data shows that although the corona
inception voltage for lighting rods has the lowest value for the three discharges (where the variation in the values is due to physical structure orientation of the rods [51]).
Table 3. Corona inception electrical field values and characteristic temperature (K) for atmospheric discharge at ground / sea level.

| Parameter | Corona discharge <br> $1-2 \mathrm{~cm}$ diameter <br> grounded lighting rods <br> $[50,51]$ | Streamer discharge [51] | Leader discharge [51] |
| :--- | :--- | :--- | :--- |
| Electric field $(\mathrm{kV} \mathrm{cm}$ <br> $1)$ | $0.2-2.7$ | $\sim 5$ | $\sim 1-5$ |
| Gas temperature $(\mathrm{K})$ | $\sim 300$ | $\sim 300$ | $\geq 3000 \mathrm{~K}$ |

Peek's formula was originally proposed as an empirical formula for coaxial cylindrical configurations, parallel wires and spheres in the 1920s [1]. Peek's empirical formula utilizes the local atmospheric condition and the surface condition of a conductor to estimate the corona visual inception voltage at local gas breakdown. For a manmade ac voltage source, see equation (9).
$g_{v}=g_{o} \delta m\left(1+\frac{k}{\sqrt{\delta r_{0}}}\right)$ Measured in units of $\mathrm{kV} \mathrm{cm}-1$


Ground and sea structures
electrically grounded
Airships
electrically isolated
Fig 3. Schematic of corona discharge boundary limits ( $g_{v}$ and $g_{o}$ ) for a single electrode. The ionic wind region is where unipolar charge carriers (for dry air, $\mathrm{N}_{2}{ }^{+}$and $\mathrm{O}_{2}{ }^{+}$ions) drift away for the corona region with insufficient energy to generate further reactions and/or ion creation.

In equation $9, g_{v}$ is the voltage gradient $\left(\mathrm{kV} \mathrm{cm}^{-1}\right)$ at the visual corona inception voltage; $g_{o}$ is the disruptive electric gradient, for an ac voltage the value varies from: $27.2 \mathrm{k} . \mathrm{cm}^{-1}$ for a sphere, 30 $\mathrm{kV} . \mathrm{cm}^{-1}$ for parallel wires, and $31 \mathrm{kV} \mathrm{cm}{ }^{-1}$ for coaxial geometries. The parameter $\delta$ is the local relative air density (at sea level, $\delta=1$ under fair weather conditions and 0.9 to 0.8 for storm conditions), $m$ is the surface roughness factor ( $m=1$ represents dry and smooth clean surface under laboratory conditions). For wet conditions, Peek found that the $g_{v}$ fell sharply and considered this as a special case for $m$ by substituting it with $g_{o}=9 \mathrm{kV} \mathrm{cm}^{-1}$. The parameter $k$ is an empirical dimension factor $(0.301$ to 0.308$)$ and $r_{o}$ is the tip geometry radius ( cm ). As energy is required to start a corona discharge the single electrode surface-to-space boundary limits requires that the surface electrical stress be raised to $g_{v}$ so that at a finite distance away in space where $k \forall r_{o}$ is $g_{o}$ air breakdown occurs. For dry air, the conducting carriers are typically $\mathrm{N}_{2}{ }^{+}$and $\mathrm{O}_{2}{ }^{+}$ions within a background of neutral gas molecules that drifts away from the corona discharge [52].

Natural occurring disruptive electric gradients formed by thunderstorms may also have a direct current voltage component [1], therefore equation 9 may be rewritten as follows:
$g_{v}=21.9 \delta m\left(1+\frac{k}{\sqrt{\delta r_{0}}}\right)$ Measured in units of $\mathrm{kV} \mathrm{cm}^{-1}$
Where 21.9 is the route mean square (RMS) of the ac disruptive electric gradient for air $\left(g_{o}\right)$. The parameters: $g_{v}, k$ and $r_{0}$ having the same meaning as in equation 3 .

Given that Peek's Law, in its different forms (equation 9 and 10), is an empirical mathematical construct, parameters $\delta m k$ and $r_{o}$ may be varied to fit the scenario of an airship entering a Cumulonimbus vertical cloud formation where the fairweather electric field is enhanced. In this scenario, the electrically isolated airship may become negatively charged with respect to the cloud were the amount of negative charge is determined by the competing effects of the rate of positive ions pulled to the charge surface as compared to the rate of electron generation by photoemission at the surface under ion bombardment conditions. Under this negative corona condition, the initial visual inception voltage generates discrete discharge points, or tufts, on airship sharp edges where the electrical stress is the greatest. These discrete discharges only grow in number to produce a uniform discharge as the voltage is increased. In addition, beyond the corona boundary ( $g_{o}$ ) electrons are propelled away from the discharge with sufficient number and energy to drive electron-impact reactions with neutral molecules [52,53]. The presence of a visible SEF glow on flying aircraft surface also appears to dependant on its airspeed. Researchers at MIT have recently demonstrated that high airspeed the SEF become detached leaving the electrical stress level to raised to it pre-visible inception voltage level [53]. Which of these two corona mechanisms (positive or negative) has the greater potential as a $\mathrm{H}_{2}$-air ignition source is of interest when considering the destruction of $\mathrm{H}_{2}$ liftgas airships?

## 7 Storm and thunderstorm activity leading to airship disasters

This section considers six notable dirigible and one non-rigid airship disaster attributed to storm and thunderstorm activity. These are LZ-40 and SL-9; The Royal Air Force (RAF) North Sea class NS.11; the French Navy Dixmude, originally built by as LZ-114 for the Imperial German Navy; the USS Akron and USS Macon; and the Hindenburg that ended the dirigible airship adventure.

### 7.1 LZ-40 (1915)

Two of the earliest known military $\mathrm{H}_{2}$ lift-gas dirigible disasters caused by natural atmosphere electrostatic disturbance were the LZ-40 and the SL-9 during WW1.

The LZ-40 took part in a number of bombing raids on England between June and September 1915 [11, 54 and 55]. On the LZ-40 last reconnaissance (commanded by Kapitänleunant Klaus Hirsch), the airship encountered a thunderstorm whilst returning to base on September 3, 1915. Robinsons [11], page $124-125$, provides details of the disaster. The following text is therefore complied from Robinson's account. In the afternoon of September 3, a radio message from LZ-40 informs Nordholz airship base that they would be returning at 3.30 pm . The metrology conditions in the local area were thunder and lighting, and at 2.30 pm , a number of eyewitness at the base saw in the direction of the town of Cuxhaven a 'large flash of flame like that of an explosion'. There accounts detailed how the explosion was red in color and the LZ-40 smothered in flames falling into the tidal region between the island of Neuwerk and Cuxhaven. Immediately rescue attempts were underway, but it was not until the following day that the salvage teams were able to recover eleven bodies out of a total of twenty on board the airship, along with the airships recording barograph. Posted on the www.wrecksite [55] is a photograph of the LZ-40 salvage operation with the airship's airframe laying in the shallow tidal sea.

In the report of the disaster (written by Kapitän Peter. Strasser; Chief of German Imperial Naval Airship Division) it was detailed that the airship was above its pressure ceiling, venting $\mathrm{H}_{2}$ lift-gas at the time of the disaster and ventures to relate the disaster to lighting even though there was no eyewitness to verify this. Strasser goes on to not: 'airships should in all circumstances try go around thunderstorms. If this is not possible, they should go through as far as possible under the pressure
height as the squalls will allow. The Airships of the Division now have such orders; also in thunderstorms they are ordered to reel in antennas'.

On a final note, D'Orcy mistakenly lists the LZ-40 destroyed at Ostend harbor on August 10, 1915 by the Royal Navy Air Service (RNAS) [20], in reality it was LZ-43 that was destroyed [11].

### 7.2 SL-9 (1917)

Commissioned into the Germany imperial Navy at Seddin in Pomerania, the SL-9 was the ninth in the series of twenty-four airships built by Schütte-Lanz. Although it wood and laminated wood airframe caused concerns to the German Imperial Navy, in particular Strasser [11], page 56. However, thirteen reconnaissance flights were made by SL-9, and in the summer of 1916, the airship bombed the port of Mariehimn, Finland (July 25, 1916) [56] and later took part in joint Army-Navy bombing raids over the South-East coast of England where airframe damage was sustained that required a month of repairs. After these raids, the SL-9 fell in flames into the Baltic Sea near Pillau on March 30, 1917 with the loss of twenty-three lives. Robinson [11], page 393 mentions that SL-9 burnt in a thunderstorm, while the Wikipedia website [57] claims the possible cause of the crash was a lightning strike. As with the LZ-40, the ignition source has not been determined. A possible explanation is that SL-9 rose above its pressure ceiling by a violent updraft causing an automatic blow off of $\mathrm{H}_{2}$ lift-gas which was then ignited by cloud electrification.
7.3 NS. 11 non-rigid airship (1919)

The NS. 11 entered service with the recently formed RAF in 1918. Based at RN airship station Longside Aberdeenshire, the airship made its record-breaking endurance flight of 400 miles in 100 hours and 50 minutes on February $9-13,1919$ [28]. The R-34 broke this endurance record some months later when flying East-to-West transatlantic flight from East Fortune, England to Mineola, long island, USA in 108 hours on July 2-6, 1919. Following this new endurance record, Captain W.K.F.G. Warneford of the NS. 11 filed a circular 48-hour flight plan over the North Sea from Pulham airship station. Thus, the NS.11, with Warneford and eight crewmembers rather than the usual nine, commenced its last journey at 9 pm on July 13, 1919 from Pulham.

Some eight minutes past midnight, a routine radio message revealed no problem with the flight, but some 15 minutes later, a Mr. E.T. Elwin from the hamlet of Newgate heard the NS. 11 aero-engines making 'a lot of noise' and thought the NS. 11 was in trouble. A few minutes later in the town of Cley, Mr. A.E Stangroom heard the NS. 11 pass overhead, again making 'a 'tremendous noise'. By 12.45 am a number, people heard the NS. 11 pass over Blakeney. At approximately 1.45 am , a violent explosion out to sea was heard, with the noise being carried as far as Wells and Cromer. The NS. 11 underwent a midair explosion and fell burning and exploding again before crashing into the shallow North Sea some 5 to 6 miles of the Norfolk coast with the loss of nine lives [58, 59]. Unlike the ZL-40 and SL-9, there were many ground-based eyewitnesses to the unfolding NS. 11 disaster. Several unsuccessful, rescue attempts were made. Most of the eyewitness heard the noise of an explosion then turned to look at where the noise came from, at which point they described what they saw. Two eyewitnesses saw the explosion; an old seaman saw the airship turning under the cloud before the explosion and then 'turned on end', whilst the other stated that, 'she 'took a header'. Both the witnesses inferring that the airship aft tail fins went up as the ship took fire. The staff at Pulham airship station unaware of the unfolding disaster until someone from Easter Daily Press (Norwich) phoned to ask if they could comment on the disaster. As for the recovery of NS. 11 crew, only the body of the second Coxswain (Sgt. C.H. Lewry) was found, it was washed up on the beach at Salthouse two weeks later on July 31.

In all of the accounts, the most notable metrological feature of the unfolding disaster was the isolated 'greasy black cloud', which the NS. 11 was approaching and then turned away when the initial explosion occurred. Importantly there is no mention of lightning. N. Peake [60] has reconstructed a plausible account of NS. 11 destruction that starts with the eyewitnesses seeing the NS. 11 turning thereby presenting its rear gasbag and tail fins to the cloud and the second explosion occurring as the remaining gasbags rupture on impact with the sea. Figure 4 graphically shows the account. In this
account the appearance of the 'greasy black cloud' is characteristic of an advancing 'cold front', Where the front is formed dry denser cold air pushes under moisture-laden clear air which is forced up where upon moisture is condenses out as water droplets to form the greasy appearance of the cloud. The condensation process also releases heat that causes a self-sustaining warm updraft leading to the formation of cumulonimbus and ultimately thunderclouds. Under these metrological conditions, the fairweather electric field is enhanced as the 'greasy black cloud' grows with a potential to induce SEF on the outer fabric of the NS. 11 envelope. This scenario in itself does not explain the explosion. However, factoring in that the caption and coxswain anticipated the updraft would force the NS. 11 above its pressure ceiling, and vent $\mathrm{H}_{2}$ lift-gas to counteract the uplift thereby creating the very distinct possibility that SEF would ionize the escaping $\mathrm{H}_{2}$ gas particularly if the $\mathrm{H}_{2}$ gas was of poor purity and withn the flammable limit. This scenario has credibility if the reports were true that Captain Warneford was attempting to break his own endurance record, by leavening Pulham with maximum fuel and minimum crew. In which case there would be no air in airship bonnets and her pressure ceiling would be much reduced. The official court of enquiry findings was inconclusive, but lighting was considered as a possible cause, despite no forthcoming evidence.


Fig 4. A schematic depiction of the NS. 11 near what the eyewitness described as a 'greasy black cloud'.

### 7.4 Dixmude dirigible (1923)

The L-72 was the third and final 1918 'Height Climber' X-class Zeppelin, designed to have a working altitude $6,000-6,400 \mathrm{~m}$ within a bombing raid duration of two-days. These airships required a significant increase in length (addition of one gasbag) and a reduction in weight. The achieved weight loss through the removal of parts of the original airframe, along with one of the original seven Maybach IVa aero-engines from the rear gondola, a reduction in fuel and water ballast capacity, as well as a reduction in machine gun armaments.

As part of the war reparations, in July 1920, the LZ-72 was turned over to France in 'perfect condition' and renamed the Dixmude. At the French naval air base Cuers-Pierrefeu near Toulon, the airship came under the command of the charismatic twenty-eight year-old naval officer: Lt.Cdr. Jean Du Plessis du Grenedan. Du Plessis supervised a three-year rebuild program of the Dixmude for extended flight duration (four-five days) at low altitude ( $2,000 \mathrm{~m}$ ). To achieved this goal, new goldbeater's skin gasbags supplied by the newly formed Astra Company, rather than the original German Company and the airframe strengthen to carry the increased fuel and water ballast plus crew
and passengers.
After a number of trial flights, the Dixmude began its last flight on December 18, 1923, a planned return flight from Toulon-France to the Algerian oasis of Ain-Salah (figure 5). At 8.00 am on Thursday (some 50 hours of flying time) the Dixmude turned north, bound form the Algerian coast, the airship encountered strong impeding winds and as the Dixmude fought against the winds, radio messages were sent reporting that fuel was running low and two aero-engines had broken down. The Dixmude was now being force east to Tunisia as a 'free-balloon' and at the mercy of the winds. The airships last radio message (02:08 am Saturday morning December 21) reported that they were following standard operating procedures to reel-in its radio antenna due to thunderstorm activity. Soon afterwards (02:30 am) railway workers and a hunter near Sciacca - Sicily reported a red flash in the Western night sky followed by burning objects falling in to the sea. On the morning of December 26, 1923, burnt wreckage of the Dixmude was found along with the charged corpse of her Commandant and the radio operator. As the news of the crash spread around the world, many newspapers speculated that lightning struck the Dixmude and was the cause of death of the fifty crew and passengers [61, 62].


Fig 5. The last flight of the Dixmude. Map redrawn from Ridley-Kitts (2010) [64].
Throughout this period, French newspaper reported the Dixmude voyage up to the last radio massage, as for the 4-6 days the Dixmude was missing reports emerge that the airship was lost in the Tunisian desert, and French, Italian and British naval ships searched for the airship in the Mediterranean Sea.

Confusion reigned in the French Ministry of Marine and international newspapers. Later the French commission of enquiry confirmed the newspaper speculation that the Dixmude was destroyed by lightning.

In 1924, Dr. Hugo Eckener (Manger of Luftschiffbau Zeppelin and later Commandant of the Graf Zeppelin) wrote in the 'Luffiahrt' on the Dixmude disaster [44]. In the article, the known German construction details of the L-72 and the subsequent conversion to the Dixmude are analyzed and the probable last four to five days timeline of the disaster presented. The following text provides a summary of his analysis. Firstly, repurposing the L-72 airframe from one of a high altitude bomber to one intended for extended flight duration by altering the distributed payload would cause excess stress on the airframe at low altitude. Secondly, the six Maybach aero-engines were pushed well beyond their military specification maintenance schedule of 1 to 2 days. Indeed, Maybach refused to guarantee more than 48 hours continuous use, especially for the crankshafts. Thirdly, as for a lightning strike being the energy source of the disaster, Dr. Eckener comments that duralumin airframe are designed to withstand routine lighting strikes by dissipating the electrical charge throughout the metal airframe, particularly at the nose and rear of the airship. [N.B. The Grafe Zeppelin and the Hindenburg are a case in-point, as both were struck by lightning many time as they voyaged between Europe and the Americas]. Fourthly, the burnt condition of the wreckage and the body parts found were consistent with a gasoline fire rather than a $\mathrm{H}_{2}$ fire that is less destructive to immediate surroundings. Fifth, automatic opening of the pressure ceiling valves due to violent updrafts may have been a contributing factor. Finally, even the radio message sent by the Dixmude build a picture of the storms it encountered; it is most likely that we will never know true cause of the airframe sudden and catastrophic failure.

A contemporary in-depth analysis of the Dixmude may be read in Ridley-Kitts three-part history of the Dixmude: published in Dirigible (2010 and 2011) [63-65].

### 7.5 USS Akron dirigible (1933)

In 1929, the USS Akron (the first purpose built flying aircraft carrier airship) was laid down and took her maiden voyage on November 2, 1931. After two ground-handling accidents, both captured on newsreels February 22, 1932 [66] and May 8, 1932 [67], a third ground handling occurred at the Lakehurst hangar 1 on August 22, 1932. On April 4, 1933, the worst airship disaster unfolded as the USS Akron crashed at sea off the coast of New Jersey with the loss of seventy-six crewmembers. The high death toll being due to drowning or hypothermia a factor being that there were no life jackets onboard the airship [68]. On this occasion, the surviving crewmembers were able to give a firsthand account of the disaster. The disaster happened whilst the airship was navigating at low altitude through a thunderstorm when her lower tail section hit the water. As with the first three accidents, the fourth and final accident provides a real-life and death example of the dangers of violent crosswinds and vertical winds to airships at or close to ground / sea level.

### 7.6 USS Macon (1935)

The USS Macron airship took to the sky on April 21, 1933, two week after the lost of her sister airship, USS Akron. In April 1934, whilst maneuvering through mountains in Arizona the USS Macron was forced to exceed its pressure ceiling height ( 910 m ) and climb to $1,800 \mathrm{~m}$ to past the mountain range which required $7,300 \mathrm{~kg}$ of ballast and fuel to be jettisoned. To gain a safe altitude it became necessary to jettison $\mathrm{H}_{2}$ lift-gas leaving the airship's ability to compensate for further changes in buoyancy greatly reduced. That is the USS Macon began take on balloon flying characteristics. With minimal ballast, and fuel, she pasted through the next mountain range in Texas, where violent up- and downdrafts could not be compensated for, resulting large aerodynamic pressures buckling the leading tail fin girder ring (\# 17.5). Subsequent repairs where made to the lower and lateral fins, but where not finished on the upper tail before her next flight on February 12, 1935. In this flight, the USS Macon encountered a storm off Point Sur-California where aerodynamic pressure at the rear of the airship caused the upper fin to shear off. The tail fin structural failure caused the USS Macon to climb above it pressure ceiling where the He lift-gas was automatically released, subsequently the USS Macon slowly glided in to the sea. Unlike the USS Akron, life jackets and rafts where on board
and SOS messages sent, resulted in only two lives being lost with the remaining eighty-one crewmembers rescued [69]. Here again the tail fins attachment appears to be a contributing factor under storm conditions. A similar tail cone problem was to plague the British R-100 airship test flights and on the airship's maiden transatlantic crossing to Canada (July 29 to August 1, 1930) where on a rival the outer fabric of the starboard elevator became ripped [70].
7.7 Hindenburg dirigible (1937)

At 18.00 local time on May 6, 1937 the second worst dirigible airship disaster, with the lost 36 lives, unfolded at Lakehurst, New Jersey when the Hindenburg commenced its tethering procedure at the airship mooring-mast. The airship had been delayed by poor weather and nearby thunderstorms as portrayed by British Pathé newsreel of the unfolding disaster [71]. Out of the sixty-two survivors, many gave testament to the disaster along with many ground witnesses. The disaster has evoked many books, journal papers [26, $72-77$ ] and aired TV programs [78]. This section considers the $\mathrm{H}_{2}$ lift-gas ignition theory based around four eyewitness accounts (Broadcaster: Herbert Morrison [79]: history Professor Mark Healed [76, 77]: photographer Arthur Cofod Jr [80]; and Helmsman Helmut Lau [81]. To aid the reader with these accounts, Figure 6 shows a sketch of the Lakehurst airfield and a schematic of the final 30 minutes of the Hindenburg flight. In addition, the approximate location of the external eyewitness is as given: H. Lau position is within the Hindenburg's lower tail fin auxiliary control room.

Positioned between Hangar \# 1 and the mooring-mast, Herbert Morrison and Charles Nehlson’s [79] sound recording of the Hindenburg disaster, transmitted on the following day of the disaster, imprinted such public reflective memories to give rises the 'Hindenburg syndrome' [72]. Morrison and Nehlson's account would not allow $\mathrm{H}_{2}$ gas to be used in public transport for many decades. The German Board of Inquiry into the Hindenburg disaster (picked out from many plausible reasons) two $\mathrm{H}_{2}$ gas ignition theories: (a) and (b).

Theory (a), proposed that due to atmospheric electric disturbances at the time of landing of the airship a corona discharge, otherwise known, as SEF or brush discharge, was the ignition source.

Theory (b), after dropping of the landing ropes, the airships outer fabric became less well grounded than the framework of the airship due to the lower conductivity of the outer fabric. Under these conditions, a spark possibly caused ignition of a $\mathrm{H}_{2}$-air mixture present over the gasbags four and five.

Professor M. Heald with his wife and son were located outside the main gate of the naval base on a trip to see the Hindenburg. From the car park lot, he records seeing a dim blue flame flickering along the Hindenburg's top ridge minutes before the fire started [76, 77]. The Heald's account from outside the airfield that gave a starboard side view of the Hindenburg against the backdrop of darkening eastern sky rather than the view from port side of the airship as told by Morrison and Nehlson. Given this information, it is generally thought that the blue color of the dim flicker might be SEF, thus supporting theory (a).

Closet to the initial ignition of the fire was Helmsman H. Lau who was (stationed with three over crewmembers) at the auxiliary control room within in the lower tail fin. In his testimony to the board of inquiry, he states the first time he notices anything wrong is when he 'hears a muffed detonation and looked up and saw from the starboard side down inside the gas cell a bright reflection on the front bulkhead of cell No. 4'. He goes on to stay: "The bright reflection in the cell was inside. I saw it through the cell. It was at first red and yellow and there was smoke in it. The cell did not burst on the lower side. The cell suddenly disappeared by the heat.... The fire proceeded further down and then it got air. The flame became very bright and the fire rose up to the side, more to the starboard side, as I remember seeing it, and I saw that with the flame aluminum parts and fabric parts were thrown up. In that same moment the forward cell and the back cell of cell 4 also caught fire [cell 3 and cell 5]. At that time parts of girders, molten aluminum and fabric parts started to tumble down from the top. The whole thing only lasted a fraction of a second.

Helmut Lau's testimony (translated by Willy von Mesiter) uses the word 'aluminum', which is assumed a simple transcription mistake as in the US Navy the names are interchangeable [29]. Given this, the Hindenburg's airframe would be expected to exhibit pronounced airframe deformation at approximately $471^{\circ} \mathrm{C}$ and produce molted duralumin (aluminum-copper alloy) $630^{\circ} \mathrm{C}$ [82].


Fig 6. Schematic of the Lakehurst airfield and the final flight of the Hindenburg.
Arthur Cofod Jr (AC) took a series of black and white photographs of the Hindenburg from a location to the north of Hangar \# 1. His most memorable photograph (Figure 7) shows the starboard and aft section of the Hindenburg 10s of seconds before crashing to the ground. With the back of the dark cloudy sky, the image photograph graphically details how the fire progresses forward with the keel just buckling aft of the rear two aero-engines, suggesting that the temperature generated by the fire is $>471^{\circ} \mathrm{C}$. Some 250 m above the Hindenburg, the updraft from the fire forms as a pyrocumulus mushroom-shaped cloud: where the upper bright region is normally associated with condensation of water vapor and the lower dark region contains burning debris of the airship with the most heaviest parts falling back down under the force of gravity. The moving $\mathrm{H}_{2}$-air flame-front is said to create a mantle effect between the patches of un-burnt outer fabric [74, 75]. It is also clear that at this stage of the fire the lower tail fin with its auxiliary control room is horizontal and still intact. Presumable, it is this aspect of the fire that enables H. Lau, (along with three other crewmembers (H. Freund, R. Kollmer and R. Sautar)) to escape the inferno when the intact lower tail fin crashes to the ground.

Alan Thomas writing in the Dirigible (2012) [26], advances the plausible theory of how the last of the three vented $\mathrm{H}_{2}$ lift-gas volumes may have be ignited. This theory may be dived into both what is known and speculation as to what may have happened. What is known is that, in the final minutes before mooring the maneuvering valves are operated to vent $\mathrm{H}_{2}$ gas to stabiles the airships neutral bouncy at the mooring-mast. In addition, with all of the Hindenburg aero-engines reversed the airship comes to a stop at the mooring-mast. At this point, the vent shaft theory may come into play that the
airship slows down and the aerodynamic extraction force at the top of the vent shaft is corresponding reduced. Thus, $\mathrm{H}_{2}$ gas slowly builds-up in the vertical vent shaft (Figure 2), with little of the $\mathrm{H}_{2}$-air mixture diluted in the airships slowing slipstream. With $\mathrm{H}_{2}$-air mixture, exiting the top of vent shaft becomes partially ionized by SEF. In addition, the ionized $\mathrm{H}_{2}$-air mixture flows-back down the vent shaft to combust the concentrated $\mathrm{H}_{2}$ gas.


Fig 7. Hindenburg seconds before dropping to the Lakehurst airfield. The photo is downloaded from Wikimedia and attributed to Arthur Cofod Jr / Public domain [80].

## 8 Discussion

Since the beginning of recorded history, St Elmo's Fire (SEF) has been widely observed at the closing stages of thunderstorm activity: both at sea level and in mountain regions. The systematic study of these naturally occurring atmospheric weather disturbances has proved difficult due to verifiable eyewitness accounts and real-time high-voltage air breakdown measurements. However, at least five non-lethal airship-SEF encounters are known to have occurred in WW1, see table 1. The five $\mathrm{H}_{2}$ liftgas airships ( 5 dirigibles and one non-rigid) disasters presented here represent the most notable storm weather related airships disasters. In contrast to these 5 airships disasters the He-lift-gas USS Akron disaster and its sister ship USS Macon had similar tail fin structural and aero-engine design faults, both of which played a significant part in their encounter with violent storms systems. Lightning, SEF or another form of static discharge did not have a role in these two airship disasters. The boards of enquiries in to each disaster indicate a combination of Pilot error and structural failures where the primary contributing factors in the destruction of the two airships

This review has looked at $\mathrm{H}_{2}$ lift-gas airship disasters where blue $\mathrm{H}_{2}$ gas is the main source of lift. In the first two discussed (LZ-40 and SL-9), all the crew of both the airship died, thus providing no firsthand evidence to the cause of either crash. However, Nordholz airfield ground crew did see the LZ-40 burst into flames as the airship prepared to land. The recovered barograph from the wreck indicted that the airship was a height of 2,400 feet and valving $\mathrm{H}_{2}$ lift-gas at the time of the disaster. As for the cause of SL-9 disaster, there can only be speculation.

This work has reviewed the anatomy of a $\mathrm{H}_{2}$ lift-gas airship fire along with cloud electrification as the ignition source using Peeks formula to describe the point of ignition. From the forgoing line of reasoning, it is hypothesized that a potential ignition source in weather related airship disasters, in part, might be due to cloud electrification and the production SEF. The mechanism of positive and negative corona discharge along the airship airspeed may also have a role in the production SEF. Notwithstanding this observation, SEF is most likely to be prevalent at high electrical stress points on the airship external surface coupled with automatic $\mathrm{H}_{2}$ blow off, or the manual operations of maneuvering valves to blow off $\mathrm{H}_{2}$ gas. Out of five $\mathrm{H}_{2}$ lift-gas airship disasters reviewed here, three airships (NS.11, Dixmude and the Hindenburg) are likely candidates as the means of the airship destruction. The scenario in which the disasters occurred is as follows.

The NS. 11 disaster was witnessed by many people and recorded in newspapers of the time where lighting strike was portrayed as the guilty party was even though a thunderstorm was not present at the time. Without clear evidence, the board of enquiry found that lightning was the most probable cause of the disaster thereby deflecting blame form unauthorized flight endurance attempts by the captain of the NS.11.

As Dixmude turned home on its endurance flight from the Algerian oasis of Ain-Salah, there was no eyewitness of unfolding disaster. The disaster was pieced together in French national newspapers from radio massages and the discovery of the airship wreckage some four to six days after the event. One year later, a detailed forensic analysis of the Dixmude disaster (by Dr. Hugo Eckener) highlights the failings of the airship's aero-engines and modification (strengthening) to the original L-72 airframe as being a major contributing factors to the lost of the airship.

In the case of the Hindenburg, Professor Heald's family provided visual evidence of SEF flickering along the upper ridge close to the tail fin of the Hindenburg moments before the disaster. This however was not given at the board of inquiry. Although late, this new evidence gives weight to the first option (a), where atmospheric electric disturbances at the time of landing of the airship, a corona discharge, otherwise known as SEF or brush discharge was the ignition source of a $\mathrm{H}_{2}$-air gas mixture. In this case as manual maneuvering valves where operated to vent the $\mathrm{H}_{2}$ lift-gas as the airship approached the mooring mast.

It may be concluded that the NS.11, Dixmude and the Hindenburg fell victim to the 'first rough draft of history' as portrayed in the newspapers where a lightning strike was stated to the likely guilty party. However, the Hindenburg disaster was the first airship disaster to be captured using real-time soundrecordings, black-and-white movie-reels and photos. Herbert Morrison's recorded radio broadcast of the Hindenburg disaster was the final death blow to Germanys dirigible travel, but in reality the Pan American Airways M-130 China Clipper scheduled flight across the Pacific on November 22, 1935, (some three months before the Hindenburg first took to the air) was the first blow. In Russia, the end of $\mathrm{H}_{2}$ lift-gas dirigible service did not end until the SSSR-V6 and SSSR-10 crashed in 1948 with a combined lost of twenty lives. Non-rigid airship service continued throughout the 'Great Patriotic War' and beyond as unpressurised $\mathrm{H}_{2}$ bulk transporters. The lost of the Patriot and Pobeda (Victory) in 1947 may be considered as the end of the $\mathrm{H}_{2}$ lift-gas airship golden age.

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# 'Dubro’ resophonic guitar: glissando gestures 

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#### Abstract

Whether in the Hawaiian, Bluegrass, Rock' n' roll, film sound track or animated cartoon genre, the swoop (glissando) sound made on a slide-guitar is one the most instantly recognizable in western music. This paper reports on the complex acoustical and perceptual glissando of the opening few seconds of Warner Brothers 'Looney Tunes' ascending glissando, and its counterpart (descending glissando), both played on a 'Dubro' resophonic guitar. The aim is to analyze these guitar themes in an attempt to provide both a historical development, as well as a technical understanding of the generated sound. With the resophonic guitar tuned to open G (D-G-D-G-B-D), the radiated sounds, includes the guitarist gestures and the glissando sound of steel and glass bottleneck, Using the toolbox within Audacity software (time-domain, standard autocorrelation, spectrogram and noise reduction), the recorded tracks are transcribed for tempo, consonant, dissonant, string squeaks, and incoherent / coherent noise. This study also attempts to map the complex psychoacoustic tonal quality of a resophonic guitar, which has been demonstrated to impact emotionally on the listener.

It is found that dynamic slide movement divides the string scale length into two coupled longitudinal vibrating segments, each producing a coherent continuous mirrored exponential varying pitch that extends to the guitar brilliance region ( 4.5 to 20 kHz ). Incoherent or 'hiss-like' noise is found within the lower psychoacoustic warm region ( 0 to 0.5 kHz ). This incoherent noise is linked to a slip-stick friction process between the slide and string. Slide material and slide direction varies the intensity of the noise that has a Voss-Clarke $1 / f$-like response with a Brownian $\sim-7 \mathrm{~dB} / 10 \mathrm{~Hz}$ roll-off. It is proposed that the guitarists fretting arm musculoskeletal system plays a role in the generation incoherent or hiss-like noise.


Keywords: Dubro, resophonic guitar, bottleneck, glass, steel, glissando, incoherent noise

## 1. Introduction

Chordophones have an important cognitive and emotional role in the development of world music [1], and western music, where the guitar is the main instrument in this classification of stringed instruments. When playing a string instrument, the listener's perceptual experience can invoke a strong psycho physiological response (chills and tears [2], change in heartbeat and respiration rate [3], pupil dilation response [4], and dance [5]. Over riding these emotions is whether the music is played in the major or minor cord: where a major cord instills joy and happiness, and calmness and sadness is found in minor cords [6]. The average tempo of a composition also influences the listener's emotions. For example, an Allegro composition of 140 beat per minute (BPM) has been found to increase listener's blood pressure and heart rate, whereas Andante of 80-82 BPM produces calmness [7]. For a human body mechanism to produce these responses, Voss and Clarke [8] proposed that the natural chemical oscillations within nerve membranes are a likely candidate. Arguably the resonant guitar (or, resophonic guitar [9,10] played with a bottleneck slide provides one of the most distinctive glissando [11] sounds within the family of string instruments. For example, Freddie Travares's crystal-clear attention grabbing opening 2 seconds for the Warner Brothers instrumental theme 'Looney Tunes' (based on the song Merry-go-round broke down) [12, 13]. Travares's credited guitar work on Elvis Presley's 'Blue Hawaii' [14] is another, if not well known, example of the guitar glissando. It is no surprise then that 'Loony Tunes' is associated with comedy and happiness and 'Blue Hawaii' is associated with mellow emotions. Ry Cooders reworking of Blind Willies Johnson slide guitar chords in the film 'Paris, Texas', goes one further by introducing vibrato at the end of each fading cord to
evoke the feeling of doubt, sadness and yearning of the American dry desert landscape [15].
Beyond the resophonic guitar patents [9,10], online commentary of partitioning the resophonic guitar psychoacoustic pitch / frequency bands [16], mechanical modal analysis of the resophonic guitar [17, 18] and the development of virtual slide guitar software [19], detailed mapping of the resophonic guitar psychoacoustic space has not be documented as played by a guitarist. The aim of this work is to analysis the radiated sound of a 'Dubro' resophonic guitar with the guitarist playing the instantaneously recognizable opening seconds of the ascending glissando of the 'Looney Tunes' instrumental theme and its counterpart associated with the instrumental theme music to the film 'Paris, Texas'. The capture and analysis of these guitar themes and guitarist gestures maps the perceived sound by the musician and nonmusicians alike thus providing a greater insight to the resophonic guitar psychoacoustics space.

This paper is organized as follows: Section 2 reviews the origins of the resophonic guitar and open G tuning. Section 3 provides the experimental. Section 4 describes Benchmarking of the guitar under steel- and glass-bottleneck at a fixed fret position when strumming a using a plastic plectrum [20]. The frequency range of the benchmark extends through three psychoacoustic regions: warm ( 0 to 2 kHz ), bright ( 2 to 4.5 kHz ) and the lower brilliance region ( 4.5 to 8 kHz ) [16]. Section 5 explores the ascending and descending glissando in these three psychoacoustic regions. In section 6 the extended brilliance range up to the human audible threshold limit ( 20 to 22 kHz ) is examined for ascending and descending glissandos. In section 7 , the background detailed in sections 4,5 and 6 , informs and identifies the incoherent noise produced by the guitarist gestures. Finally, section 8 provides a discussion on this work.

## 2. Development of resophonic guitar

### 2.1 Pythagoras string instrument theory

The employment of music in the treatment of disease dates back to the earliest times including when David strummed his harp before Saul [21]. Later Pythagoras [ca, 570-495 BC] became interested in understanding the notes and scales used in Greek music for the healing of disease. In particular, the use of the stringed instrument, called the lyre. It is from this time the use of a mathematical approach to help achieve a greater understanding of western music became established. Pythagoras studies found that when two strings with the same length, tension, and thickness, sounded the same when they were plucked, or picked. This means they have a unison sound to the human ear (or consonant), when played together. He also found that if the strings have different lengths (keeping the tension and thickness the same); the strings have a different sound and generally sounds bad (or dissonant) when played together. He also noted that strings having different lengths produce sounds but were consonant rather than dissonant. Pythagoras called the relationship between two notes an interval. Since these discoveries, music containing consonant tones has treated disorders of the ear and epilepsy, sciatic gout and a range of mental disorders [21]. Today, when two strings of the same length are plucked, or picked we say they have the same pitch and, if one string is plucked, or, picked at exactly one-half of the length of the other string, the pitch is doubled and are consonant when played together. This interval is called an octave (harmonic). Furthermore, if one string has a length that is two-thirds the length of the other, the strings again sound consonant when played together and this interval is called a Perfect Fifth. Finally, if one string has a length that is three-quarters the length of the other, the strings again sound consonant, when played together and this interval is a Perfect forth. Hence, the length of the strings being a certain ratio defines interval. Musically speaking the intervals discussed have ratios of: unison (1:1), octave ( $2: 1$ ), a perfect fifth (3:2) and perfect forth (4:3) and so on. The frequency response of the human ear however can only spatially differentiate a limited number of tones within an octave, which are 12 half tones or semitones.

### 2.2 Vincenzo Galilei’s fret fingerboard

In the late Renaissance period, the composer, experimentalist, mathematician, and father to Galileo Galilei, Vincenzo Galilei [ca, 1520-1591], developed Pythagoras linear ideas for string instrument to
perhaps the first non-linear theory of stringed instruments. From his work we get the rule of eighteen i.e., the division of the active vibrational length of the string (string length $(S L)$ ) by 18, to obtain the first fret position (fret ${ }_{1}$ ) on the string fingerboard (equation 1) and dividing the remaining string length by 18 again to get the second fret, and so on. The distance from front of the nut at the headstock to the bridge is defined as $S L$. Today we use the more precise calculation of 17.817 , although the rule of eighteen is still commonly used. This rule places the string octave at the twelfth fret', thus providing an equal temperament between each fret. However, the exact overall length from nut to bridge varies slightly with each string, due to the different mass of each string. In this case, the bridge is orientated at an angle to make a slightly longer sounding length for the lower strings and a shorter one for the high strings, thereby, altering each string scale length minutely to improve intonation across all strings in relation to each other for more accurate tuning when playing up the neck. Equations 1 and 2 help to demonstrate this relationship [22].
fret $_{1}=\frac{S L}{17.817}$
Equation 2 computes the $n^{\text {th }}$ fret position from the front of nut at the headstock.
$\operatorname{fret}_{n}=S L-\left(\frac{S L}{\left(2^{n / 12}\right)}\right)$

The posture of the guitarist is in the seated position with the finger board held by the left hand at about 45 degrees to the horizontal. The area between the thumb Interphalangeal joint and the Metacarpophalangela joints of the left hand are warped around the neck two allow the bottleneck (placed on the ring finger) to act a mobile fret. In this position the left hand is moved up- and -down the fingerboard using the musculoskeletal arm system (with minimal wrist flexion). To produce the rich and complex Delta and bluegrass sound, the index, second and fourth finger do not mute (dampen) the strings. In addition, the guitarist uses a 0.5 mm thick plastic plectrum held in the righthand to down-stroke the cord string while the palm and lower fingers mute (dampen) the remaining strings.


Fig 1 Fret distance from nut plotted on a $\log _{10}$-linear scale with the data points represented by opencircles, fitted with a Microsoft Excel exponential trend-line.

Using equation 2, the fret-offset distance to the nut (fret $=0$ ) can be plotted as log to the base 10 on the horizontal axis against the Fret number, as shown in Figure 1. This example is for a Dubro guitar that has a 19 fret fingerboard with a fixed $S L$ value of 61.2 cm , see experimental section. The exponential trend-line fitted to the data points is associated with the fitting parameter. The trend-line deviation towards fret $=0$ and fret $=19$, indicates that equal temperament is not directly achieved. In practice however, equal temperament is achieved by altering the bridge orientation and string tension as mentioned above.

Unlike non-fretted instruments (violin), the guitar fretted fingerboard allows people with limited musical knowledge to know when to stop at a given target pitch. This is because the additional tactile and visual cues add to the audible cues to provide an all round cognitive feedback system between the guitarist and the sound of the guitar strings when plucked or picked.

### 2.3 Origins of the six-string acoustic guitar

The six-string acoustic guitar as we know it today has its origins in post Braque Europe, in particular in Spain where Antonio de Torres Jurado [1817 to 1892] developed the classical hour glass look and the introduction of the evolutionary "fan" bracing pattern within the guitar's body. Using a circular aperture (hole) in the top plate as the principle mode of acoustic amplification and sound projection (see Helmholtz equation 3) [23], his design improved the volume and tone of the guitar when using the rapidly accepted standard guitar tuning of (lowest pitch, thickest string) E-A-D-G-B- E (highest pitch, thinnest string).
$f_{0} \approx \frac{v}{2 \pi} \sqrt{\frac{A}{V_{o L_{e q}}}}$
Where $f_{o}$ is the resonant frequency of aperture in the guitar top plat, v is the speed of sound (at $20^{\circ} \mathrm{C}, v$ $\approx 343 \mathrm{~m} . \mathrm{s}^{-1}$ ), $A$ is the area of the aperture, $V_{o}$ is the volume of the guitar body and $L_{e g}$ is the equivalent length of the neck plus end correction.
$f$-holes were originally developed for the violin in the Braque period and Antonio Stradivari [1644 to 1737], is widely regarded as having produced the best design in sound projection and pleasing appearance. Later the physicist Félix Savart [1791 to 1841] brought this innovation to the guitar, thereby helping to separate the guitar from its classical roots and gain a new audience in the form of country and jazz. In 2015, a study of $f$-hole sound projection revealed that the axial-length of the $f$ hole rather than its area that determines the acoustic power projection [23].

### 2.4 The steel-string acoustic guitar

The first steel-strings for the banjo and guitar are generally considered to have been offered by Christian F Martin [1796- to 1867] in the mid 1920s, when Hawaiian music became popular in the USA. The union of the steel-strings with the guitar produced a brighter and louder sound that could complete with horns, pianos and drums at mid west American barn dances. Here it's worth noting that a direct and contemporary comparison between the 5 steel-string banjo and the 6 steel-string guitar can be found in the 1972 film Deliverance [24]. The emerging expressive music (Cajon, country, Folk and Bluegrass music) also meant that standard guitar tuning had to change to an open $G$ (lowest pitch, thickest string first) D-G-D-G-B-D (highest pitch, thinnest pitch last) to enable the G major chord (G-B-D) to be strummed on all six strings without the use of the guitarists fret hand, or a capo.

As open $G$ tuning only requires the re-tensioning of only three strings, this new tuning style was readily adopted in bands with a wide spread of music genre. Open $G$ tuning requires the sixth and five strings pitch to be lowered in to $\mathrm{D}_{2}$ and $\mathrm{G}_{2}$ respectively. The next three strings $(4,3$, and 2$)$ remain the same while the first string (1) with the highest pitch and thinnest string is lowered in pitch from $\mathrm{E}_{4}$ to $\mathrm{D}_{4}$. Table 1 tabulates this process, where the last row provides the comparative frequency compression (brightness) of open G tuning with respect to standard tuning.

Table 1 Standard and open G tuning of a guitar with pitch values rounded to the nearest whole number. The shaded rows (string 4, 3, and 2) have no change of tuning.

|  | Standard |  | Open G |  |
| :--- | :--- | :--- | :--- | :--- |
| String | SPN | Pitch $(\mathrm{Hz})$ | SPN $^{*}$ | Pitch (Hz) |
| 6 | $\mathrm{E}_{2}$ | 82 | $\mathrm{D}_{2}$ | 73 |
| 5 | $\mathrm{~A}_{2}$ | 110 | $\mathrm{G}_{2}$ | 98 |
| 4 | $\mathrm{D}_{3}$ | 147 | $\mathrm{D}_{3}$ | 147 |
| 3 | $\mathrm{G}_{3}$ | 196 | $\mathrm{G}_{3}$ | 196 |
| 2 | $\mathrm{~B}_{3}$ | 247 | $\mathrm{~B}_{3}$ | 247 |
| 1 | $\mathrm{E}_{4}$ | 330 | $\mathrm{D}_{4}$ | 294 |
| Frequency range <br> (center point) |  | 248 |  | 221 |
| $(110.5)$ |  |  |  |  |

* Scientific pitch notation (based on 400 Hz ), subscript denotes the octave in which the note is played.


### 2.5 Lap-steel-guitar and Slide-guitar

It is said 'that in the 1890s, Joseph Kekuku [1873 to 1932], accidently strummed a Spanish guitar with a discarded bolt and from that day Kekuku become the inventor of the Hawaiian 'lap-steel-guitar'. This guitar music requires the guitar to be played in a flat and horizontal position across the guitarist's knee. Bolts, nails, back of a pocketknife and steel combs all give a pleasing descendingglissando sound that invokes a vision of Hawaiian palm beaches and rolling surf. Around the turn of the $19^{\text {th }}$ century, the Steel guitar began to be held against the body as in the Spanish style with the guitarist using a metal, or glass cylindrical object worn on the fretting finger. These fretting techniques, known as 'Slide-guitar' in the Mississippi Delta: where in the Deep South, Blind Willie Johnson [25], Elmore James [26] and others developed and popularized Gospel Delta blues and Bluegrass. By the early 1920s, the term bottleneck came in use, due to a common idea that the remnants of broken glass bottles left over from bar room fights were picked-up and played on the guitar frets, and if not up to the task than another bottleneck could be picked-up from the floor and used.

In practices, the bottleneck divides the guitar string into two coupled vibrating string-lengths, with the extreme ends of the two sting lengths fixed and the opposing ends coupled through the damping action point of the bottleneck. When it comes to the sound quality 'slide-guitar' guitarists consider that glass slides offer a smoother playing feel, and produces a warmer and thicker sound that emphasizes the low to mid overtones within the harmonic series compared to metal slides that give a longer sustain that is also brighter and harsher [16].

## 3. Experimental

This study firstly investigates the sound generated by the Dubro DM-33 Hawaiian resonator guitar (figure 2a). The name 'Dubro' is a portmanteau of 'Dopyera Brothers' who invented this type of resonant guitar. The guitar has a chrome-plated brass metal body with sandblasted palm trees, two rolled $f$-holes (axial length $112 \mathrm{~mm} \times 15 \mathrm{~mm}$ at their widest point), and a 19 -fret rosewood fingerboard with pearl dot inlays (figure 2a). The average scale length of the strings is 61.2 cm , and the string action is 3.5 mm , to minimize accidental fret notes. The Dubro is a relatively complex string instrument, compared to the classical acoustic guitar, where the primary mechanical sound amplification is produced by a 26.7 cm diameter outward facing resonator cone / diaphragm, at the top of which is attached a biscuit bridge (figure 2 b and R. Dopyera US patterns [ 9,10$]$ ). The purpose of the cone is twofold, 1), to project the string vibrational sound out and away from the guitar and 2), send part of the sound in to sound-well and out via body ports. The two main ports being rolled $f$ holes that are set symmetrically set either side of the strings. Using this arrangement, the cone produces a harsh mid-high frequency range ( 1 kHz and above) while the $f$-holes project sound energy in the low to mid frequency range 70 to 150 Hz . The mechanical complexity of the Dubro does mean regular carful maintenance and regular tune-up is required. A range of short videos of resophonic guitar tuning can be found on YouTube, see for example [27, 28].

### 3.1 Recording of the Dubro guitar

For this study, the digital recordings of the acoustic resophonic guitar were made during a performance in Kastollos, Crete in August 2020. The guitar radiated sound is recorded using a Zoom H 4 n handy recorder (frequency response $\sim 30 \mathrm{~Hz}$ to 22 kHz ) positioned one meter in front of the guitar. The sound levels were set using an alto ZMX122FX mixer and the track recordings saved in waveform audio files on a SD card. The choice of a microphone rather than an electric pick-up is deliberate as this gives both acoustical and perceptual information of the guitar sound as played by the guitarist.


Fig 2a-b Photo of the Dubro DM-33 resonator guitar a). Cross-section of Dubro cone / resonator diaphragm based on R. Dopyera's 1932 US patent b).

The posture of the guitarist is in the seated position with the fingerboard held by the left hand at about 45 degrees to the horizontal. The area between the thumb Interphalangeal joint and the Metacarpophalangela joints of the left hand are warped around the neck two allow the bottleneck (placed on the ring finger) to act a mobile as the left hand is moved up- and -down the fingerboard using the musculoskeletal arm system (with minimal wrist flexion). The bottleneck divides the strings into two vibrating portions that are designated as string bridge $\left(S_{b}\right)$ and string nut $\left(S_{n}\right)$ respectively. The guitarist may choose to mute (dampen) $\mathrm{S}_{\mathrm{n}}$ to generate a crystal-clear tone as in the case of Freddie's swoop in the opening seconds of 'Looney Tunes' (figure 3a) or un-mute (figure 3b) to provide a rich and complex sound that is character of Delta blues. Steel and glass bottleneck slides are used in the recordings. The steel has a length $=51 \mathrm{~mm}$, inside diameter $=19 \mathrm{~mm}$ and outside diameter $=26.5 \mathrm{~mm}$ and glass has a length $=70 \mathrm{~mm}$, inside diameter $=20 \mathrm{~mm}$ and outside diameter $=2.5 \mathrm{~mm}$ ). In the following text the slides are designates as s-slide and g-slide. In addition, the guitarist uses a 0.5 mm thick plastic plectrum held in the right-hand to down-stroke the cord string while the palm and lower fingers mute (dampen) the remaining strings


Fig 3a-b Cross-section schematic of slide and fingers in the $\mathrm{S}_{\mathrm{n}}$ muted position a). Cross-section schematic of slide and fingers in the $\mathrm{S}_{\mathrm{n}}$ un-muted position b).

Track transcription is performed within a Lenovo laptop running Microsoft Windows 10, therefore the xxx.wav files are fully combatable with Microsoft's Resource Interchange File format (RIFF) specification. Audacity ${ }^{\circledR}$ version 2.4.2 (a free, open-source, cross platform audio software) is used to transcribe the guitar sound recordings [29]. The software uses a sampling rate of 44100 Hz with a dynamic range of 32 -bit float to provide a coupled time-domain and spectrogram (3-D plot of sound intensity (color) as a function of frequency and time) of the selected audio track recording. Frequency spectrum analyzer is also available. Table 2 provides the basic metadata for these displays.

Generally, open-access spectrometer software is limited in its ability to provide real-time frequency analysis due to the latency within the software. The latency is because of the lack of processing power to handle the large amount of time-series data that is needed to be converted into the frequencydomain using a Fast Fourier Transform (FFT) algorithm. To overcome this problem Audacity toolbox contains a set autocorrelation algorithms used to identify the SPN frequencies. This option measures how many times SPNs are repeated within the selected waveform record length. This is achieved by taking two copies of the waveform data set, and moving one waveform data set piecewise ( $\mathrm{n}=1$ ) followed by multiplying the two waveform data sets together. The piecewise process is repeated, up to the selected size option. This mathematical noise reduction tool is one of many embedded delay timeseries analysis tools used in chaos theory to extract periodic signals (overtones, octave, and harmonics) out of incoherent noise [30].

Table 2 Audacity software project rate and display information.

|  | Lower <br> Frequency <br> $(\mathrm{Hz})$ | Upper <br> Frequency <br> $(\mathrm{kHz})$ | Sampling rate <br> $\left({\left.\mathrm{s} . \sec ^{-1}\right)}\right.$ | Display video <br> bandwidth <br> $(\mathrm{Hz})$ | Screen shot |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Waveform | N/A | N/A | 44100 <br> 32 -float | N/A | Yes |
| Spectrum <br> analyzer | 30 | 22 | 44100 <br> 32 -float | N/A |  |
| Spectrogram <br> default <br> maximum | 30 | 8 | 44100 <br> $32-f l o a t ~$ | 50 | Yes |

In the case of the spectrogram, a noise reduction algorithm (NRA) uses the FFT with a Hann window to sample the local neighborhood noise to obtain an incoherent noise profile. Subtracting the noise profile from the whole of spectrogram leaves the coherent acoustic signature of the resophonic guitar. Three operator parameters (amplitude, sensitivity, and frequency smoothing bands) settings determine the impact upon the guitars acoustic signature and the surrounding noise floor. The RNA is used here to estimate of the specific incoherent noise contribution for ascending and descending glissandos, rather than to clean-up the musical signature of the guitar. In mathematical terms this noise reduction technique is called spectral noise gating [31] and is used the compare the SPN and glissando modes of
the bottlenecks. Other pixel thresholding methods may be applied using different software platforms, such as LabVIEW [32].

## 4. Benchmarking

Viewing the stereo sound tracks from the recordings, revealed that there was no different in X - and Y tracks presumably this because the closeness of the microphones to each other $(0.01 \mathrm{~m})$, with respect to the guitar position $(1.5 \mathrm{~m})$, even though microphones have an XY orientation. Given this, only the X-tracks are used. The purpose of the Benchmarking the plectrum down-stroke is to establish both the acoustic signature of the guitar and the guitarist's gesture. Two Benchmarks are made, 1) strumming open $G$ and 2 ) the first-string triad (strings: $3,2,1$ ). The first-string triad is frequently used in the Rock ' $n$ ' roll genre [33] and therefore is included in this study. The knowledge gained from the Benchmarking informs the identification process of ascending and descending glissandos and incoherent noise.

### 4.1 Open G tuning Benchmark

To establish the plastic plectrum down-stroke Benchmark, the guitar is strummed, and recorded, for 35 seconds. An initial analysis of the total waveform record-length yielded an average BMP of 144. A more detailed standard autocorrelation of a 2.2-seconds period encompassing both down and up cords yields the tones and overtones. To produce the greatest definition, the autocorrelation algorithm is set with a Hann window and sample size of 2048.

Figure 4 provides the computation where the correlation delay time is on the horizontal axis and SPN level on the vertical axis. In this representation and figures 7 and 8 , frequency decreases to the right, therefore the root tones are to the right and the higher overtones progress to the left. Note, the delay time 0.01 and 0.025 corresponds to the at-rest human heart beat range ( 60 to 100 BPM ). Using this representation, the tones $G_{1}, B_{1}$, and overtones $C_{2}, D_{2}$ and $G_{2}$ fall within the at-rest heat beat range, and the higher overtones $\left(\mathrm{B}_{2}\right.$, and $\left.\mathrm{D}_{3}\right)$ are in the +38 BPM elevated / stressed human heart beat range.


Fig 4 Standard autocorrelation of the 2.2 second Benchmark. The root tones are to the left and the higher overtones to the rights. The human heart beat range is between $G_{1}$ and $D_{3}$.

### 4.2 First-string triad Benchmark

The plastic plectrum down-stroke of the first-string triad (strings: 3, 2 and 1), with the bottleneck slides damping the fifth fret produces corresponding values of $s_{b} \sim 45 \mathrm{~cm}$ and $S_{n} \sim 15.3 \mathrm{~cm}$, respectively. In this style of strumming, the second-string triad (strings: 6,5 and 4) are damped by the guitarist palm. This procedure changes the open G cord by five semitones without changing the original open G tuning.

Figure 5a-d shows two typical strumming cord acoustic waveforms (5a-b) and their associated default spectrogram ( $5 \mathrm{c}-\mathrm{d}$ ) for both s-slide and g-slide positioned on the fifth fret. In the case of the waveforms, there are two features of note. First is the truncation of the first and second envelopes by the third envelope that fades out to completion. Note also and that the s-slide fades by an additional 0.5 seconds compared to the $g$-side. Second, for both s-slide and g-slide, a string speak caused by an involuntary guitarist gesture is present in the first envelope (plectrum annotation in figure 5a-b). Note also, string squeaks are not found in the second or third envelope. In addition, at the end of the completed cord, a 10 to 20 Hz non-complex resonance is present. Not shown in these figures, but shown in figure 6, are further examples of complex resonances that appear at the start of additional recorded cords. The resonances have similar timestamps to the guitarist applied bottleneck pressureon (p-on) and release (here called pull-off), thus bracketing the cord.


Fig 5a-d Waveforms and spectrograms obtained from the first-string triad with bottleneck damping on the fifth fret: s-slide $4 \mathrm{a}-\mathrm{c}$, g -side 4 b -d.

The two default spectrograms in figure 5 c -d provide a greater insight to the fifth fret bottleneck position. To aid the reader's eye, the psychoacoustic terms: warm ( 0 to 2 kHz ), bright ( 2 to 4.5 kHz ) and brilliance ( 4.5 to 8 k Hz , which extend to 22 kHz , see section 6) are located on the right-hand
frequency axis of the spectrograms. Within the two spectrograms there are five features of note, these are listed as follows:

1. The bottleneck p-off points at the end of the cord are located at the lower-end of the warm region.
2. An intermodulation of tones are observed in the psychoacoustic brilliance region that are caused by energy being transferred up- and down- in frequency range where addition and subtraction of consonant and dissonant tones result in fading in-and-out in the higher frequency range. Unlike electromagnetic signals, the origin of the acoustic energy (in this case the strings and body of the guitar) is directly altered by the vibration mode of the strings and body, and the medium that the sound is traveling through. Thus, each pitch has a non-zero bandwidth [34, 35] that periodically fads when subtraction occurs.
3. A series short rising tones of approximately +0.5 kHz (blue arrow annotation) that have an initial timestamp corresponding the picking of the strings. A second set of descending glissandos of approximately -2 kHz (red arrow annotation) are launched after the raised tones are also present.
4. There is a marked difference in the $s$-slide and the $g$-slide sustain periods within the warm and bright regions. In the warm region, the s-slide produces a stronger spectral density compared to g -slide. However both slide produce similar short sustain periods.
5. The string squeaks caused by an involuntary guitarist gesture appear mixed and muddled within the warm region.


Fig 6 Typical bottleneck pressure-on and pull-off signatures.
Figure 7 and 8 shows the standard autocorrelation (Hann window and 2084-sample size) of the waveform for the s -slide and g -slide damping the fifth fret. Note for clarity, the sound level of the second and third envelopes are offset by +100 , and +200 , respectively. For each envelope, the major overtones and the root along with prominent flat (dissonant) overtones are labeled.

In figure 7 (s-slide), the first envelope overtones $\mathrm{E}_{3-4}$ and $\mathrm{G}_{3-2}$ and $\mathrm{C}_{3-2}$ are clearly defined, along with the flat (dissonant) $\mathrm{F} \#_{4-5}$ and $\mathrm{A}_{3}$. Note also $\mathrm{G}_{2}$ transposes up in pitch to $\mathrm{A}_{2}$ within the duration of the envelope period. In the second envelop the $D_{4-2}$ and the $B_{3-2}$ are present, in addition $A \#_{1}$ transposes up in pitch to the root $\mathrm{B}_{1}$, and $\mathrm{D}_{2}$ transposes up in pitch to $\mathrm{E}_{2}$ within the envelope period. The $\mathrm{F} \# 5$ is also
present. In contrast, the third envelope contains the major overtones $\mathrm{C}_{4-2}, \mathrm{E}_{3}, \mathrm{~A}_{2}$ and $\mathrm{F}_{2}$, with the flat (dissonant) tones less prominent.


Fig 7 First-string triad (3, 2, and 1) with s-slide damping the fifth fret.
Figure 8 reveals that the first envelope contains the overtones $G_{5-2}, \mathrm{E}_{2-3}$, and the roots $\mathrm{B}_{1}$ and $\mathrm{A}_{1}$, along with the flat (dissonant) tones $\mathrm{F}_{4}$ and $\mathrm{GH}_{2}$. In the second envelope, $\mathrm{D}_{4-2}$ and $\mathrm{B}_{3-1}$ are present along with $\mathrm{GH}_{5}$ and $\mathrm{F}_{3}$, but at reduced amplitude compared to the s-slide. Again, in contrast, the first two envelopes, the envelope exhibit the major tones $\mathrm{C}_{4-2}, \mathrm{E}_{3}, \mathrm{~A}_{2}$ and $\mathrm{F}_{2}$ ), with the flat (dissonant) tones less prominent.


Fig 8 First-string triad (3, 2, and 1) with g-slide damping the fifth fret.

## 5. Ascending and descending glissandos

This section looks at the glissando sound production between the seventh and twelfth fret for open G tuning and different bottleneck material (steel and glass). Using equation $2, \mathrm{~S}_{\mathrm{b}}$ therefore varies between approximately 30 and 40 cm , and $S_{n}$ varies between approximately 20 and 30 cm . For ease of comparison, spectrograms of a first-string triad ascending glissando using the s-slide is presented followed by two pairs of comparative 'Looney Tunes' and it counterpart tracks.

Figure $9 \mathrm{a}-\mathrm{b}$, depicts the default spectrograms for first-string-triad with steel and glass bottleneck for the descending glissando (twelfth to seventh fret). Annotated on the right-hand vertical axis is the warm, bright and brilliance regions and the horizontal dashed-lines (at 2 and 4.5 kHz ) delineate the regions. Within these two spectrograms, three contrasting features are observed and are listed as:

1. The root and overtones within the warm psychoacoustic region have differing sustain lengths, where the s-slide produce longer and stronger tones compared to the g-side.
2. As the slides physically moves perpendicular across the strings (at rate of approximately 50 $\mathrm{mm} . \mathrm{s}^{-1}$ ) a mirrored bifurcation occurs where the glissandos have an exponential trajectory with a frequency shift of approximately 2.2 kHz with time. These mirrored glissandos extend through the bright region and fades into the brilliance region.
3. There is a marked and contrasting noise floor between the two bottlenecks? In the case of the gslide, a greater incoherent (hiss-like [19]) noise is present at the lower end of the warm region ( 0 to 0.5 kHz ) as compared to the s-slide bottleneck. Section 6 further quantifies these noise features.

Time (s)
Time (s)


Fig 9a-b First-string triad: s-slide descending glissando (twelfth to seventh fret) for s-slide 9a, and gslide 9 b .

Figure 10a-b depicts the default spectrogram for the 'Looney Tunes' ascending glissando (seventh to twelfth fret, 10a) and its counterpart descending glissando (twelfth to the seventh fret, 10b). Both spectrograms are for s-slide. Again, the psychoacoustic regions are annotated on the right-hand vertical axis. The spectrograms reveal a number of contrasting features.


Fig 10a-b 'Looney Tunes's-slide ascending glissando (seventh to twelfth fret) 10a; and s-slide descending glissando (twelfth to seventh fret) $10 b$.

1. The inclusion of the thicker strings $(4,5$, and 6$)$ generates a high-frequency content that ranges to the top of the bright psychoacoustic region.
2. Now as the slides physically moves perpendicular across the strings at a rate of approximately 50 $\mathrm{mm} . \mathrm{s}^{-1}$ mirrored bifurcation of the glissando occurs. As in figure $9 \mathrm{a}-\mathrm{b}$, the frequency shift is some 2.2 kHz , however in this case the glissando extend through the bright and well in to the brilliance region. To separate apart these mirrored glissandos it is reasonable to assigned the string ascending glissando with increasing fret number, hence the mirrored glissando is assigned to the slip-stick friction process of the slide.
3. The noise floor at the lower-end of the warm ( 0 to 0.5 kHz ) region is raised with incoherent, or hiss-like, noise. For comparison, see figure 9 a .

Figure 11a-b provides the default spectrogram for the 'Looney Tunes' ascending glissando (seventh to twelfth fret, 10a), and its counterpart (twelfth to the seventh fret, 10b). Both spectrograms are for the g-slide. Again the right-hand vertical axis depicts the psychoacoustic regions. Main features of note are:

1. As in figure $10 a-b$, mirrored bifurcation of the glissandos produce exponential trajectories as the slide moves perpendicularly across the strings at a rate of approximately $50 \mathrm{~mm} \cdot \mathrm{~s}^{-1}$.
2. Incoherent, or hiss-like, noise is prominent has marked increase in lower-end of the warm region ( 0 to 0.5 kHz ) as compared to the s-slide (figure 10a-b).
3. Taking figures $9 \mathrm{a}-\mathrm{b}, 10 \mathrm{a}-\mathrm{b}$ and $11 \mathrm{a}-\mathrm{b}$ together, psychoacoustic feature of ascending and descending seventh to twelfth glissando may be summarized. Firstly, the sound of the first-string triad slide extends to the bright region, whereas the thicker strings extend the guitar response in to brilliance region. Secondly, pronounced mirrored glissandos are produced when all six strings are played with the slides. Third, incoherent, or hiss-like, noise in the lower-end of the warm region is produced by the g-slide first-string triad mode, and when all strings are played using
either the s-slide or g-slide.
4. For a slide acoustic guitar, Pakarinen et al [19] has demonstrated that slide divides the damped string into two longitudinal excited string segments, where the main sound originates from the slide to bridge string segment (here labeled $\mathrm{S}_{\mathrm{b}}$ ) and second excitation originates from the slide to nut (here labeled $S_{n}$ ) segment. This excitation process appear to hold in the resophonic guitar, although a string portion from bridge to the tailpiece must vibrate due to the up-and-down motion of bridge, albeit a smaller bandwidth of $S_{b}$ and $S_{n}$. Given this scenario, vibrational energy is continuously flowing between the string segments as slide moves across the string. Following this, it is reasonable to assign the origin of the mirrored exponentially varying pitch glissandos. Hence, an ascending glissando associated with increasing fret number (7-12) originates in $S_{b}$, whilst the mirrored descending glissando has it origin in $\mathrm{S}_{\mathrm{n}}$.

Pakarinen et al [19] has also identified incoherent, or hiss-like, noise in the steel-string acoustic guitar and assigned this noise to contact points as the side moves across the string. When they synthesized this form of noise they used a noise pulse train thereby evoking an impact and friction modal, otherwise known as slip-stick friction between the surface of the string and slide / Bow [36]. The lowfrequency nature of the noise also suggests there is Voss-Clarke flicker noise ( $1 / f^{\alpha}$ noise) content [8]. Section 6 further explores this psychoacoustic noise for the resophonic guitar.

Time (s)
Time (s)


Fig 11a-b 'Looney Tunes' g-slide ascending glissando (seventh to twelfth fret) 11a; and g-slide descending glissando (twelfth to seventh fret) 11 b .

## 6. Resophonic guitar upper psychoacoustic brilliance ( 0 to 22 kHz ) region

Given the lack of full range psychoacoustic data for the resophonic guitar, this section looks at the 'Dubro' resophonic guitar's radiated sound in the 0 to 22 kHz frequency range to understand the interaction and delineation of each psychoacoustic region. This is achieved by using the Audacity spectrogram with a selected full frequency range ( 22 kHz ) for 'Looney Tunes' ascending s-slide (11a) and g-slide glissando (11b).


Fig 12a-b. Extended frequency range of 'Looney Tunes' g-slide ascending glissando (seventh to twelfth fret) 12a, and g-slide ascending glissando (twelfth to seventh fret) 12b.

Figure 12a-b provides the comparison between the s-slide and g-slide ascending glissando. In the case of s-slide (12a), the ascending glissando overtones extend through the warm region with typical sustain periods of 3 seconds and to a lesser extent ( 0 to 2 seconds) in the bright region. Whereas the initial plastic plectrum attack overtones have sustain periods of typically 0.5 seconds throughout the 4.5 to 22 kHz brilliance region. In addition there is some evidence of weak glissando overtones with typical sustain periods of 1 second. In comparison, the $g$-slide (12b) produces weaker sustain periods in all three psychoacoustic regions. The least marked being in the brilliance region where the initial plectrum attack overtones have sustain periods decaying from 1 seconds at 4.5 kHz to 0.25 second at 22 kHz . Within the decay process, the ascending glissandos overtones also become less pronounced.

## 7. Noise reduction

Figure $9 \mathrm{a}-\mathrm{b}$ has revealed, that for a first-string triad ascending glissando, the g -slide induces more incoherent (or hiss-like) noise at the lower-end of the warm region, when compared the s-slide. The aim of this section is therefore threefold: First to isolate and remove the incoherent noise to, or below, the noise floor of s-side glissando, thereby providing an estimate of the noise contribution. The second is to extend the noise reduction knowledge to the 'Looney Tunes' (figure 10a and 11a) and the counterpart descending glissandos (figures 10 b and 11b). Third, identify and map the characteristic morphology of the noise [8].

### 7.1 First-string triad bottleneck noise reduction

The first step in estimating the incoherent noise contribution is to identify and isolate the noise. This is performed by first defining the noise profile ( np ) within the spectrogram (13b). The selection criterion is based on that incoherent noise contains random pixel variables with a well-defined statistical characteristic as compared to the coherence pixel regions of glissando.

The removal step uses three parameters to control the noise reduction process. These parameters are noise reduction level, sensitivity, and frequency-smoothing band. The noise reduction controls the volume reduction (in dB ) applied to the noise. The sensitivity controls the amount the signal to be considered as noise (using a scale of 0 to 24 ). In addition, the frequency-smoothing band controls the
spread of the smoothing in neighboring bands, therefore altering the original sampling rate (using a scale of 0 to 12 and is set to zero so that direct comparison between the original and modified dataset is made). [N.B. Further details on how the three parameters are used, see Audacity software [29, 30, 31]]. A series of iteration processes follows, where the noise reduction value and sensitively value is changed, with the aim of reducing the incoherent noise with minimal damage to the coherent glissando feature within the spectrogram. Figure 13a-d shows the overall process in spectrogram format where figure 13a is the first-string triad for the s-slide (taken from figure 9a). Figure 13b is the first-string triad for the g -slide along with the incoherent noise profile region selected. Figure 13 c is the noise-reduced image using a noise reduction value of 12 dB and a sensitivity value of eight. A comparison of figure 13 c with the s-slide (13a) reveals similar coherent features and the incoherent spectral densities for both slides is in the 0 to 0.4 kHz range. Thus the indicating the g -slide incoherent noise contribution is in the order of 12 dB .

Figure 13d, depicts the removed residue noise spectrogram in the low-frequency region of the acoustic spectrogram. It is noted that the isolated noise inevitably captures part of the overtone structure, and therefore some of overtone in figure 13 c is lost. The overall discrimination process may not be perfect, but it is more beneficial in this case when compared to a low-pass filter that would remove higher frequency noise in the bright and brilliance regions [29-32].


Fig 13a-d First-string triad ascending glissando spectrogram for s-slide 13a. First-string triad ascending glissando for g -slide 13 b . The g -slide reduced noise profile after noise reduction ( -12 dB ) 13 c . The g -slide residue noise spectrogram 13 d .

### 7.2 Incoherent noise reduction

To quantify the visible incoherent noise in the audio spectrogram figures 10a-b and 11a-b, the same attenuation process as in figures 13b-d is undertaken. To allow a direct comparison throughout, only the noise attenuation (dB) is altered, whist the sensitivity or frequency-smoothing band values are fixed at eight and zero, respectively.

Table 3 depicts the required incoherent noise reduction values to achieve the desired noise floors for each spectrogram. The results support the general concept that a g -slide produces more ( 3 dB ) incoherent, hiss-like, noise than an s-slide. In addition, a descending glissando (twelfth to seventh
fret) also produces more incoherent. This finding suggests that the guitarist gesture whether emotional or and musculoskeletal (movement of the upper extremity as the left moves away from the body when play a descending glissando) may also have a role in the production of slide noise.

Table 3 Incoherent noise reduction algorithm variable parameter values.

| Spectrogram figure | Noise reduction $(\mathrm{dB})$ | Sensitivity level | Frequency-smoothing <br> band |
| :--- | :--- | :--- | :--- |
| 10a (s-slide: $7-12$ fret) | 9 | 8 | 0 |
| 10b (s-slide: $12-7$ fret) | 12 | 8 | 0 |
| 11a (g-slide 7-12 fret) | 9 | 8 | 0 |
| 11b (g-slide: $12-7$ fret) | 12 | 8 | 0 |

### 7.3 Characteristic noise morphology

Using the Audacity FFT algorithm, the five-residue noise datasets obtained in section 7.1 and 7.2 are analyzed for their spectral morphology (color). Figure 14 depicts the FFT results as $\log -\log$ plot, where frequency $(\mathrm{Hz})$ plotted on the horizontal-axis and the sound amplitude $(\mathrm{dB})$ plotted on the vertical-axis. In this representation, all five datasets exhibit a $1 / f$-like response: e.g. -6 dB per 10 Hz in the 10 to 20 Hz frequency band and -7 dB per 10 Hz in the 30 to 150 Hz frequency band. Note the 6 dB roll-off in 10 to 20 Hz band is most likely an artifact of the microphone cut-off frequency. In addition, the structures above 150 Hz are the captured coherent portions of the glissandos and are not considered further as they are not the primary interest here.


Fig 14 Frequency spectra of all five residue datasets (see figure 13d and Table 3).
Using the first-string triad (Fst, solid black line) as a comparative control, remaining four six-string triads are partitioned around the control. Where the descending glissando for both steel and glass produce the greatest residue noise and therefore are above the control. The opposing ascending glissandos produce the least noise and therefore are positioned below the control. The limited measurements present here appear to indicate that the direction of the slide movement along the fingerboard determines the relative residue noise level, also. One possible cause for this differentiation in the musculoskeletal locomotion force required to extend and retract the guitarist fret arm [37, 38], as similarly observed in violinist [39]. In the case of guitar descending glissando, the guitarist musculoskeletal systems extends the left arm, hand and hence bottleneck from the twelfth to seventh fret so altering the body's center of gravity from the seated position (and vice-versa for the ascending glissando). These varying locomotion conditions are known to induce ulnar nerve entrapment, and therefore merit further investigation.

## 8. Discussion

This paper has presented a study of the 'Dubro' resophonic guitar psychoacoustic response to a plastic plectrum applied in the down-stroke for ascending and descending glissando where both steel and glass slide are used. The 'Dubro' is chosen for its enhanced mechanical sound amplification as compared the classic acoustic and electrical guitar. The slide is placed on the ring finger on of the left hand with the index, second and fourth finger not used to mute (dampen) the strings. This style of slide play provides the rich and complex guitar Delta bluegrass sound. The work has focused on a guitarist's gesture (rather than a mechanical modal based method [17, 18]) to help provide the human psychoacoustic perception of the 'Dubro'.

The measured radiated sound recordings extend through the psychoacoustic warm ( 2 to 4.5 kHz ), bright ( 4.5 to 8 kHz ) and brilliance region up to a frequency of 22 kHz where the initial attack, rather than chord overtones are present. It is worth noting that online commentary declares that the resophonic guitar brilliance may extend to 20 kHz [16]. As the guitarist musculoskeletal system physically moves the slide perpendicular across the full six strings (at approximately $50 \mathrm{~mm} . \mathrm{s}^{-1}$ ) between the seventh to twelfth fret, glissando overtones are generated that instantly undergo mirrored bifurcation forming two exponential trajectories: one decreasing in pitch and the other increasing in pitch.

The glissando overtones extend throughout the bright psychoacoustic region and the lower brilliance region for both s -slide and g -slide. In the case of the s -slide, the overtones are weakly present in the mid brilliance region ( 8 to 15 kHz ). The exponential trajectory of the glissandos as the slide traverses perpendicularly across the strings demonstrates Vincenzo Galilei's non-linear theory of fretted string instruments (equation 1 and 2 and graph 1).

Fading in the brilliance psychoacoustic region is also indentified, and is attributed to intermodulation (or, addition and subtraction) of overlaying consonant and dissonant tones. Due to the inner ear's inability to separate high pitch overtones, a listener may perceive the fading process as roughness or timbre in the guitar overall sound. Incoherent, hiss-like, noise is identified and shown to be associated with the slip-stick friction processes between the moving slides and the vibrating strings, where the intensity of the noise is more pronounced on the thicker (wound) steel strings. The g -slides produce a higher intensity (some 3 dB ) incoherent, hiss-like, noise than the s -slide.

The direction of slide movement is observed to produce a variation in the amplitude of the incoherent, hiss-like, noise. Slide movements away from the gustiest body centre of gravity (associated with a descending glissando) produce an increase in noise amplitude. As musculoskeletal pain and stress in string instrument players is common [37-39] the measured noise may be a significant finding. This aspect of the work requires further research both in guitarist gesture and in mechanical based modals.

The time-domain and frequency-domain information presented here provides control-data (slide contact gestures) for improved slide-music and slide-noise synthesis within virtual slide guitar systems as reported by Pakarinen, Puputti and Välimäki 2008 [19]. In their work an Omni-directional contact-noise building block is used that did not differentiate the direction of slide movement. Our new work (this paper) indicates that a bidirectional contact-noise building block should be used to synthesize possible differences in musculoskeletal induce noise.

To conclude this work, the opening seconds of the original Warner Brothers instrumental theme 'Looney Tunes' as played by F. Travares is used as a control. In the original Travares recording, muting (damping) of the strings is performed to make the non-complex (crystal-clear) sound. Our findings reveal that the mirrored bifurcation is present when the strings are not muted. This finding supports the two vibrating string portion mechanism when the slide employ with muting and damping.

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## Isoscattering chains of graphs and networks

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#### Abstract

In a recent paper by Michał Lawniczak, Adam Sawicki, Małgorzata Białous, and Leszek Sirko, Scientific Reports 11, 1575 (2021) [1] the isoscattering chains of quantum graphs possessing $n$ units and $2 n$ infinite external leads were identified and discussed. It was shown that their isoscattering properties are preserved for $n \rightarrow \infty$. The theoretical predictions were confirmed experimentally using $n=2$ units, four-leads microwave networks. Here we extend the experimental analysis presented in Ref. [1] to higher frequency range $\nu=0.01-2 \mathrm{GHz}$. The studied problem generalizes a question of Mark Kac "Can one hear the shape of a drum?", originally addressing isospectral dissipationless systems, to the case of infinite chains of open graphs and networks with dissipation.


## I. INTRODUCTION

The famous question of Marc Kac "Can one hear the shape of a drum?" [2] was posed to addresses the problem of isospectral drums having the same area and perimeter. From the mathematical point of view the question is reduced to the uniqueness of spectra of the Laplace operator on the planar domains with Dirichlet boundary conditions. The negative answer to the above question was given by Gordon, Webb, and Wolpert [3, 4] who using Sunada's theorem [5] constructed different in shape pairs of isospectral dissipationless domains in $\mathbb{R}^{2}$. These important theoretical findings were confirmed experimentally by Sridhar and Kudrolli [6] and Dhar et al. [7] using specially designed pairs of isospectral microwave cavities. The isospectral properties of neutrino billiards which are known to be isospectral in the nonrelativistic limit have been recently investigated numerically in Ref. [8]. It was shown that the isospectrality of the billiards was lost when changing from the nonrelativistic to the relativistic case.

In the case of quantum graphs, i.e. the union of vertices connected by one-dimensional bonds [9], the problem of isospectrality was analyzed by Gutkin and Smilansky [10]. They proved that the spectrum uniquely identifies the graph if the lengths of its bonds are incommensurable. However, in the case of commensurate lengths of bonds there do exist graphs with different topological properties which are isospectral. A general method of construction of isospectral dissipationless graphs was presented in Refs. [11, 12]. It uses representation theory arguments and the transplantation technique which assigns to every eigenfunction of the first graph an eigenfunction of the second one with the same eigenvalue.

In open physical systems, including quantum graphs with leads [13] and microwave networks, one deals with dissipation of energy due to, e.g., internal absorption and coupling to the outside world. In such a case one can pose a more general question whether the geometry of a graph can be determined in scattering experiments. Again the question was answered in negative. Band, Sawicki and Smilansky [14, 15] analyzed finite isospectral quantum graphs with attached two infinite leads and constructed pairs of graphs which are called isoscattering. This theoretical finding was experimentally confirmed in the series of papers [16-18] with two isoscattering microwave networks simulating a pair of isoscattering quantum graphs with two external infinite leads. The experimental results were based on characteristics of graphs such as the cumulative phase and the structures of resonances and
poles of the determinant of the two-port scattering matrices.
In this article we discuss the chains of isoscattering open quantum graphs and microwave networks that are constructed from $n$ building blocks (units), each one possessing two external leads, where $n$ changes from 1 to infinity. The theoretical predictions are confirmed experimentally for $n=2$, i.e., for four-leads microwave networks.

## II. ISOSCATTERING GRAPHS

In Ref. [1] it was proved that the two chains of open graphs $\Gamma_{1,2 n}$ and $\Gamma_{2,2 n}$ given in Fig. 1(a) and Fig. 1(b) are isoscattering. To do this the transplantation matrix $\hat{T}_{2 n}$ was constructed that transforms a wave function $\hat{\Psi}_{1,2 n}$ with the frequency $\nu$ defined on $\Gamma_{1,2 n}$ to a wave function $\hat{\Phi}_{2,2 n}$ with the same frequency $\nu$ defined on $\Gamma_{2,2 n}$

$$
\left(\begin{array}{c}
\Phi_{2,1}  \tag{1}\\
\vdots \\
\Phi_{2,2 n}
\end{array}\right)=\hat{T}_{2 n}\left(\begin{array}{c}
\Psi_{1,1} \\
\vdots \\
\Psi_{1,2 n}
\end{array}\right)
$$

If we put

$$
\begin{gather*}
\Phi_{2,2 n}=\Psi_{1,2 n-1}+\Psi_{1,2 n}, \quad \Phi_{2,1}=\Psi_{1,1}-\Psi_{1,2},  \tag{2}\\
\Phi_{2, k}=\Psi_{1, k-1}-\Psi_{1, k+1}, \quad \text { for } k \in\{2, \ldots, 2 n-1\}, \tag{3}
\end{gather*}
$$

then $\hat{\Phi}_{2,2 n}$ satisfies the vertex conditions of $\Gamma_{2,2 n}$ provided that $\hat{\Psi}_{1,2 n}$ does the same on $\Gamma_{1,2 n}$. In such a case the transplantation matrix $\hat{T}_{2 n}$ is given by

$$
\hat{T}_{2 n}=\left(\begin{array}{cccccccc}
1 & -1 & 0 & & & & &  \tag{4}\\
1 & 0 & -1 & 0 & & & & \\
0 & 1 & 0 & -1 & 0 & & & \\
& 0 & 1 & \ddots & \ddots & \ddots & & \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \\
& & & 0 & 1 & 0 & -1 & 0 \\
& & & & 0 & 1 & 0 & -1 \\
& & & & & 0 & 1 & 1
\end{array}\right)
$$

The existence of transplantation warrants that the $2 n \times 2 n$ scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$ of the open graphs $\Gamma_{1,2 n}$ and $\Gamma_{2,2 n}$ are conjugated through the matrix $\hat{T}_{2 n}$,

$$
\begin{equation*}
\hat{S}^{(I)}=\hat{T}_{2 n}^{-1} \hat{S}^{(I I)} \hat{T}_{2 n} \tag{5}
\end{equation*}
$$

## III. MICROWAVE NETWORKS

The simulation of quantum graphs by microwave networks is possible because of a direct analogy between the telegraph equation describing a microwave network and the Schrödinger equation of the corresponding quantum graph [19-21]. Microwave networks allow for the simulation of quantum systems described by three ensembles in the random matrix theory (RMT): the Gaussian orthogonal ensemble (GOE) [16, 19, 22-25] and the Gaussian symplectic ensemble (GSE) [26] characterized by $T$-invariance as well as the Gaussian unitary ensemble (GUE) [19-21, 27] without $T$-invariance. Microwave networks can be successfully used to investigate properties of quantum graphs with complex topology and large absorption [20, 22, 24, 26, 28].

Microwave networks which are applied to simulate quantum graphs with preserved time invariance symmetry consist of microwave vertices connected by edges - coaxial cables. Each vertex $i$ of a network is connected to the other vertices by $v_{i}$ edges, $v_{i}$ is called the valency of the vertex $i$. The coaxial cables (SMA-RG402) used in the construction of microwave networks consist of an inner conductor of radius $r_{1}$ surrounded by a concentric conductor of inner radius $r_{2}$. The space between the conductors is filled with a material having the dielectric constant $\varepsilon=2.06$. The $\mathrm{TE}_{11}$ mode cut-off frequency for the SMA-RG402 coaxial cable is $\nu_{\text {cut }} \simeq \frac{c}{\pi\left(r_{1}+r_{2}\right) \sqrt{\varepsilon}} \simeq 33 \mathrm{GHz}$ [29], where $c$ is the speed of light in the vacuum. Below the onset of the $\mathrm{TE}_{11}$ mode inside a coaxial cable only the fundamental TEM mode can propagate.

## IV. EXPERIMENTAL SETUP

The isoscattering chains $\Gamma_{1,4}$ and $\Gamma_{2,4}$ obtained from the two elementary units shown in Fig. 1(c) and Fig. 1(d) are shown in Fig. 2(a) and Fig. 2(b), respectively. In Fig. 2(c) we present the isoscattering chain of microwave networks $\Gamma_{2,4}$ used in the experiment.

In the case of the discussed graphs we considered the two typical physical vertex boundary conditions, the Neumann and Dirichlet ones. The Neumann boundary condition imposes the continuity of waves propagating in edges meeting at the vertex $v_{i}$ and vanishing of the sum of their derivatives calculated at $v_{i}$. The Dirichlet boundary condition demands vanishing of the waves at the vertex.

The chain $\Gamma_{1,4}$ of the $n=2$ graphs in Fig. 2(a) consists of $V=8$ vertices connected by $B=8$ edges. The valency of the vertices $3-6$ including leads is $v=4$ while for the other ones $v=1$. The vertices with numbers $3-8$ satisfy the Neumann boundary conditions, while for the vertices $1-2$ we have the Dirichlet ones. The second chain $\Gamma_{2,4}$ in Fig. 2(b) consists of $V=6$ vertices connected by $B=7$ edges. The vertices with the numbers $2-6$ satisfy the Neumann boundary conditions while for the vertex 1, the Dirichlet condition is imposed.

The edges of the chains of the microwave networks $\Gamma_{1,4}$ and $\Gamma_{2,4}$ have the following optical lengths:

$$
\begin{aligned}
& b / 2=0.0537 \pm 0.0005 \mathrm{~m}, \\
& c / 2=0.0508 \pm 0.0005 \mathrm{~m}, \\
& a=0.1597 \pm 0.0005 \mathrm{~m}, \\
& b=0.1074 \pm 0.0005 \mathrm{~m}, \\
& c=0.1016 \pm 0.0005 \mathrm{~m}, \\
& 2 a=0.3194 \pm 0.0005 \mathrm{~m} .
\end{aligned}
$$

In contrast to the systems investigated in Ref. [16] which consisted only $L=2$ external leads here we study more complex isoscattering microwave chains having $L=4$ external leads. Therefore, the chains of the networks $\Gamma_{1,4}$ and $\Gamma_{2,4}$ are described by $L \times L$ scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$, respectively. The relationship between both matrices is given by

$$
\begin{equation*}
\hat{S}^{(I)}=\hat{T}_{4}^{-1} \hat{S}^{(I I)} \hat{T}_{4}, \tag{6}
\end{equation*}
$$

where $4 \times 4$ transplantation matrix $\hat{T}_{4}$ is defined by Eq. (4)

$$
\hat{T}_{4}=\left(\begin{array}{cccc}
1 & -1 & 0 & 0  \tag{7}\\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{array}\right)
$$

The matrix $\hat{T}_{4}$ does not depend on the frequency and the equation (6) is valid for all values of $\nu$.

One should point out here that the application of a standard measure of isoscattering such as a phase of the scattering matrix determinant

$$
\begin{equation*}
\operatorname{Im}\left[\log \left(\operatorname{det}\left(\hat{S}^{(I)}\right)\right)\right]=\operatorname{Im}\left[\log \left(\operatorname{det}\left(\hat{S}^{(I I)}\right)\right)\right] \tag{8}
\end{equation*}
$$

is from the experimental point of view very inconvenient since for each $4 \times 4$ scattering matrix $\hat{S}^{(I)}$ or $\hat{S}^{(I I)}$ it requires measurements of 16 matrix elements.

Therefore, in Ref. [1] we introduced a new measure of isocattering which is the trace of scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$. Using the properties of the trace function from the formula (5) one obtains

$$
\begin{equation*}
\operatorname{tr} \hat{S}^{(I)}=\operatorname{tr} \hat{S}^{(I I)} . \tag{9}
\end{equation*}
$$

Both functions $\operatorname{tr} \hat{S}^{(I)}$ and $\operatorname{tr} \hat{S}^{(I I)}$ are complex and depend on microwave frequency $\nu$. The application of the trace function simplifies the experimental procedure because now the measurement of only 4 diagonal elements of each scattering matrix in a function of frequency $\nu$ is required.

The measurements of the diagonal elements of the scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$ were performed using the vector network analyzer (VNA) Agilent E8364B. The experimental procedure is demonstrated in the case of the microwave chain $\Gamma_{2,4}$ presented in Fig. 2(c). To measure the diagonal elements $S_{33}^{(I I)}$ and $S_{44}^{(I I)}$ of the scattering matrix $\hat{S}^{(I I)}$ the flexible 50 Ohm test port cables HP 85133-60016 and HP 85133-60017 of the VNA were connected to the vertices 4 and 5 of the microwave network shown in Fig. 2(c). To the vertices 3 and 2, 50 Ohm loads were attached as the realization of the two additional leads $L_{2}^{\infty}$ and $L_{1}^{\infty}$. The connection of the VNA to a microwave network (see Fig. 2c) is equivalent to attaching of two infinite leads $L_{3}^{\infty}$ and $L_{4}^{\infty}$ to a quantum graph. The diagonal elements $S_{11}^{(I I)}$ and $S_{22}^{(I I)}$
were measured similarly. In this case the VNA was connected to the vertices 2 and 3, while to the vertices 4 and $5,50 \Omega$ loads were connected.

## V. EXPERIMENTAL RESULTS

The measurements of the diagonal elements of the scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$ were performed in the frequency range $\nu=0.01-2 \mathrm{GHz}$. In Fig. 3(a) we show that the amplitudes of the trace function $\left|\operatorname{tr} \hat{S}^{(I)}\right|$ and $\left|\operatorname{tr} \hat{S}^{(I I)}\right|$ of the scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$ of the networks $\Gamma_{1,4}$ and $\Gamma_{2,4}$, marked by red open circles and black full circles, respectively. They are close to each other, proving that we are dealing with the isoscattering networks. For the frequency range $0.01-1 \mathrm{GHz}$ the agreement between the results obtained for both networks is almost perfect. However, for the frequency range $1-2 \mathrm{GHz}$ small discrepancies arise, caused by small differentiation of the vertex boundary conditions of the networks in a function of frequency $\nu$ and by small differences in the cables' lengths. The modulus of the trace function of the scattering matrices can be alone treated as a proper measure of the isocattering properties of the networks and graphs with dissipation. However, Eq. (9) deals with the full trace function which is a complex number. Therefore, the isoscattering properties of the networks should be also observed in the phases of the trace functions, regardless of the absorption strength

$$
\begin{equation*}
\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I)}\right)\right]=\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I I)}\right)\right] \tag{10}
\end{equation*}
$$

In Fig. 3(b) we present the comparison of the phases $\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I)}\right)\right]$ (red empty circles) and $\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I I)}\right)\right]$ (black full circles) of the trace function of the scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$, respectively. The agreement between the results obtained for different networks $\Gamma_{1,4}$ and $\Gamma_{2,4}$, especially in the frequency range $0.01-1 \mathrm{GHz}$, is very good, demonstrating that we deal with the chains of the isocattering networks.

In summary, we demonstrated that there exist isoscattering chains of graphs possessing $n$ units and $2 n$ infinite external leads. Their isoscattering properties are preserved for $n \rightarrow \infty$. The theoretical predictions were confirmed experimentally using $n=2$ units, four-leads microwave networks. In the analysis of the networks a new measure of isoscattering - the trace function was successfully used.

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FIG. 1: Schemes of the isoscattering chains of the $\Gamma_{1,2 n}$ and $\Gamma_{2,2 n}$ graphs. (a) The isoscattering chain $\Gamma_{1,2 n}$ with $n-1$ loops, $2 n$ leads, and $V=2 n+4$ vertices. (b) The isoscattering chain $\Gamma_{2,2 n}$ with $n$ loops, $2 n$ leads, and $V=2 n+2$ vertices. The Neumann and Dirichlet boundary conditions are marked by N and D capital letters. The restriction of the wave function $\hat{\Psi}_{1,2 n}$ to a segment $k$ of the chain $\Gamma_{1,2 n}$ is denoted by $\Psi_{1, k}$. The wave function $\Phi_{2, k}$ restricted to a segment $k$ of $\Gamma_{2,2 n}$ can be expressed by the components of the wave function $\hat{\Psi}_{1,2 n}$ using the formulas (2-3). (c-d) The elementary units of the isoscattering chains $\Gamma_{1,2 n}$ and $\Gamma_{2,2 n}$. The vertices of the internal units possess the Neumann boundary conditions while the vertices of the last units of the chains from the right hand side fulfil the Dirichlet boundary conditions.


FIG. 2: The schemes of the isoscattering chains $n=2$ graphs $\Gamma_{1,4}$ and $\Gamma_{2,4}$ with 4 leads. (a) The isoscattering chain of graphs $\Gamma_{1,4}$ with 4 leads $L_{1}^{\infty}, \ldots, L_{4}^{\infty}$ and $V=8$ vertices. (b) The isoscattering chain of graphs $\Gamma_{2,4}$ with 4 leads $L_{1}^{\infty}, \ldots, L_{4}^{\infty}$ and $V=6$ vertices. The Neumann and Dirichlet boundary conditions are marked by N and D capital letters, respectively. (c) The experimental realization of the chain of graphs $\Gamma_{2,4}$. The microwave cables of the VNA are connected to the vertices 5 and 4 in order to measure the diagonal elements $S_{44}^{(I I)}$ and $S_{33}^{(I I)}$ of the scattering matrix $\hat{S}^{(I I)}$. The connection of the VNA to a microwave network is equivalent to attaching of two infinite leads $L_{4}^{\infty}$ and $L_{3}^{\infty}$ to a quantum graph. To the vertices 3 and 2, 50 Ohm loads were attached as the realization of the two additional leads $L_{2}^{\infty}$ and $L_{1}^{\infty}$.


FIG. 3: The amplitudes $\left|\operatorname{tr} \hat{S}^{(I)}\right|$ and $\left|\operatorname{tr} \hat{S}^{(I I)}\right|$ (panel (a)) and the phases $\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I)}\right)\right]$ and $\operatorname{Im}\left[\log \left(\operatorname{tr} \hat{S}^{(I I)}\right)\right]$ (panel (b)) of the trace function of the scattering matrices $\hat{S}^{(I)}$ and $\hat{S}^{(I I)}$ obtained for the isoscattering chains of microwave networks $\Gamma_{1,4}$ with $V=8$ vertices (red open circles) and $\Gamma_{2,4}$ with $V=6$ vertices (black full circles), respectively.

# On the Origin of the Universe: Chaos or Cosmos? 

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#### Abstract

I would like to consider the Universe according to the standard Big Bang model, including various quantum models of its origin. In addition, using the theory of nonlinear dynamics, deterministic chaos, fractals, and multifractals I have proposed a new hypothesis, Ref. [12]. Namely, I have argued that a simple but possibly nonlinear law is important for the creation of the Cosmos at the extremely small Planck scale at which space and time originated. It is shown that by looking for order and harmony in the complex real world these modern studies give new insight into the most important philosophical issues beyond classical ontological principles, e.g., by providing a deeper understanding of the age-old philosophical dilemma (Leibniz, 1714): why does something exist instead of nothing? We also argue that this exciting question is a philosophical basis of matters that influence the meaning of human life in the vast Universe.


Keywords: Chaos, Cosmos, Universe, Creation.

Chaos is the score on which reality is written.
Henry Miller (1891-1980)

## 1 Introduction

In science the evolution the Universe is based on the Big Bang model, which has now become a standard scenario. However, very little is known about the early stages of this evolution, where we should rely on some models, because the required quantum gravity theory is still missing. On the other hand, creation of the Universe is usually an important issue of philosophy. Hence, one should return to great philosophers starting from the Greeks asking the questions about the origin of existence of the world [7], including

- Plato's creation: a Demiurg transformed an initial chaotic stuff into the ordered Cosmos.

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- Aristotle's universe is eternal: the world always existed, but needed the possibly atemporal Prime Mover or the First Cause.

In this paper, we would like to consider the origin of the Universe in view of the modern science, including quantum models of creation, and the recent theory of nonlinear dynamics, deterministic chaos, and fractals, see Ref. [12]. We hope that these modern studies give also new insight into the most important philosophical issues exceeding the classical ontological principles, e.g., providing a deeper understanding of the age-old philosophical question:

## Why does something exist instead of nothing?

Gottfried Wilhelm von Leibniz (1646-1716)

## 2 The Universe in Modern Science

Here we discuss the Standard Model of the Evolution of the Universe based on the Standard Model of Forces together with selected models of the creation of the world based on quantum theory and modern mathematics [12, ch. 2].

A veritable revolution in understanding of the evolution of the Universe was achieved only a century ago owing to the foundation of general relativity by Albert Einstein in 1916. This theory is based on the principle of relativity insisting that physical laws should be independent of the observer, even in the case of a noninertial frame of references (i.e., moving with acceleration).

### 2.1 The Geometry of Spacetime

According to general relativity, gravitation is revealed by the curvature of local spacetime, as schematically shown in Figure 1. Instead of the flat fourdimensional Minkowski spacetime we should involve a non-Euclidean spacetime with positive (elliptic type) or negative (hyperbolic) curvatures, respectively, as formulated by Georg F. B. Riemann (1826-1866). Minkowski geometry (corresponding to four-dimensional Euclidean pseudo-space) is only a special case of Riemannian geometry. General theory of relativity can well be applied even in the case of strong gravitational fields. Therefore, one should conclude that spacetime and matter cannot be independent. We may briefly state that mass (energy) tells spacetime geometry about its curvature, but curved spacetime tells the mass how to move.

### 2.2 Gravitational Waves

Since the formulation of the theory of general relativity, it was expected that strong gravitational waves which are actually distortions of spacetime, can arise during the merger of two massive black holes. Figure 2 shows computer simulations of a possible generation mechanism of gravitational waves in the vicinity of black holes. On the one-hundredth anniversary of this theory, we can now confirm its important implications. In fact, the measurements of experimental


Fig. 1. Gravitation and geometry.


Fig. 2. The generation of gravitational waves (LIGO).
signals by two independent detectors of the Laser Interferometer GravitationalWave Observatory (LIGO) in Hanford and Livingston (separated by $\sim 3000 \mathrm{~km}$ ) are consistent with observations of a gravitational-wave strain, which is of the order of the amplitude of a gravity wave, with a relative amplitude of $\sim 10^{-21}$ ) [1]. For the first time this proves that the international experiment LIGO directly detected gravitational waves originating several billions years ago from the merging of two black holes (of masses about 30 times larger than the mass of the Sun) in the rotating binary system GW150914. Therefore, a large fraction of energy ( $\sim 5 \%$, corresponding to three solar masses) has been released in this process in form of gravitational waves. In 2017 the Nobel Prize in Physics was awarded to the American experimental and theoretical physicists Rainer Weiss, Kip Thorne, and Barry Barish for their role in the detection of gravitational waves.

### 2.3 The Big Bang Model



Fig. 3. Schematic of the evolution of the Universe, credit: NASA / WMAP Science Team.

According to the Big Bang model, the Universe expanded from an extremely dense and hot state and continues to expand today. It is worth noting that space itself is expanding, carrying galaxies with it. A representation of the Universe's evolution is schematically shown in Figure 3, based on the best available measurements of the Wilkinson Microwave Anisotropy Probe (WMAP) operating from 2001 to 2010. The far left depicts the earliest moment we can now probe: size is depicted by the vertical extent of the grid in this graphic. The original state of the Universe began around 13.8 billion years ago, when the Big Bang occurred. This was possibly followed by 'inflation', producing a burst of exponential growth in the size of the Universe. The first microsecond, consisting of electroweak, quark, and hadron epochs, together with the lepton epoch (until 3 minutes of its existence) was decisive for further evolution, leading to the nucleosynthesis of helium from hydrogen. Only after 70 thousand years was light separated from matter. The afterglow light seen by WMAP was emitted about 400 thousand years after the beginning (when the electrons and nucleons were combined into atoms, mainly hydrogen) and has traversed the Universe largely unimpeded since then. The conditions of earlier times are imprinted on this light; it also forms a backlight for later developments of the Universe. The first stars appeared about 400 million years later.

Also the Planck mission launched in 2009 (deactivated in 2013) has become the most important source of information about the early Universe by providing
unique data at microwave and infra-red frequencies with high sensitivity and small angular resolution. The Planck data suggest that the Dark Ages (before the first star appeared) ended somewhat later, i.e, 550 million years after the Big Bang. This mission has also provided a new catalog of more than 1500 clusters of galaxies observed in the Universe. More than 400 of these galaxy clusters have large masses ranging between 100 to 1000 times that of our Milky Way galaxy.

After the formation of galaxies, and finally, our solar system, about 4.5 billion years ago, for the next several billion years the expansion of the Universe gradually slowed down as the matter in the Universe pulled on itself by gravity. One can ask whether the present expansion will continue forever or if it might eventually stop, thereby allowing a subsequent contraction. Even though we cannot give a definitive answer to this question, recently it has appeared that the expansion has begun to speed up again, as the repulsive effects of mysterious dark energy have come to dominate the expansion of the Universe. The Planck data also support the idea of dark energy acting against gravity. At present this accounts for about $70 \%$ of the entire mass of the Universe, and it will presumably increase in the future.

### 2.4 The Birth and Evolution of the Universe



Fig. 4. The Grand Unification Theory (GUT) for the Universe.

The role of the elementary interactions during the evolution ${ }^{1}$ is depicted in Figure 4. One can see that the splitting of one force after the Big Bang

[^3]into the four kinds of forces that we know today, after $1.38 \times 10^{10}$ years of the evolution, happened in a very tiny fraction of the first second. Strong forces should be limited only to the scales (nucleon size of $\sim 10^{-15} \mathrm{~m}$ ) in the microworld, while general relativity models long-range gravitational interactions on very large scales of up to the size $\left(\sim 10^{27} \mathrm{~m}\right)$ of the observed Universe. It is interesting that timescales are from $10^{-24} \mathrm{~s}$ in atomic nuclei to nearly $10^{18} \mathrm{~s}$ of the experimentally confirmed age of the Universe. This means a range of 42 orders of magnitude is the same as for spacescales; the masses span the range of about 83 orders of magnitude, between $10^{-30} \mathrm{~kg}$ for the electron mass and about $10^{53} \mathrm{~kg}$ for the of mass of the whole world ( $\sim 10^{80}$ baryons, mainly nucleons: protons and neutrons with mass of $\sim 10^{-27} \mathrm{~kg}$ ); this range is roughly twice as large as the time or space scale range.

Because the Universe has already expanded to that extremely huge size, gravitational forces (basically about 40 orders of magnitude weaker than strong nuclear forces) dominate the evolution of the Universe at present. However, at early stages of its evolution both forces resulted from an unknown simple law and could have been of a similar strength. The other long-range electromagnetic interactions between charged particles have already been unified with the shortrange weak interactions responsible for the decay of nuclei (electroweak forces). Of course, the Grand Unification Theory (GUT) in Figure 4 describing the unknown primordial force responsible for the creation of the Universe at a Planck scale of $10^{-43} \mathrm{~s}$ will facilitate a better understanding of the physical processes at very early stages of the history of our world.

### 2.5 Quantum Models for the Creation of the Universe

Using the three available universal physical constants - namely the gravitational constant $G$, the speed of light $c$, and the Planck constant $h$, we can construct a quantity called a Planck length $l_{\mathrm{P}}=\sqrt{G \hbar / c^{3}}$, where $\hbar=h /(2 \pi)$. Another quantity $l_{\mathrm{P}} / c$ is the respective Planck time scale, $t_{\mathrm{P}}$. Because we do not have a quantum theory of gravitation quantum gravity a number of models for the creation creation of the Universe with the following characteristics have been proposed, including:

- The quantum model [2] creation from 'nothing', ex nihilo
- Noncommutative geometry [4]
beginning is everywhere
- String theory, M-theory [30]
collision of branes
- Cyclic (ekpyrotic) model [26,27]
big bangs and crunches
- Eternal chaotic inflation [5]
bubble of universes
The concept of the quantum wave function of the primordial Universe was put forward in Ref. [2]. This point of view was illustrated in a simple minisuperspace model with an invariant scalar field as the only gravitational degree
of freedom. The authors of this model focus on the ground state with minimum excitation of an initial Universe on extremely small scales. Providing that the time is changed to imaginary values it, spacetime with a four-dimensional geometry becomes positive-defined. This allows us to obtain the path integral of the respective Euclidean action. In this way, the authors obtained finite nonzero probabilities of propagating from the ground (vacuum) state to the spectrum of possible excited states.

It is worth noting that below the Planck threshold $l_{\mathrm{P}}=1.6 \times 10^{-35} \mathrm{~m}$ $\sim 10^{-35} \mathrm{~m}$ and $t_{\mathrm{P}}=5.4 \times 10^{-44} \mathrm{~s} \sim 10^{-43} \mathrm{~s}$, in space and time, respectively, any time could be formally eliminated in the quantum model. In this scenario the Universe interpreted without any boundary conditions [2]. Moreover, because one can obtain the excited state from the vacuum state, they argue for the creation out of nothing, even ex nihilo. However, one should bear in mind that a quantum vacuum state is not actually 'nothingness' - indeed it could be interpreted as a 'sea' of various possibilities [3].

An alternative interesting solution for the origin of spacetime on extremely small scales has been proposed in Ref. [4], where it was suggested that these critical values would correspond to a phase transition from a smooth commutative geometry to a rather singular noncommutative régime, with no space points and no time instances. Hence, noncommutative algebra is the other quantum gravity counterpart of the observable in the standard quantum theory, which can help in the application of quantization methods to the origin of the primordial Universe. Therefore, as one can paradoxically put it: the beginning is everywhere.

Following the M theory [30], in the context of an initial universe resulting from a collision of branes, Another interesting non-standard cosmological scenario has been proposed in Ref. [26,27]. According to their proposed model, the Universe undergoes a sequence of cosmic epochs each of which begins with a created world with a standard big bang event, followed by a slowly accelerating expansion with radiation and matter domination periods, but ends by contraction with a crunch. This model is called ekpyrotic, because in ancient Greece's Stoic philosophy ecpirosi means 'escape from fire'. This endless cycle of big bangs and crunches would avoid any particular singularity, but is able to explain the approximate homogeneity of distribution of mass, instead of a hypothetical inflation following the Planck epoch. It is worth noting that the model produces the recently observed flatness of spacetime geometry, providing the energy needed to restore the Universe from the same vacuum state in the next cycle. These authors also assure us that, owing to acceleration, this continuously repeating cyclic solution is an attractor [27].

Taking the wave function of the Universe [2], it can be shown that the large scale fluctuations of the quantum scalar field can generate an infinite process of self-reproducing primordial mini-universes. Therefore, one can suggest an eternally existing chaotic inflationary scenario, describing the Universe as a selfgenerating fractal that springs up from the multiverse [5]. Because it seems improbable that only one such Universe is chosen in reality by compactification during the expansion, it is argued that there exists a bubble of all possible universes that is always growing until a new universe is created by chaotic
inflation in the bubble [5]. Therefore, there should exist an exponentially large number of causally disconnected mini-universes corresponding to all possible vacuum states followed by inflations. Admittedly, in the last two models time is eternal, but it is difficult to verify these models according to the criterion of falsifiability required for any scientific theory of Popper [24].

## 3 Nonlinear Dynamics and Fractals

In the second part of this paper we focus on nonlinear chaotic dynamics and fractals in a search for implications of an unknown nonlinear law related to a hidden order responsible for the creation of the Cosmos at the Planck epoch, see [12, ch. 3].

### 3.1 Deterministic Chaos

 Stewart [28]):

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

More precisely, we say that a bounded solution $\mathbf{x}(t)$ of a given dynamical system, $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$, is SEnsitive to initial conditions if there is a finite fixed distance $d>0$ such that for any neighborhood $\|\Delta \mathbf{x}(0)\|<\delta$, where $\delta>0$, there exists (at least some) other solutions $\mathbf{x}(t)+\Delta \mathbf{x}(t)$ for which for some time $t \geq 0$ we have $\|\Delta \mathbf{x}(t)\| \geq d$. This means that there is a fixed distance $d$ such that, no matter how precisely one specifies an initial state, there exists a solution of a dynamical system starting from a nearby state (at least one) that gets a distance $d$ away.

Given $\mathbf{x}(\mathbf{t})=\left\{x_{1}(t), \ldots, x_{N}(t)\right\}$, any positive finite value of Lyapunov exponents (or equivalently metric entropy)

$$
\begin{equation*}
\lambda_{k}=\lim _{t \rightarrow \infty} \frac{1}{t} \ln \left|\frac{\Delta x_{k}(t)}{\Delta x_{k}(0)}\right| \tag{1}
\end{equation*}
$$

where $k=1, \ldots N$, implies chaos.
One example comes from the dynamics of irregular flow in viscous fluids, which is still not sufficiently well understood. It appears that the behavior of such systems can be rather complex: from equilibrium or regular (periodic) motion, through intermittency (where irregular and regular motions are intertwined) to nonperiodic behavior. Two types of such nonperiodic flows are possible, namely chaotic and hyperchaotic motions. As discovered by Lorenz (1963) deterministic chaos exhibits sensitivity to initial conditions leading to the unpredictability of the long-term behavior of the system (the 'butterfly effect') [6].

### 3.2 Hyperchaos

Hyperchaos is a more complex nonperiodic flow, which was discovered by Macek and Strumik (2010) [15] in the generalized Lorenz system previously proposed in Ref. [14]. Mathematical and physical aspects of this new low-dimensional model of hydromagnetic convection together with the detailed derivation from the basic partial differential equations, including the magnetic diffusion equations and naturally the anisotropic tension of the magnetic field lines, has been addressed in detail in Ref. [11].

Within the theory of dynamical systems transitions from fixed points to periodic or nonperiodic flows often occur in a given system through bifurcations, intermittency, resulting in a turbulent irregular behavior of the nonlinear system. In fact, we have identified type I and III intermittency [23] in the generalized Lorenz model of hydromagnetic convection, as also discussed in the papers $[14,15,10]$. It would be interesting to look for the remaining basic type II intermittency and the respective Hopf bifurcation in this model.

The following ordinary differential equations are obtained in the generalized Lorenz system [14]:

$$
\left.\begin{array}{rl}
\dot{X} & =-\sigma X+\sigma Y-\omega_{0} W  \tag{2}\\
\dot{Y} & =-X Z+r X-Y \\
\dot{Z} & =X Y-b Z \\
\dot{W} & =\omega_{0} X-\sigma_{\mathrm{m}} W
\end{array}\right\}
$$

In this simplified system, $X(t)$ denotes a time amplitude of the potential of the velocity of a viscous horizontal fluid layer in the vertical gravitational field heated from below, with the normalized (dimensionless) Rayleigh number $r$, proportional to an initial temperature gradient $\delta T_{0}$, which is a control parameter of the system. Similarly, $Y(t)$ and $Z(t)$ correspond to the two lowest-order amplitudes of the deviation from the linear temperature profile of the layer (of height $h$ ) during the convection. The other parameter $\sigma=\nu / \kappa$ is the ratio of the kinematic viscosity $\nu$ to thermal conductivity $\kappa$ (the Prandtl number) characterizing the fluid and $b=4 /\left(1+a^{2}\right)$ is a geometric factor related to the aspect ratio $a$ of the convected cells.

Admittedly, Lorenz (1963) only took three of several coefficients appearing in the lowest-order of the bispectral Fourier expansion, cf. [25]. In addition to the standard Lorenz system [6], a new time dependent variable $W$ in Equations (2) describes the profile of the magnetic field induced in the convected magnetized fluid. We have also introduced the second control parameter proportional to an initial horizontal magnetic field strength $B_{0}$ applied to the system, more precisely defined here as a basic dimensionless magnetic frequency $\omega_{0}=v_{\mathrm{A} 0} / v_{0}$, which is the ratio of the Alfvén velocity $v_{\mathrm{A} 0}=B_{0} /\left(\mu_{0} \rho\right)^{1 / 2}$, with a constant magnetic permeability $\mu_{0}$ and mass density $\rho$, to a characteristic speed $v_{0}=4 \pi \kappa /(a b h)$. Naturally, besides $\sigma=\nu / \kappa$, the magnetized viscous fluid is characterized by an analogue parameter $\sigma_{\mathrm{m}}=\eta / \kappa$, defined as the ratio of the magnetic resistivity $\eta$ to the thermal conductivity $\kappa$ (related to the magnetic Prandtl number, $\left.\operatorname{Pr}_{\mathrm{m}}=\sigma / \sigma_{\mathrm{m}}\right)$.

The results of the more recent paper illustrate how all these complex motions can be studied by analyzing this simple model [15, Fig. 1]. For example,


Fig. 5. Color-coded dependence of the long-term asymptotic solutions of the generalized Lorenz system on the control parameters $\omega_{0}$ and $r$ parameters (for $\sigma_{\mathrm{m}}=3$ ). Equilibria (fixed points) (with a negative largest Lyapunov exponent, $\lambda_{1}<0$ ) are shown in black, periodic solutions $\left(\lambda_{1}=0\right)$ - in violet/blue, and (nonperiodic) chaotic solutions $\left(\lambda_{1}>0\right)$ - in a color, on the color bar scale, from violet to yellow. Fine structures are shown in the inset, as taken from (Macek and Strumik, 2014).
for a chosen value of $\sigma_{\mathrm{m}}=3$ (other parameters have the same values as for the classical Lorenz model, $\sigma=10, b=8 / 3$ ), Figure 5 plots the largest Lyapunov exponent, calculated according to Equation (1), depending on the control parameters $\omega_{0}$ and $r$. Convergence of the asymptotic solutions of Equations (2) to equilibria described by fixed points ( $\lambda_{1}<0$ ) is shown in black, to periodic (limit cycles) solutions ( $\lambda_{1}=0$ ) - in violet/blue color (see the color bar for $\lambda_{1}=0$ ), to chaotic (nonperiodic) solutions ( $\lambda_{1}>0$ ) - in a color, consistently with the color bar scale, from violet/blue to yellow. For the panel an enlargement of the region bounded by black lines is shown in the right-bottom part of plots. Fine structures are shown in the inset. This proves that various kinds of complex behavior are closely neighbored in the space of control parameters $\omega_{0}$ and $r$.

Convection appears naturally in plasmas, where electrically charged particles interact with the magnetic field. Therefore, the obtained results could be important for explaining dynamical processes in solar sunspots, planetary and stellar fluid interiors, and possibly for plasmas in nuclear fusion devices. Generally speaking, nonlinear differential equations or iterated discrete maps are
useful models of some phenomena appearing naturally in the contexts in biology (e.g., animal population), economics, including finance theory, e.g., [22], and social sciences.

## 4 Fractals and Multifractals

Let us now move on to a basic concept of a fractal coined from the Latin adjective fractus and the corresponding verb frangere, which means 'to break into irregular fragments', see p. 4 of Ref. [20]; Mandelbrot (1982) always argued that fractal geometry is important for understanding the structure of nature describing, for example clouds, mountains, and coastlines, e.g. p. 1 of Ref. [20]. We can say that a fractal is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally self-similar and independent of scale, described by a fractal dimension.

Namely, fractal structure is obtained recursively using a simple rule. The initial stages of the construction of two typical fractals in one-dimensional and two-dimensional space are schematically illustrated in Figure 6 for a middle Cantor (a) and a Koch triangle (b) sets, respectively, which are also discussed in many textbooks, e.g. [29,21]. First, as proposed by the German mathematician Georg Cantor in 1883, let us take a unit closed interval on a onedimensional line and remove its open middle third, but necessarily leaving the endpoints behind. Second, we remove the open middle thirds of both closed smaller intervals, and in each of the following $k$-th step this produces $2^{k}$ closed (more and more narrower) intervals of length $(2 / 3)^{k}$, where $k=1, \ldots, n$. Now imagine that the repetitions never end, one obtains the limiting set that consists of the intersection of all such closed intervals. Provided that $n \rightarrow \infty$, the resulting set has structure at arbitrarily small scales; the remaining elements during the construction are separated by various gaps. Surprisingly enough, two paradoxically opposite topological properties of the Cantor set (called also a dust) can be reconciled: the set itself is totally disconnected (without any closed intervals), but arbitrarily close to each elements one can always find another neighboring element (there are no isolated points).

Further, it is worth noting that each element of this set is specified by its location at successive steps, in the left (denoted by zero) or right (marked by one) fragment. One now sees that elements of the Cantor set are equivalent to various infinite sequences of zeros and ones, and can be put into one-to-one correspondence with the elements of the entire initial interval (in binary representation). Because common sense has some difficulty in comparing countable with uncountable infinity, this is somewhat strange that the Cantor set is uncountable, notwithstanding its total length equal to zero (the length of all the removed parts is equal one). Mainly because of this paradox, such sets are commonly called strange fractals, even though one can also construct fractals with length or in general volume (strictly a Lebesgue measure) different than zero. Similar fractal sets with zero Lebesgue measures constructed starting from a triangle or a full square on a two-dimensional plane were proposed by the Polish mathematician Wacław Sierpiński (1882-1969) in 1916.
(a)

(b)


Fig. 6. Self-similar fractals of the Cantor (a) and Koch (b) sets.

Figure 6 (b) shows another interesting snowflake curve obtained on a plane by adding onto sides of an initial equilateral triangle additional triangles that are three times smaller, after removing as before open middle thirds of any side. Blowing up this van Koch curve by a factor of three results in its length four times as large, and hence the length of perimeter of the triadic Koch island increases and becomes ultimately infinite, despite the fact that the area of course remains finite. Surprisingly, the arc length between any two elements of such a Koch set is also infinite. Therefore, because every element of this set is located infinitely far from any other element, the length cannot be used to identify the elements of such a strange fractal.

Mandelbrot (1982) noted that a fractal (Hausdorff) dimension ${ }^{2}$, which plays a central roles in case of fractal sets, exceeds the topological dimension, $D_{\mathrm{T}}$ [20]. Anyway, the concept of dimension should be modified as compared with a standard topological dimension useful in the Euclidean linear geometry. However, a somewhat different definition of a fractal set is generally accepted. The capacity dimension $D_{\mathrm{F}}$, which takes into account how many elements (cubes) of size $l$ in phase space is needed to cover the set, is defined by

$$
\begin{equation*}
D_{\mathrm{F}}=\lim _{l \rightarrow 0} \frac{\ln N(l)}{\ln 1 / l} \tag{3}
\end{equation*}
$$

This means that fractal dimension is calculated by taking the limit of the quotient of the logarithm change in object size and the logarithm in scale as the limiting scale approaches zero. For example, the fractal dimensions of the Cantor and the Koch sets are $D_{\mathrm{F}}=\ln 2 / \ln 3 \approx 0.63$ (this means $D_{\mathrm{F}}>0$ ) and $D_{\mathrm{F}}=\ln 4 / \ln 3 \approx 1.26(>1)$ i.e., greater than the respective topological dimensions, $D_{\mathrm{T}}=0$ and 1. As is known, the later non-integer dimension describes sufficiently well the length of the rocky western coast of Great Britain as a function of diminishing scale size; in reality the lowest scale is admittedly limited.

### 4.1 Multifractal Models for Turbulence

A deviation from a strict self-similarity is also called intermittency, and that is why a generalized two-scale weighted Cantor set has been applied for modeling intermittent turbulence in fluids [8,9].

In fact, this complex process can be described by the generalized weighted Cantor set, as illustrated in Figure 7 taken from Ref. [8]. In the first step of the two-scale model construction, we have two eddies of sizes $l_{1}$ and $l_{2}$ satisfying $p_{1} / l_{1}+p_{2} / l_{2}=1$. Therefore, the initial energy flux $\varepsilon_{0}$ is transferred to these eddies with the different proportions: $\varepsilon_{0} p_{1} / l_{1}$ and $\varepsilon_{0} p_{2} / l_{2}$. In the next step the kinetic or magnetic energy flux is divided between four eddies in the following way: $\varepsilon_{0}\left(p_{1} / l_{1}\right)^{2}, \varepsilon_{0} p_{1} p_{2} /\left(l_{1} l_{2}\right), \varepsilon_{0} p_{2} p_{1} /\left(l_{2} l_{1}\right)$, and $\varepsilon_{0}\left(p_{2} / l_{2}\right)^{2}$. At $n$th step we have $N=2^{n}$ eddies and partition of energy $\varepsilon$ can be described by the binomial formula, e.g., [9]:

$$
\begin{equation*}
\varepsilon=\sum_{i=1}^{N} \varepsilon_{i}=\varepsilon_{0} \sum_{k=0}^{n}\binom{n}{k}\left(\frac{p_{1}}{l_{1}}\right)^{(n-k)}\left(\frac{p_{2}}{l_{2}}\right)^{k} \tag{4}
\end{equation*}
$$

For any real number $-\infty<q<+\infty$, one obtains the generalized dimension defined by $D_{q}=\tau(q) /(q-1)$ by solving numerically the transcendental equation, e.g., [21],

$$
\begin{equation*}
\frac{p_{1}^{q}}{l_{1}^{\tau(q)}}+\frac{p_{2}^{q}}{l_{2}^{\tau(q)}}=1 \tag{5}
\end{equation*}
$$

[^4]

Fig. 7. The generalized two-scale weighted Cantor set model for turbulence.
which is only somewhat more general than the analytical solution. In particular, for the one-scale multifractal model with $l_{1}=l_{2}=\lambda$, we have $D_{q}=$ $-\ln \left(p_{1}^{q}+p_{2}^{q}\right) / \ln \lambda$, and a special case for $\lambda=1 / 2$ is called P-model, as classified on the right side of Figure 7. We see that only for equal scales together with equal weights ( $p_{1}=p_{2}=1 / 2$ ) there is no multifractality, and we have a monofractal with a fractal dimension given by Equation (3).

## 5 Implications for Cosmology and the Creation of the Universe

This method was extensively used in various situations in solar wind magnetized plasmas based on space missions penetrating various regions of the solar system, see Refs $[9,13,16,17]$. In this way, based on a wealth of data acquired from Helios in the inner heliosphere and especially from deep space Voyager 1 and 2 spacecraft in the outer heliosphere, we have shown that turbulence is intermittent in the entire heliospheric system, even at the heliospheric boundaries [19]. However, it appears that the heliosphere is immersed in a relatively quiet very local interstellar medium. Therefore, after crossing the heliopause (on 25 August 2012), which is the ultimate boundary separating the heliospheric and interstellar plasmas, Voyager 1 only detected smoothly varying magnetic fields. As expected this change in the behavior of plasma parameters (with a frozen-in magnetic field) was confirmed by the crossing of the heliopause by Voyager 2 in 2018.

Moreover, based on scientific experience, I have argued that a simple but possibly nonlinear law [7], within the theory of chaos and (multi-)fractals, can describe a hidden ORDER for the creation of the COSMOS, at the Planck epoch, when space (at a scale of $10^{-35} \mathrm{~m}$ ) and time $\left(10^{-43} \mathrm{~s}\right)$ originated, see Ref. [12, p. 3.4].

## 6 Conclusions

To summarize, based on space, astrophysical, and even cosmological applications, one can say that

- Nonlinear systems exhibit complex phenomena, including bifurcation, intermittency, and chaos.
- Fractals can describe complex shapes in the real word.
- Strange chaotic attractors have fractal structure and are sensitive to initial conditions.
- Within the complex dynamics of the fluctuating intermittent parameters of turbulent media there is a detectable, hidden order described by a generalized Cantor set that exhibits a multifractal structure.

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# Bifurcation theory of dynamical chaos in Hamiltonian and conservative systems 

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#### Abstract

It is considered a new bifurcation approach to the analysis of solutions of perturbed Hamiltonian and conservative systems, which implies the construction of an approximating extended two-parameter dissipative system whose stable solutions (attractors) are arbitrarily exact approximations to solutions of the original conservative system. It is shown on the basis of numerical experiments for several Hamiltonian and conservative systems such as conservative Croquette equation, Yang-Mills-Higgs and Mathieu-Magnitskii Hamiltonian systems that, in all these systems, transition to chaos takes place not through the destruction of two-dimensional tori of the unperturbed system in accordance with KAM (Kolmogorov-Arnold-Moser) theory, but, conversely, through the generation of complicated two-dimensional tori around cycles of the extended dissipative system and through an infinite cascades of bifurcations of the generation of new cycles and singular trajectories in accordance with the universal bifurcation FShM (Feigenbaum-Sharkovskii-Magnitskii) theory. Keywords: Hamiltonian and conservative systems, dynamical chaos, FShM-theory.


## 1 Introduction

The divergence of the right-hand side of the conservative system of ordinary differential equations is equal to zero. Consequently, a conservative system of ordinary differential equations cannot have attractors, since it preserves volume while moving along its trajectories. Therefore, the study of dynamical chaos in conservative systems is a more difficult task compared to the analysis of chaotic dynamics in dissipative systems, the attractors of which can be described by the universal bifurcation Feigenbaum-Sharkovsky-Magnitskii (FShM) theory (Magnitskii [2-4]). A special case of a conservative system is a Hamiltonian system with $n$ degrees of freedom, that is $2 n$-dimensional autonomous system of ordinary differential equations

$$
\dot{q}_{i}=\frac{\partial H\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}\right)}{\partial p_{i}}, \dot{p}_{l}=-\frac{\partial H\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}\right)}{\partial q_{i}}, i=1, \ldots, n
$$

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The function $H\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}\right)$ is called the Hamiltonian of the system, the variables $q_{i}$ are called generalized coordinates, and the variables $p_{i}$ are called generalized momenta. The movement in such a system occurs along a $2 n-1$ dimensional energy surface, specified by condition $H\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}\right)=\varepsilon$. $\varepsilon=$ const. Typical view of pictures arising in Poincare sections of the phase space of the Hamiltonian system for small and sufficiently large values of the parameter $\varepsilon$ are shown in Fig. 1 taken from the author's work (Magnitskii [1]) describing the transition to chaos in the famous system of Henon-Heiles equations with two degrees of freedom and with the Hamiltonian

$$
\begin{equation*}
H(x, y, z, r)=\frac{\left(x^{2}+y^{2}+z^{2}+r^{2}\right)}{2}+x^{2} z-\frac{z^{3}}{3}=\varepsilon \tag{1}
\end{equation*}
$$



Fig.1. Projections of Poincaré sections of the Henon-Heiles system at $\varepsilon=1 / 24$ (a) and $\varepsilon=1 / 8$ (b).

At small values of the parameter $\varepsilon$, there is no chaos in the system, and the energy surface of the system is divided into regions filled with two-dimensional tori around two pairs of elliptic cycles of the system. The boundaries of the regions are separatrix surfaces (manifolds) passing through hyperbolic cycles. For sufficiently large values of the parameter $\varepsilon$, chaotic dynamics is observed in the system. In (Magnitskii [1-5]), it was shown using numerous examples that the development of chaos in conservative and, in particular, Hamiltonian systems does not occur in accordance with the Kolmogorov-Arnold-Moser (KAM) theory as a result of the destruction of some mythical tori of an unperturbed system, but on the contrary, through bifurcation cascades of birth of complex tori around cycles, the birth bifurcations of which occur in extended dissipative systems in accordance with the FShM theory.
For generally nonlinear conservative system of autonomous ordinary differential equations with a smooth right part

$$
\begin{equation*}
\dot{x}=f(x), \quad x \in R^{n}, \quad \operatorname{div} f(x)=0 \tag{2}
\end{equation*}
$$

which variables are connected by some equation $H\left(x_{1}, \ldots x_{n}\right)=\varepsilon$, an extended dissipative system is two-parametrical system of ordinary differential equations

$$
\begin{equation*}
\dot{x}=g(x, \varepsilon, \mu), \quad x \in R^{n}, \tag{3}
\end{equation*}
$$

the only solutions of which are the solutions of system (2) with initial conditions $H\left(x_{10}, \ldots x_{n 0}\right)=\varepsilon$ at $\mu=0$. As the dissipation parameter $\mu$ tends to zero, the stability regions of stable cycles of the extended dissipative system (3) turn into tori of the conservative (Hamiltonian) system (2) around its elliptic cycles, into which stable cycles pass. The tori of a conservative (Hamiltonian) system touch through hyperbolic cycles, into which the saddle cycles of the extended dissipative system pass. Areas of stability of complex cycles, singular attractors and also heteroclinic separatrix manifolds of dissipative system generate chaotic solutions in conservative system. In this regard, it is clear that only a small number of cycles of the FShM cascade of an extended dissipative system with large stability regions can generate elliptic cycles and tori of a conservative system. In papers (Ryabkob [6], Dubrovskii [8]) the given approach has been applied and strictly proved by continuation along parameter of solutions from dissipative into conservative areas by means of the Magnitskii method of stabilization of unstable periodic orbits at research bifurcations and chaos in the Duffing-Holmes equation

$$
\ddot{x}+\mu \dot{x}-\delta x+x^{3}-\varepsilon \cos (\omega t)=0 .
$$

and in the model of a space pendulum

$$
\ddot{x}+\mu \dot{x}+k x+\varepsilon \sin (2 \pi x)=h \cos (\omega t) .
$$

Corresponding bifurcation diagrams in a plane $(\varepsilon, \mu)$ of existence of cycles of various periods down to a conservative case at $\mu=0$ are received.
The aim of this work is the direct numerical detection of bifurcations of cycles from the FShM cascade, when increasing values of the parameter $\varepsilon$, in the conservative Crockett equation, in the Hamiltonian system of Yang-MillsHiggs equations with two degrees of freedom and in Hamiltonian system of Mathieu-Magnitskii equations generated by the conservative generalized system of Mathieu equations. An additional difficulty in solving this problem is that, in contrast to dissipative systems, the initial conditions play an essential role in the numerical determination of the elliptic cycles of conservative systems. It is necessary to get into a small neighborhood of the cycle that lies inside the torus formed by the cycle, the turns of the narrow tubes of which can quite densely fill the phase space.

## 2. Conservative Croquette equation

Let's consider as the first example the classical conservative Croquette equation

$$
\begin{equation*}
\ddot{x}+\alpha \sin x+\beta \sin (x-\omega t)=0 \tag{4}
\end{equation*}
$$

modeling a magnet rotary fluctuations in an external magnetic field in absence of friction. It is easy to see, that the equation (4) is equivalent to twodimensional conservative system with periodic coefficients (so called Hamiltonian system with one and a half degrees of freedom) and also to fourdimensional conservative (not Hamiltonian) autonomous system of the equations

$$
\begin{equation*}
\dot{x}=y, \quad \dot{y}=-(\alpha+r) \sin x+z \cos x, \quad \dot{z}=\omega r, \quad \dot{r}=-\omega z \tag{5}
\end{equation*}
$$

with a condition $H=z^{2}+r^{2}=\varepsilon^{2}, \quad z_{0}=z(0)=0$. Extended dissipative system for the system of (5) will be

$$
\begin{equation*}
\dot{x}=y, \quad \dot{y}=-(\alpha+r) \sin x+z \cos x-\mu y, \quad \dot{z}=\omega r, \quad \dot{r}=-\omega z \tag{6}
\end{equation*}
$$

It is easy to check up numerically, that the two-parametrical system (6) with initial conditions $z_{0}=z(0)=0, r_{0}=r(0)=\varepsilon$ has the subharmonic cascade of bifurcations at each value of parameter $\varepsilon$ and at reduction of values of parameter $\mu$. For some cycles of the cascade in a plane of parameters $(\varepsilon, \mu)$ it is possible to construct monotonously increasing bifurcation curve $\mu(\varepsilon)$ of births of the given cycle. Boundary values of such curves at $\mu=0$ are bifurcation values of the subharmonic cascade of bifurcations in conservative Croquette system (5) for parameter $\varepsilon>0$ (see Fig. 2 for some cycles of the cascade).


Fig. 2. Projections on the plane $(x, y)$ of the cycle (a) for $\varepsilon=0.45$, its cycle of double period (b) for $\varepsilon=0.48$ and its cycle of quadruple period (c) for $\varepsilon=0.497$ in conservative Croquette system (5) for $\alpha=\omega=1$.

## 3. Hamiltonian Yang-Mills-Higgs system with two degrees of freedom.

The analysis of the chaotic dynamics of classical nonabelian Yang-Mills gauge fields is considered to be very important from the viewpoint of the solution of the well-known confinement problem in quantum chromodynamics (QCD), which, in turn, plays a key role in the construction of the standard model of
elementary particle physics. In (Magnitskii [5]), it was considered the transition to chaos in the simplest case of a Hamiltonian system of homogeneous Yang-Mills-Higgs fields with two degrees of freedom, which is a system of YangMills equations taking into account the interaction of the gauge field with the so-called Higgs vacuum. The Hamiltonian of this interaction has the form

$$
\begin{equation*}
H(x, y)=\frac{\dot{x}^{2}+\dot{y}^{2}}{2}+\frac{x^{2} y^{2}}{2}+\frac{\gamma\left(x^{2}+y^{2}\right)}{2}=\varepsilon \tag{7}
\end{equation*}
$$

The system of equations corresponding to the Hamiltonian (7) has the form

$$
\begin{equation*}
\ddot{x}+x\left(\gamma+y^{2}\right)=0, \quad \ddot{y}+y\left(\gamma+x^{2}\right)=0 \tag{8}
\end{equation*}
$$

passing into the classical system of the Yang-Mills equations at $\gamma=0$. It was shown in (Magnitskii [5]) that the structure of solutions of the Hamiltonian system (7) - (8) is completely determined by cascades of bifurcations of cycles of the extended dissipative system

$$
\begin{equation*}
\dot{x}=u, \quad \dot{u}=-x\left(\gamma+y^{2}\right)-\mu u, \quad \dot{y}=v+(\varepsilon-H), \quad \dot{v}=-y\left(\gamma+x^{2}\right) \tag{9}
\end{equation*}
$$

as the dissipation parameter $\mu$ tends to zero. The stable cycles of the dissipative system generated as a result of cascades of bifurcations are transformed into elliptic cycles of the Hamiltonian system, and their stability regions - into tori around these elliptic cycles. The resulting tori of the Hamiltonian system touch along hyperbolic cycles, into which the corresponding unstable cycles of the dissipative system pass. Unstable cycles are born either together with stable cycles as a result of saddle-node bifurcations, or as a result of fork or period doubling bifurcations. In the vicinity of separatrix surfaces of hyperbolic cycles, new, more complex hyperbolic and elliptic cycles are formed in accordance with the nonlocal effect of multiplication of cycles and tori in conservative systems (see Magnitskii [1-3]). It is shown that it is the latter effect that plays a key role in the system of Yang-Mills-Higgs equations at the initial stage of the transition from regular to chaotic motion. However, let's show now that in transition to chaos an important role is also played by subharmonic cascades of bifurcations in accordance with the Sharkovsky order, occurring in the extended dissipative system and partially preserved in the Hamiltonian system as the dissipation parameter tends to zero.
For $H=1$ the system (8) has four sets of periodic solutions to which there correspond four sets of basic cycles in phase space

$$
\begin{gathered}
C_{x}: y \equiv 0, \quad \dot{x}^{2}+\gamma x^{2}=2 ; \quad C_{y}: x \equiv 0, \quad \dot{y}^{2}+\gamma y^{2}=2 \\
C^{ \pm}: y= \pm x, \quad \dot{x}^{2}+\gamma x^{2}+x^{4} / 2=1
\end{gathered}
$$

Fig. 3 shows the projections onto the plane $(x, u)$ of sections by the plane $y=0$ of two-dimensional tori of the Hamiltonian system (8) around its four main
elliptic cycles $C_{x}, C_{y}$ and $C^{ \pm}$. The cycle $C_{y}$ corresponds to the point $(0,0)$ in the projection, the cycle $C_{x}$ corresponds to the outer contour of the figure, and to the cycles $C^{ \pm}$there correspond two elliptic points lying on the u-axis, and each of the two points in the projection correspond to two intersection points of the plane $y=0$ of two cycles $C^{ \pm}$. Hyperbolic cycles correspond in Fig. 3 to two points lying on the $x$-axis of intersection of the projections of the separatrix surfaces along which different two-dimensional tori touch. Moreover, each of the two points in the projection corresponds to two intersection points of the plane $y=0$ of two cycles. Note that the cycles $C^{ \pm}$which mainly determine the dynamics of system (8) exist and different from the cycles $C_{x}, C_{y}$ for any $\gamma>0$.


Fig. 3. Projections on the plane $(x, u)$ of Poincare sections by the plane $y=0$ of two-dimensional tori of the Hamiltonian system (8) around its four main elliptic cycles for $\gamma=2$ (a) , $\gamma=1$ (b) and for $\gamma=0.55$ (c).

It can be seen from Fig. 3 that for sufficiently large values of the parameter $(\gamma=2)$ there is no chaos in the system (8), and its topology is completely determined by four main cycles and tori around these cycles. For smaller values of the parameter ( $\gamma=1$ ), the topology of system (8) is determined by the main tori of the system around its four main elliptic cycles and by a set of tori around more complex elliptic cycles that were born in the dissipative system (9) initially stable as a result of the saddle-node bifurcations in the vicinity of hyperbolic cycles, when the values of the parameter $\mu$ tend to zero. As a result of computational errors, the movement of the trajectory along the surfaces of such complex multi-turn tori creates the illusion of chaotic movement, which is well observed in various Poincaré sections (Fig. 3b). The tangency of various turns of complex multi-turn tori occurs along hyperbolic cycles that are born in a dissipative system together with stable cycles as a result of saddle-node bifurcations. Since in the vicinities of hyperbolic cycles of the system there are located, intertwining and touching each other, various systems of multi-turn complex tori (internal with respect to the separatrix contour, external and mixed), the global stability of solutions of the system (8) cannot be preserved. At the same time, the example of system (8) shows that hyperbolic and elliptic cycles of a Hamiltonian or conservative system can be the limiting case of stable and saddle cycles of an extended dissipative system which were born not only as
a result of saddle-node bifurcations, but also as a result of fork-type bifurcations and period-doubling bifurcations from the subharmonic cascade of Sharkovsky bifurcations. So, for $\gamma=0.594$ in the dissipative extended system (9) for $\mu \rightarrow 0$, the stable cycle $C_{y}$ loses its stability as a result of a fork-type bifurcation, becoming a saddle one, and two stable cycles are born in its vicinity, passing into elliptic cycles of the Hamiltonian system (8) for $\mu=0$.
The tori arising around these two elliptic cycles are tangent along the hyperbolic cycle $C_{y}$. The same thing happens with the $C_{x}$ cycle. This is clearly seen in Fig. 3c at $\gamma=0.55$. Period-doubling bifurcations of the cycles $C^{ \pm}$occur in the dissipative extended system (9) when $\mu \rightarrow 0$ at $\gamma \approx 0.73$. Twodimensional tori formed in the system (8) around period-doubling cycles that have become elliptic cycles are clearly visible along the diagonals in Fig. 3c for $\gamma=0.55$. The projections onto the plane $(x, u)$ of the cycle $C^{+}$of the Hamiltonian system (8) and cycles of its double and quadruple periods at $\gamma=0.73$ (a), $\gamma=0.55$ (b) and $\gamma=0.534$ (c) are shown in Fig. 4.


Fig.4. Projections of cycles of Hamiltonian system (8) on a plane $(x, u)$ : an initial cycle $C^{+}$at $\gamma=0.73$ (a), its cycle of double period at $\gamma=0.55$ (b) and its cycle of quadruple period at $\gamma=0.534$ (c).

Further, the process continues with the birth of infinitely folded heteroclinic separatrix manifold, stretched over separatrix Feigenbaum tree, both in extended dissipative system (9), and in Hamiltonian system (8) close to it. Accordion of corresponding heteroclinic separatrix zigzag fill the whole phase space of the system, however the limited accuracy of numerical methods does not allow to track this process up to the value $\gamma=0$, corresponding to the system of the Yang-Mills equations.

## 3. Hamiltonian system of Mathieu-Magnitskii equations.

Consider the Hamiltonian system with two degrees of freedom

$$
\begin{equation*}
\dot{x}=y, \quad \dot{y}=-(\delta+z) x-x^{3}, \quad \dot{z}=r, \quad \dot{r}=-z-\frac{x^{2}}{2} \tag{10}
\end{equation*}
$$

System (10), obtained by adding the term $-x^{2} / 2$ to the last equation of a system equivalent to the generalized conservative Mathieu equation, was first
considered in (Magnitskii [1]) and called the Mathieu-Magnitskii system of equations in (Korolkova [7]). Hamiltonian of the system (10) has a kind

$$
H(x, y, z, r)=\frac{\left(\delta x^{2}+y^{2}+z^{2}+r^{2}\right)}{2}+\frac{z x^{2}}{2}+\frac{x^{4}}{4}=\varepsilon
$$

For $\varepsilon>0$ the solution of the system is the cycle $x=y=0$, given by the condition $H=z^{2}+r^{2}=2 \varepsilon$. The system (10) linearized in the vicinity of the cycle $x=y=0$ has the form of the classical Mathieu equation

$$
\begin{equation*}
\ddot{x}+(\delta+\sqrt{2 \varepsilon} \cos t) x=0 . \tag{11}
\end{equation*}
$$

Analysis of cycle multiplicators using the Floquet theory applied to the system (11) makes it possible to obtain the stability conditions for cycle $x=y=0$. The result is the presence of an infinite number of alternating regions of cycle hyperbolicity, numbered in Fig. 5. Outside the numbered areas, the cycle $x=y=0$ is elliptical. The boundaries of the regions with odd numbers are the bifurcation lines of doubling period of the elliptic cycle. When crossing such a boundary, the cycle itself becomes hyperbolic, and an elliptic cycle of a doubled period is born in its vicinity. The boundaries of areas with even numbers are fork type bifurcation lines of the elliptic cycle. When crossing such a boundary, the cycle itself becomes hyperbolic, and in its vicinity two elliptic cycles $C_{1}$ and $C_{2}$ of the same period are born. The region of hyperbolicity of the cycle $x=y=0$ with a number $n$ touches the axis $\delta$ at a point $\delta=(n / 2)^{2}$. With large numbers $n$, the corresponding region approaches the axis $\delta$ with a narrow tongue, the width of which sharply decreases with growth $n$.


Fig.5. Diagram of bifurcations of cycle $x=y=0$ of system (10).

In (Magnitskii [1]) and (Korolkova [7]) it was analyzed further bifurcations of elliptic cycles $C_{1}$ and $C_{2}$ in region with number two for the fixed parameter value $\delta=1$ and when increasing the parameter values $\varepsilon$. For this research the extended dissipative three-parameter system can have the form

$$
\begin{gather*}
\dot{x}=y, \quad \dot{y}=-(\delta+z) x-x^{3}-\mu y, \quad \dot{z}=r \\
\dot{r}=-z-\frac{x^{2}}{2}+(\varepsilon-H(x, y, z, r)) r \tag{12}
\end{gather*}
$$

where dissipation parameter $\mu$ tends to zero. The divergence of the right-hand side of system (12) on solutions of system (10) is equal to $-\mu-r^{2}$ and hence is negative for all $\mu>0$.
For small $\varepsilon>0$, chaotic dynamics is absent in the Hamiltonian system (10), and the topology of the system is specified by basic tori of the system, which include tori around the basic cycle $x=y=0$, pairs of tori around the originally stable basic cycles $C_{1}$ and $C_{2}$ (which are generated in the dissipative system (12) for $\mu>0$ and also generated in Hamiltonian system (10) when crossing the bifurcation line of region 2 in Figure 5), and the transition tori implementing the passage from tori around the basic cycle $x=y=0$ of the system to pairs of tori around the basic cycles $C_{1}$ and $C_{2}$ (Fig. 6a for $\varepsilon=0.125$ ).


Fig. 6. Projections on the plane $(x, y)$ of Poincare sections of two-dimensional tori of the Hamiltonian system (10) around its two main elliptic cycles by the plane $r=0$ for $\varepsilon=0.125$ (a), $\varepsilon=0.5$ (b) and by the plane $x=0$ for $\varepsilon=0.5$ (c).

It is important to note that the tori existing in the Hamiltonian system (10) for any $\varepsilon>0$ have no relation to classical tori of the unperturbed system in KAM (Kolmogorov-Arnold-Mozer) theory for any value of the parameter $\delta$, i.e., for any (rational or irrational) relation of frequencies of the unperturbed system. Moreover, in system (10), as in all other Hamiltonian and simply conservative systems, the tori of the unperturbed system are completely absent, which are considered in the KAM theory as Cartesian products of several (in this case, two) circles. Consequently, contrary to the assertions of KAM theory, the tori of the perturbed system (10) are not any of tori of the unperturbed system for any
$\varepsilon>0$, and the complication of the dynamics of the perturbed system (10), when values of the parameter $\varepsilon$ grow, is not a consequence of the destruction of any tori of the unperturbed system. Obviously, the notions of a perturbed and an unperturbed Hamiltonian systems with two or more degrees of freedom should be re-considered.
For large values of the parameter $\varepsilon$, the situation becomes much more complicated. Now the topology of system (10) is specified by basic tori of the system (these are tori around basic cycles of the system), transition tori, and families of tori around more complex originally stable cycles generated being in the dissipative system (12) in the vicinity of the separatrix surface of the hyperbolic cycle $x=y=0$ as the parameter $\mu$ decays to zero. The chaotic dynamics is absent in the system for now, although the motion along the tori around complex cycles can readily be mistaken for a chaotic one if, for example, it is observed in the Poincare section (Fig. $6 \mathrm{~b}, \mathrm{c}$ for $\varepsilon=0.5$.). The pseudochaos appearing in a neighborhood of the cycle $x=y=0$ is not homoclinic chaos and is only a consequence of computational errors caused by the instability of the considered problem. The chaotic effect is induced by points lying in the Poincare section on surfaces of tori, which are interlaced and lie around different complex cycles formed in the dissipative system (12) for $\mu>0$.
For $\varepsilon=1$, the situation becomes even more complicated. Just as in the previous case, the topology of system (10) is characterized by tori around basic cycles of the system, by transition tori, and by families of tori around more complex originally stable cycles that are generated in the dissipative system (12) in the vicinity of the separatrix surface of the hyperbolic cycle $x=y=0$ as the parameter $\mu$ decays to zero. (Fig. 7a,b).


Fig. 7. Projections of Poincare sections of two-dimensional tori of the Hamiltonian system (10) around its elliptic cycles by the planes $r=0$ and $x=0$ for $\varepsilon=1(\mathrm{a}, \mathrm{b})$, and by the plane $x=0$ for $\varepsilon=4(\mathrm{c})$.

In addition, the phase space of system (10) contains an essential, at first glance, domain of chaotic motion. However, it is reasonable to assume that the chaos shown in Fig. 7a,b is pseudochaos, since it is not generated by cascades of bifurcations of stable cycles of a dissipative system but either is a result of
errors in the computation of the trajectory on surfaces of complex many-turn interlacing tori formed by stability domains of complex cycles or is induced by stability domains of complex cycles of the dissipative system (12) themselves, which do not become tori around these cycles for $\mu=0$. It was not possible to detect infinite cascades of bifurcations of the FShM theory in the dissipative system (12) as the dissipation parameter tends to zero.
But such cascades of bifurcations were found at large values of the parameter $\varepsilon$ in the extended dissipative system (12) and then, as a consequence, in the original Hamiltonian system (10). In (Korolkova [7]) this method was used to find bifurcations of doubling periods of elliptic cycles $C_{1}$ and $C_{2}$ of the Hamiltonian system (10), generated by the hyperbolic cycle $x=y=0$, occurring at $\varepsilon \approx 1.25$. However, at values of the parameter $\varepsilon \approx 1.47$, there are reverse bifurcations of the degeneration of a cycles of a doubled periods into the cycles of period one. Let us show that in system (10) there is a region of values of the parameter $\varepsilon$ in which bifurcations occur of the birth of not only cycles of period two, but also cycles of periods four and eight from the cascade of the Feigenbaum period-doubling bifurcations. Figure 8 shows the projections onto the plane $(x, r)$ of cycles of periods 1,2 and 4 from the cascade of the Feigenbaum period-doubling bifurcations for Hamiltonian system (10) for $\delta=1, \varepsilon=3.21$ (a), $\varepsilon=3.468$ (b) and $\varepsilon=3.666$ (c).


Fig. 8. Projections on the plane ( $x, r$ ) of cycles of the Hamiltonian system (10) of period 1 for $\varepsilon=3.21$ (a), its cycle of double period for $\varepsilon=3.468$ (b) and its cycle of quadruple period for $\varepsilon=3.666$ (c).

Thus, for $\varepsilon=4$, the case of which is shown in Fig. 7 c , the Hamiltonian system (10) necessarily has chaotic dynamics since when the parameter $\mu$ is decreasing, then the cycles $C_{1}$ and $C_{2}$ undergo a subharmonic and homoclinic cascades of bifurcations in full accordance with the FShM theory and generate infinitely many regular and singular attractors in the closures of their separatrix surfaces. Then two strips (separatrix surfaces) of singular attractors merge, and for $\mu \approx 0.208$ two symmetric stable cycles are generated and undergo cascades of subharmonic bifurcations. All cycles of all cascades and singular attractors become unstable but do not vanish and become solutions of the Hamiltonian system for $\mu=0$. For $\mu=0.13$, there appears a symmetric stable cycle generating two symmetric cycles by a fork-type bifurcation for $\mu \approx 0.05$. These
cycles remain stable in system (12) for all smaller values of $\mu>0$ and generate tori of the Hamiltonian system (10) around them for $\mu=0$. These last tori are shown in Fig. 8c.
Therefore, the topology of the Hamiltonian system (10) for $\varepsilon=4$ is determined by tori around basic cycles of the system, by tori around cycles that are generated stable and remain so in the dissipative system (12) for all values of the parameter $\mu>0$, and by the domain of chaotic motion containing all cycles and nonperiodic originally stable trajectories that are generated in the dissipative system (12) for $\mu>0$ and lose their stability as a result of some subharmonic or homoclinic cascade of bifurcations in accordance with the FShM theory.

## Conclusions

Using the exampleы of a numerical analysis of the bifurcations of elliptic cycles of the nonlinear conservative Crockett equation, the Hamiltonian system of Yang-Mills-Higgs equations with two degrees of freedom, and the Hamiltonian system of Mathieu-Magnitskii equations, it is directly demonstrated that the transition to chaos in conservative and Hamiltonian systems occurs as a result of the creation of new complex of multiturn tori in the vicinity of hyperbolic cycles, as well as a result of cascades of bifurcations of elliptic cycles in accordance with the universal bifurcation scenario of Feigenbaum-SharkovskyMagnitskii, and not as a result of the destruction of some mythical tori of unperturbed systems in accordance with the Kolmogorov-Arnold-Moser theory, as is commonly believed in modern Hamiltonian mechanics.

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# Nonlinear Feedback Controller For Adaptive Generalized Hybrid Projective Synchronization Between Two Identical Chaotic Systems 

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#### Abstract

Synchronization of chaotic systems is an important research problem in chaos theory. In this research work, a novel generalised hybrid synchronization of identical uncertain chaotic systems is investigated, where the master system is synchronized by the sum of hybrid state variables for the slave system. According to the Lyapunov stability theorem, a new adaptive nonlinear controller for the synchronization is designed, and some parameter update laws for estimating the unknown parameters of these systems are also gained. The synchronization between two identical Zeraoulia systems and two identical Vaidynathan systems are studied to show the effectiveness of the proposed method.


Keywords: Adaptive Control, Chaotic systems, Hybrid Synchronization, Chaos.

## 1 introduction

Since the idea of synchronising chaotic systems was first introduced in 1990, by pecora and caroll [1], chaos synchronisation has been widely explored in a variety of fields including cryptography [2], neural network [3], ecological systems [4], secure communications [5,6] etc.
Various control schemes have been developed to investigate the synchronisation problem such as OGY method [7], Nonlinear feedback control method[8], timedelay feedback method [9], backstepping method [10], adaptive design method [11], etc.
Up to now different type of synchronization have been pesented such as: complete synchronization [12], anti-synchronization [13], phase synchronization [14], projective synchronization [15,16], where the master system can synchronize with the slave system by the scaling factor in the traditional projective synchronization.
In generalized hybrid projective synchronization with uncertain parameters, the master system is synchronized by the sum of hybrid state variables for the slave system, in ref [17], a new scheme is designed for this type of synchronization, and the advantage of this scheme is that the master and slave system are not required to have the same number of uncertain parameters, which made the research of great importance due to the possibility of applying it to most types of dynamic systems.
In the same research [17], this scheme was applied to two different master and slave systems, in this paper, the scheme is applied to achieve generalized
hybrid projective synchronization when the master and slave systems are identical. The rest of the paper is organized as follows: In Section 2, we explain the scheme that is used to achieve general hybrid projective synchronization. In Section 3, the general hybrid projective synchronization between two identical Zeraoulia systems and two identical Vaidynathan systems are studied with a numerical simulation given to demonstrate the effectiveness of the proposed scheme. The Section 4 concludes the paper.

## 2 The scheme of adaptive generalized hybrid projective synchronization

In this section, we present the scheme designed in research [17] to achieve adaptive generalized hybrid projective synchronization of uncertain chaotic systems.

Consider the master and slave systems respectively:

$$
\begin{gather*}
\dot{x}=f(x)+F(x) \alpha  \tag{1}\\
\dot{y}=g(y)+G(y) \beta+u \tag{2}
\end{gather*}
$$

Where $f, g: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ are two continuous vector functions, $F: \mathbb{R}^{n} \longrightarrow$ $\mathbb{R}^{n \times m}, G: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n \times p}$ are two continuous matrix functions, $\alpha=$ $\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right)^{T} \in \mathbb{R}^{m}, \beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{p}\right)^{T} \in \mathbb{R}^{p}$ are the unknown constants parameters vectors of the system. According to [17], the master and slave systems can be written as:

$$
\begin{gather*}
\dot{x_{i}}=f_{i}(x)+\sum_{j=1}^{m} F_{i j} \alpha_{j}, 1 \leq i \leq n  \tag{3}\\
\dot{y_{i}}=g_{i}(y)+\sum_{j=1}^{p} G_{i j} \beta_{j}+u_{i}, 1 \leq i \leq n \tag{4}
\end{gather*}
$$

The error dynamics of adaptive generalized hybrid projective synchronization can be described as:

$$
\begin{equation*}
e_{i}=x_{i}+\sum_{j=1, j \neq i}^{n} d_{i j} y_{j} \tag{5}
\end{equation*}
$$

The scaling matrix $D$ contains the scaling factors $d_{i j}$ is given by:

$$
D=\left(\begin{array}{ccccc}
0 & d_{12} & \cdots & d_{1 n-1} & d_{1 n}  \tag{6}\\
d_{21} & 0 & & & d_{2 n} \\
\vdots & & \ddots & & \vdots \\
d_{n-1} & & & \ddots & d_{n-1 n} \\
d_{n 1} & d_{n 2} & \cdots & d_{n n-1} & 0
\end{array}\right)
$$

Therefore, the goal of control is to design an appropriate controller $u$ for the slave system and the update laws for the parameter estimates to achieve
the generalized hybrid projective synchronization.
According to [17], if $|D| \neq 0$, we select the controller $u$ in the following form:

$$
\begin{equation*}
u=D^{-1} A \tag{7}
\end{equation*}
$$

ie

$$
\begin{equation*}
u_{i}=\sum_{j=1}^{n} a_{i j} A_{j} \tag{8}
\end{equation*}
$$

Where $D^{-1}=\left(a_{i j}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ is the invers of the matrix $D$, and $A=\left(A_{1}, A_{2}, \ldots, A_{n}\right)^{T}$ is given as follows:

$$
\begin{equation*}
A_{i}=-f_{i}(x)-\sum_{j=1}^{m} F_{i j}(x) \tilde{\alpha}_{j}-\sum_{j=1, j \neq i}^{n} d_{i j}\left[\left(g_{j}(y)+\sum_{k=1}^{p} G_{j k}(y) \tilde{\beta}_{k}\right)\right]-e_{i} \tag{9}
\end{equation*}
$$

And the update laws are given by:

$$
\begin{gather*}
\dot{\alpha_{i}}=\sum_{j=1}^{n} F_{j i}(x) e_{j}+\left(\alpha_{i}-\tilde{\alpha}_{i}\right)  \tag{10}\\
\dot{\tilde{\beta}_{i}}=\sum_{j=1}^{n}\left(\sum_{k=1, k \neq j}^{n} d_{j k} G_{k i}(y)\right) e_{j}+\left(\beta_{i}-\tilde{\beta}_{i}\right) \tag{11}
\end{gather*}
$$

## 3 Application of scheme for two identical chaotic systems

In this section, the scheme described in the previous paragraph will be applied to two examples to demonstrate the effectiveness of the method.

### 3.1 Synchronization between Two Identical Zeraoulia Chaotic Systems

In this subsection, as the master system, we consider the Zeraoulia [18] chaotic system

$$
\left\{\begin{array}{l}
\dot{x_{1}}=\alpha_{1}\left(x_{2}-x_{1}\right)+x_{2} x_{3}  \tag{12}\\
\dot{x_{2}}=\alpha_{2} x_{2}-x_{1} x_{3} \\
\dot{x_{3}}=x_{1} x_{2}-\alpha_{3} x_{3}
\end{array}\right.
$$

The slave system is also taken as the Zeraoulia chaotic system with controllers attached and given by

$$
\left\{\begin{array}{l}
\dot{y_{1}}=\beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+u_{1}  \tag{13}\\
\dot{y_{2}}=\beta_{2} y_{2}-y_{1} y_{3}+u_{2} \\
\dot{y_{3}}=y_{1} y_{2}-\beta_{3} y_{3}+u_{3}
\end{array}\right.
$$

Where $x_{1}, x_{2}, x_{3}$ and $y_{1}, y_{2}, y_{3}$ are the states of the two systems and $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are unknown constant parameters of the the master system, $\beta_{1}, \beta_{2}, \beta_{3}$ are unknown constant parameters of the the slave system, and $u_{1}, u_{2}, u_{3}$ are the controllers to be found

The Zeraoulia system depicts a strange chaotic attractor when the constant parameter values are taken as

$$
\alpha_{1}=36, \alpha_{2}=25, \alpha_{3}=3
$$

Compare systems (12) and (13) with Eqs. (3) and (4) we know that

$$
\begin{array}{c|l|l|l}
f_{1}(x)=x_{2} x_{3} & F_{11}(x)=x_{2}-x_{1} & F_{12}(x)=0 & F_{13}(x)=0 \\
f_{2}(x)=-x_{1} x_{3} & F_{21}(x)=0 & F_{22}(x)=x_{2} & F_{23}(x)=0 \\
f_{3}(x)=x_{1} x_{2} & F_{31}(x)=0 & F_{32}(x)=0 & F_{33}(x)=-x_{3} \\
\begin{array}{c|l|l|l}
g_{1}(y)=y_{2} y_{3} & G_{11}(y)=y_{2}-y_{1} & G_{12}(y)=0 & G_{13}(y)=0 \\
g_{2}(y)=-y_{1} y_{3} & G_{21}(y)=0 & G_{22}(y)=y_{2} & G_{23}(y)=0 \\
g_{3}(y)=y_{1} y_{2} & G_{31}(y)=0 & G_{32}(y)=0 & G_{33}(y)=-y_{3}
\end{array}
\end{array}
$$

The hybrid synchronization error is defined by

$$
\left\{\begin{array}{l}
e_{1}=x_{1}+d_{12} y_{2}+d_{13} y_{3} \\
e_{2}=x_{2}+d_{21} y_{1}+d_{23} y_{3} \\
e_{3}=x_{3}+d_{31} y_{1}+d_{32} y_{2}
\end{array}\right.
$$

where $d i j$ are the scaling constants.
From (12) and (13), the error dynamics can be obtained as follows

$$
\left\{\begin{array}{l}
\dot{e_{1}}=\alpha_{1}\left(x_{2}-x_{1}\right)+x_{2} x_{3}+d_{12}\left(\beta_{2} y_{2}-y_{1} y_{3}\right)+d_{13}\left(y_{1} y_{2}-\beta_{3} y_{3}\right)+d_{12} u_{2}+d_{13} u_{3} \\
\dot{e_{2}}=\alpha_{2} x_{2}-x_{1} x_{3}+d_{21} \beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+d_{23}\left(y_{1} y_{2}-\beta_{3} y_{3}\right)+d_{21} u_{1}+d_{23} u_{3} \\
\dot{e_{3}}=x_{1} x_{2}-\alpha_{3} x_{3}+d_{31} \beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+d_{32}\left(\beta_{2} y_{2}-y_{1} y_{3}\right)+d_{31} u_{1}+d_{32} u_{2}
\end{array}\right.
$$

According to (9)

$$
\left\{\begin{array}{l}
A_{1}=-\tilde{\alpha_{1}}\left(x_{2}-x_{1}\right)-d_{12}\left(\tilde{\beta}_{2} y_{2}-y_{1} y_{3}\right)-d_{13}\left(y_{1} y_{2}-\tilde{\beta}_{3} y_{3}\right)-e_{1} \\
A_{2}=-\tilde{\alpha}_{2} x_{2}+x_{1} x_{3}-d_{21}\left(\tilde{\beta}_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}\right)-d_{23}\left(\tilde{y}_{1} y_{2}-\tilde{\beta}_{3} y_{3}\right)-e_{2} \\
A_{3}=-x_{1} x_{2}+\tilde{\alpha}_{3} x_{3}-d_{31}\left(\tilde{\beta}_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}\right)-d_{32}\left(\widetilde{\beta}_{2} y_{2}-y_{1} y_{3}\right)-e_{3}
\end{array}\right.
$$

Let the control law be as follows:

$$
\left\{\begin{array}{l}
u_{1}=\frac{1}{|D|}\left[\left(-d_{32} d_{23}\right) A_{1}+\left(d_{32} d_{13}\right) A_{2}+\left(d_{12} d_{23}\right) A_{3}\right]  \tag{14}\\
u_{2}=\frac{1}{|D|}\left[\left(d_{31} d_{23}\right) A_{1}+\left(-d_{31} d_{13}\right) A_{2}+\left(d_{13} d_{21}\right) A_{3}\right] \\
u_{3}=\frac{1}{|D|}\left[\left(d_{21} d_{32}\right) A_{1}+\left(d_{12} d_{31}\right) A_{2}+\left(-d_{21} d_{12}\right) A_{3}\right]
\end{array}\right.
$$

By (10) and (11) the update laws for unknown parameters are given as following

$$
\left\{\begin{array}{l}
\dot{\alpha_{1}}=\left(x_{2}-x_{1}\right) e_{1}+\left(\alpha_{1}-\tilde{\alpha_{1}}\right)  \tag{15}\\
\dot{\alpha_{2}}=x_{2} e_{2}+\left(\alpha_{2}-\tilde{\alpha_{2}}\right) \\
\dot{\tilde{\alpha_{3}}}=-x_{3} e_{3}+\left(\alpha_{3}-\tilde{\alpha_{3}}\right)
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\dot{\beta_{1}}=d_{21}\left(y_{2}-y_{1}\right) e_{2}+d_{31}\left(y_{2}-y_{1}\right) e_{3}+\left(\beta_{1}-\tilde{\beta}_{1}\right)  \tag{16}\\
\dot{\tilde{\beta}_{2}}=d_{12} y_{2} e_{1}+d_{32} y_{2} e_{3}+\left(\beta_{2}-\tilde{\beta}_{2}\right) \\
\dot{\tilde{\beta_{3}}}=-d_{13} y_{3} e_{1}-d_{23} y_{3} e_{2}+\left(\beta_{3}-\tilde{\beta_{3}}\right)
\end{array}\right.
$$

The identical Zeraoulia system (12) and (13) are asymptotically hybridsynchronized for all initial conditions with the adaptive controller $u$ defined by (14) and the update laws (15) and (16).

Numerical Results For numerical simulations, we take the parameters of the Zeraoulia systems as in the chaotic case, i.e.

$$
\alpha_{1}=36, \alpha_{2}=25, \alpha_{3}=3
$$

The initial values of the master system (12) are taken as

$$
x_{1}(0)=5, \quad x_{2}(0)=8, \quad x_{3}(0)=4
$$

The initial values of the slave system (13) are taken as

$$
y_{1}(0)=1, \quad y_{2}(0)=7, \quad y_{3}(0)=3
$$

The initial values for the parameter estimates of the master system are taken as

$$
\tilde{\alpha}_{1}(0)=28, \quad \tilde{\alpha}_{2}(0)=12, \quad \tilde{\alpha}_{3}(0)=46
$$

The initial values for the parameter estimates of the slave system are taken as

$$
\tilde{\beta}_{1}(0)=17, \quad \tilde{\beta}_{2}(0)=10, \quad \tilde{\beta}_{3}(0)=30
$$

The errors have initial conditions

$$
e_{1}(0)=1, \quad e_{2}(0)=6, \quad e_{3}(0)=-8
$$

we choose the scaling factors such that $|D| \neq 0$

$$
d_{12}=3, \quad d_{13}=2, \quad d_{21}=-2.5, \quad d_{23}=3, \quad d_{31}=-1.5, \quad d_{32}=1
$$

The simulation results are shown in Figs. 1-3. Fig 1 and Fig 2 show that the estimates $\tilde{\alpha}_{1}, \quad \tilde{\alpha}_{2}, \quad \tilde{\alpha}_{3}$ and $\tilde{\beta}_{1}, \quad \tilde{\beta}_{2}, \tilde{\beta}_{3}$ of the unknown parameters can converge to $\alpha_{1}=36, \alpha_{2}=25, \alpha_{3}=3$ and $\beta_{1}=36, \beta_{2}=25, \beta_{3}=3$. Fig 3 shows synchronization errors between the slave system (13) and the master system (12).


Fig. 1. Estimated unknown parameters of the master Zeraoulia chaotic system


Fig. 2. Estimated unknown parameters of the slave Zeraoulia chaotic system

### 3.2 Synchronization between Two Identical Vaidyanathan Chaotic System

In this subsection, as the master system, we consider the Vaidyanathan [19] chaotic system which is described by

$$
\left\{\begin{array}{l}
\dot{x_{1}}=\alpha_{1}\left(x_{2}-x_{1}\right)+x_{2} x_{3}  \tag{17}\\
\dot{x_{2}}=\alpha_{2} x_{1}+\alpha_{3} x_{2}-x_{1} x_{3} \\
\dot{x_{3}}=x_{1}^{2}-\alpha_{4} x_{3}
\end{array}\right.
$$

The slave system is also taken as the Vaidyanathan chaotic system with controllers attached and given by

$$
\left\{\begin{array}{l}
\dot{y_{1}}=\beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+u_{1}  \tag{18}\\
\dot{y_{2}}=\beta_{2} y_{1}+\beta_{3} y_{2}-y_{1} y_{3}+u_{2} \\
\dot{y_{3}}=y_{1}^{2}-\beta_{4} y_{3}+u_{3}
\end{array}\right.
$$



Fig. 3. Synchronization errors $e_{1} ; e_{2} ; e_{3}$ between identical Zeraoulia systems (12) and (13)

Where $x_{1}, x_{2}, x_{3}$ and $y_{1}, y_{2}, y_{3}$ are the states of the two systems and $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ are unknown constant parameters of the the master system, $\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}$ are unknown constant parameters of the the slave system, and $u_{1}, u_{2}, u_{3}$ are the controllers to be found

The Vaidyanathan system depicts a strange chaotic attractor when the constant parameter values are taken as

$$
\alpha_{1}=25, \alpha_{2}=33, \alpha_{3}=11, \alpha_{4}=6
$$

Compare systems (17) and (18) with Eqs. (3) and (4) we know that

$$
\begin{array}{l|l}
f_{1}(x)=x_{2} x_{3} & g_{1}(y)=y_{2} y_{3} \\
f_{2}(x)=-x_{1} x_{3} & g_{2}(y)=-y_{1} y_{3} \\
f_{3}(x)=x_{1}^{2} & g_{3}(y)=y_{1}^{2}
\end{array}
$$

$$
\begin{array}{l|l|l|c}
F_{11}(x)=x_{2}-x_{1} & F_{12}(x)=0 & F_{13}(x)=0 & F_{14}(x)=0 \\
F_{21}(x)=0 & F_{22}(x)=x_{1} & F_{23}(x)=x_{2} & F_{24}(x)=0 \\
F_{31}(x)=0 & F_{32}(x)=0 & F_{33}(x)=0 & F_{34}(x)=-x_{3}
\end{array}
$$

$$
\begin{array}{l|l|l|c}
G_{11}(y)=y_{2}-y_{1} & G_{12}(y)=0 & G_{13}(x)=0 & G_{14}(x)=0 \\
G_{21}(y)=0 & G_{22}(y)=y_{1} & G_{23}(y)=y_{2} & G_{24}(y)=0 \\
G_{31}(y)=0 & G_{32}(y)=0 & G_{33}(y)=0 & G_{34}(y)=-y_{3}
\end{array}
$$

The hybrid synchronization error is defined by

$$
\left\{\begin{array}{l}
e_{1}=x_{1}+d_{12} y_{2}+d_{13} y_{3} \\
e_{2}=x_{2}+d_{21} y_{1}+d_{23} y_{3} \\
e_{3}=x_{3}+d_{31} y_{1}+d_{32} y_{2}
\end{array}\right.
$$

where $d i j$ are the scaling constants.

It is easy to see from (17) and (18) that the error dynamics can be obtained as follows

$$
\left\{\begin{array}{l}
\dot{e_{1}}=\alpha_{1}\left(x_{2}-x_{1}\right)+x_{2} x_{3}+d_{12}\left(\beta_{2} y_{1}+\beta_{3} y_{2}-y_{1} y_{3}+u_{2}\right)+d_{13}\left(y_{1}^{2}-\beta_{4} y_{3}+u_{3}\right) \\
\dot{e_{2}}=\alpha_{2} x_{1}+\alpha_{3} x_{2}-x_{1} x_{3}+d_{21}\left(\beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+u_{1}\right)+d_{23}\left(y_{1}^{2}-\beta_{4} y_{3}+u_{3}\right) \\
\dot{e_{3}}=x_{1}^{2}-\alpha_{4} x_{3}+d_{31}\left(\beta_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}+u_{1}\right)+d_{32}\left(\beta_{2} y_{1}+\beta_{3} y_{2}-y_{1} y_{3}+u_{2}\right)
\end{array}\right.
$$

According to (6) let

$$
|D|=d_{12} d_{31} d_{23}+d_{13} d_{21} d_{32}
$$

The invers of the matrix $D$ is given by

$$
D^{-1}=\frac{1}{|D|}\left(\begin{array}{ccc}
-d_{32} d_{23} & d_{32} d_{13} & d_{12} d_{23} \\
d_{31} d_{23} & -d_{31} d_{13} & d_{13} d_{21} \\
d_{21} d_{32} & d_{12} d_{31} & -d_{21} d_{12}
\end{array}\right)=\left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)
$$

According to (9)

$$
\left\{\begin{array}{l}
A_{1}=-\left(x_{2}-x_{1}\right) \tilde{\alpha_{1}}-x_{2} x_{3}-d_{12}\left(\tilde{\beta}_{2} y_{1}+\tilde{\beta}_{3} y_{2}-y_{1} y_{3}\right)-d_{13}\left(y_{1}^{2}-\tilde{\beta}_{4} y_{3}\right)-e_{1} \\
A_{2}=-\tilde{\alpha_{2}} x_{1}-\tilde{\alpha_{3}} x_{2}+x_{1} x_{3}-d_{21}\left(\tilde{\beta}_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}\right)-d_{23}\left(\tilde{y}_{1}^{2}-\tilde{\beta}_{4} y_{3}\right)-e_{2} \\
A_{3}=-x_{1}^{2}+\tilde{\alpha_{4}} x_{3}-d_{31}\left(+\widetilde{\beta}_{1}\left(y_{2}-y_{1}\right)+y_{2} y_{3}\right)-d_{32}\left(\tilde{\beta}_{2} y_{1}+\widetilde{\beta}_{3} y_{2}-y_{1} y_{3}\right)-e_{3}
\end{array}\right.
$$

Let the control law be as follows:

$$
\left\{\begin{array}{l}
u_{1}=a_{11} A_{1}+a_{12} A_{2}+a_{13} A_{3} \\
u_{2}=a_{21} A_{1}+a_{22} A_{2}+a_{23} A_{3} \\
u_{3}=a_{31} A_{1}+a_{32} A_{2}+a_{33} A_{3}
\end{array}\right.
$$

ie

$$
\left\{\begin{array}{l}
u_{1}=\frac{1}{|D|}\left[\left(-d_{32} d_{23}\right) A_{1}+\left(d_{32} d_{13}\right) A_{2}+\left(d_{12} d_{23}\right) A_{3}\right]  \tag{19}\\
u_{2}=\frac{1}{|D|}\left[\left(d_{31} d_{23}\right) A_{1}+\left(-d_{31} d_{13}\right) A_{2}+\left(d_{13} d_{21}\right) A_{3}\right] \\
u_{3}=\frac{1}{|D|}\left[\left(d_{21} d_{32}\right) A_{1}+\left(d_{12} d_{31}\right) A_{2}+\left(-d_{21} d_{12}\right) A_{3}\right]
\end{array}\right.
$$

By (10) and (11) the update laws for unknown parameters are given as following

$$
\left\{\begin{array}{l}
\dot{\tilde{\alpha}}_{1}=F_{11}(x) e_{1}+F_{21}(x) e_{2}+F_{31}(x) e_{3}+\left(\alpha_{1}-\tilde{\alpha}_{1}\right) \\
\dot{\tilde{\alpha}}_{2}=F_{12}(x) e_{1}+F_{22}(x) e_{2}+F_{32}(x) e_{3}+\left(\alpha_{2}-\tilde{\alpha_{2}}\right) \\
\dot{\tilde{\alpha}}_{3}=F_{13}(x) e_{1}+F_{23}(x) e_{2}+F_{33}(x) e_{3}+\left(\alpha_{3}-\tilde{\alpha}_{3}\right) \\
\dot{\tilde{\alpha}}_{4}=F_{14}(x) e_{1}+F_{24}(x) e_{2}+F_{34}(x) e_{3}+\left(\alpha_{4}-\tilde{\alpha_{4}}\right)
\end{array}\right.
$$

ie

$$
\left\{\begin{array}{l}
\dot{\tilde{\alpha}}_{1}=\left(x_{2}-x_{1}\right) e_{1}+\left(\alpha_{1}-\tilde{\alpha_{1}}\right)  \tag{20}\\
\tilde{\tilde{\alpha}}_{2}=x_{1} e_{2}+\left(\alpha_{2}-\tilde{\alpha_{2}}\right) \\
\dot{\tilde{\alpha}}_{3}=x_{2} e_{2}+\left(\alpha_{3}-\tilde{\alpha_{3}}\right) \\
\dot{\tilde{\alpha}}_{4}=-x_{3} e_{3}+\left(\alpha_{4}-\tilde{\alpha_{4}}\right)
\end{array}\right.
$$

and

$$
\left\{\begin{array}{c}
\dot{\tilde{\beta}_{1}}=\left(d_{12} G_{21}(y)+d_{13} G_{31}(y)\right) e_{1}+\left(d_{21} G_{11}(y)+d_{23} G_{31}(y)\right) e_{2} \\
\quad+\left(d_{31} G_{11}(y)+d_{32} G_{21}(y)\right) e_{3}+\left(\beta_{1}-\widetilde{\beta}_{1}\right) \\
\dot{\tilde{\beta}_{2}}=\left(d_{12} G_{22}(y)+d_{13} G_{32}(y)\right) e_{1}+\left(d_{21} G_{12}(y)+d_{23} G_{32}(y)\right) e_{2} \\
\quad+\left(d_{31} G_{12}(y)+d_{32} G_{22}(y)\right) e_{3}+\left(\beta_{2}-\widetilde{\beta}_{2}\right) \\
\dot{\tilde{\beta_{3}}}=\left(d_{12} G_{23}(y)+d_{13} G_{33}(y)\right) e_{1}+\left(d_{21} G_{13}(y)+d_{23} G_{33}(y)\right) e_{2} \\
\quad+\left(d_{31} G_{13}(y)+d_{32} G_{23}(y)\right) e_{3}+\left(\beta_{3}-\widetilde{\beta}_{3}\right) \\
\dot{\tilde{\beta}_{4}}=\left(d_{12} G_{24}(y)+d_{13} G_{34}(y)\right) e_{1}+\left(d_{21} G_{14}(y)+d_{23} G_{34}(y)\right) e_{2} \\
+\left(d_{31} G_{14}(y)+d_{32} G_{24}(y)\right) e_{3}+\left(\beta_{4}-\widetilde{\beta}_{4}\right)
\end{array}\right.
$$

ie

$$
\left\{\begin{array}{l}
\dot{\tilde{\beta_{1}}}=d_{21}\left(y_{2}-y_{1}\right) e_{2}+d_{31}\left(y_{2}-y_{1}\right) e_{3}+\left(\beta_{1}-\tilde{\beta}_{1}\right)  \tag{21}\\
\dot{\tilde{\beta}_{2}}=d_{12} y_{1} e_{1}+d_{32} y_{1} e_{3}+\left(\beta_{2}-\tilde{\beta}_{2}\right) \\
\dot{\tilde{\beta}_{3}}=d_{12} y_{2} e_{1}+d_{32} y_{2} e_{3}+\left(\beta_{3}-\tilde{\beta}_{3}\right) \\
\dot{\tilde{\beta^{4}}}=-d_{13} y_{3} e_{1}-d_{23} y_{3} e_{2}+\left(\beta_{4}-\tilde{\beta}_{4}\right)
\end{array}\right.
$$

The identical Vaidynathan systems (17) and (18) are globally and asymptotically general hybrid projective synchronized for all initial conditions $x(0), y(0)$ in $\mathbb{R}^{3}$ with the novel adaptive controller $u$ defined by (19), and the update laws defined by (20) and (21).


Fig. 4. Estimated unknown parameters of the master Vaidynathan chaotic system


Fig. 5. Estimated unknown parameters of the slave Vaidynathan chaotic system


Fig. 6. Synchronization errors $e_{1} ; e_{2} ; e_{3}$ between identical Vaidynathan systems (17) and (18)

Numerical Results For the numerical simulations, the fourth-order RungeKutta method with time-step 0.001, is used to solve the Vaidynathan systems (17) and (18) with the adaptive controller $u$ given by (19) and the update laws (20) and (21). We take the parameters of the Vaidynathan systems as in the chaotic case, i.e.

$$
\alpha_{1}=25, \alpha_{2}=33, \alpha_{3}=11, \alpha_{4}=6
$$

The initial values of the master system (17) are taken as

$$
x_{1}(0)=5, \quad x_{2}(0)=7, \quad x_{3}(0)=1
$$

The initial values of the slave system (18) are taken as

$$
y_{1}(0)=1, \quad y_{2}(0)=4, \quad y_{3}(0)=2
$$

The initial values for the parameter estimates of the master system are taken as

$$
\tilde{\alpha}_{1}(0)=5, \quad \tilde{\alpha}_{2}(0)=10, \quad \tilde{\alpha}_{3}(0)=20, \quad \tilde{\alpha}_{4}(0)=30
$$

The initial values for the parameter estimates of the slave system are taken as

$$
\tilde{\beta}_{1}(0)=5, \quad \tilde{\beta}_{2}(0)=10, \quad \tilde{\beta}_{3}(0)=20, \quad \tilde{\beta}_{4}(0)=30
$$

The errors have initial conditions

$$
e_{1}(0)=1, \quad e_{2}(0)=2, \quad e_{3}(0)=3
$$

we choose the scaling factors such that $|D| \neq 0$

$$
d_{12}=-3, \quad d_{13}=1, \quad d_{21}=1, \quad d_{23}=-2, \quad d_{31}=4, \quad d_{32}=-3
$$

The simulation results are shown in Figs.4-6. Fig 4 and Fig 5 show that the estimates $\tilde{\alpha}_{1}, \quad \tilde{\alpha}_{2}, \quad \tilde{\alpha}_{3}, \quad \tilde{\alpha}_{4}$ and $\tilde{\beta}_{1}, \tilde{\beta}_{2}, \tilde{\beta}_{3}, \tilde{\beta}_{4}$ of the unknown parameters can converge to $\alpha_{1}=25, \alpha_{2}=33, \alpha_{3}=11, \alpha_{4}=6$ and $\beta_{1}=25, \beta_{2}=33$, $\beta_{3}=11, \beta_{4}=6$ respectively. Fig 6 shows synchronization errors between the slave system (18) and the master system (17).

## 4 Conclusions

In this paper, a scheme for generalized hybrid projective synchronization between two identical chaotic systems is presented. This scheme has been successfully applied between two identical Zeraoulia chaotic systems and two identical Vaidynathan chaotic systems. Based on Lyapnov stability theory and the adaptive control theory, an adaptive nonlinear feedback control is designed. The numerical simulation demonstrated the effectiveness of the control used in this scheme.

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[^0]:    $14^{\text {th }}$ CHAOS Conference Proceedings, 8-11 June 2021, Athens, Greece

[^1]:    Abstract
    2.1 Introduction
    2.2 Main equations
    2.3 Action of harmonic signal on the linearized Bullard dynamo
    2.4 Action of the Gaussian delta-correlated noise and the Langevin stochastic process on the linearized Bullard dynamo
    2.5 Conclusion

    References

[^2]:    14 ${ }^{\text {th }}$ CHAOS Conference Proceedings, 8 - 11 June 2021, Athens, Greece

[^3]:    ${ }^{1}$ From http://web.williams.edu/Astronomy/Course-Pages/330/images/forces.jpg.

[^4]:    ${ }^{2}$ Strictly speaking, the Hausdorff dimension is more involved that a usual fractal capacity dimension. The boxes needed to cover a set may vary in sizes and one needs to take a supremum of the cover of the set.

